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SOME RESULTS ON CONNECTED REGULAR DOMINATION OF ZERO-DIVISOR GRAPHS

V. MAHESWARI[1], Y. KAYATHIRI, R. RAJESWARI, AND J. ANUF SARUMATHI

ABSTRACT. The concept of connected regular domination number of $\Gamma(\mathbb{Z}_n)$ was introduced by K. Ananthi, J. Ravi Sankar and N. Selvi. A connected regular domination of a graph $G(V, E)$ is defined as the vertex set $D \subseteq V$ satisfying the following conditions, (i) D is Dominating set (ii) D is regular and (iii) D is connected. In this paper, we extend the notion of connected regular domination number for some product and line graph of zero-divisor graphs.

1. INTRODUCTION

Let R be a commutative ring and let $Z(R)$ be its set of zero-divisors. We consider a graph $\Gamma(R)$ with vertices $Z^*(R) = Z(R) - \{0\}$, the set of non-zero zero-divisors of R and for distinct $x, y \in Z^*(R)$, the vertices x and y are adjacent iff $xy = 0$. The first instances of associating graph with various algebraic structures is due to Beck [4] who introduced the idea of zero-divisor graph of a commutative ring with unity and later on Anderson [3], Akbari and Mohammadian [1] continued the study of zero-divisor graph by considering only the non-zero zero-divisors. The connected regular dominating set of $\Gamma(\mathbb{Z}_n)$ was suggested by K. Ananthi, J. Ravi Sankar and N. Selvi [2]. In this paper, we have extended their work to find connected regular domination number of $\Gamma(\mathbb{Z}_n)$ of some product and line graph on commutative rings.

[1]*corresponding author*

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Definition 1.1. *Let R be a ring. A non-zero element $a \in R$ is said to be a zero-divisor if there exists a non-zero element $b \in R$ such that $ab = 0$ or $ba = 0$.*

Definition 1.2. *A dominating set is a set of vertices such that each vertex of V is either in D or has atleast one neighbour in D. The minimum cardinality of such a set is called the domination number of the graph G and it is denoted by $\gamma(G)$.*

Definition 1.3. *A connected dominating set D is a set of vertices of a graph G such that every vertex in $V - D$ is adjacent to atleast one vertex in D and the subgraph $< D >$ induced by the set D is connected. The connected domination number $\gamma_c(G)$ is the minimum cardinality of the connected dominating set of G.*

Definition 1.4. *A regular dominating set is a dominating set D of $V(G)$ if $< D >$ is regular. The minimum cardinality of a regular dominating set is called regular domination number of G and is denoted by $\gamma_r(G)$.*

Definition 1.5. *Let G be a graph, V is a vertex set of G and $D \subseteq V$, then D is said to be connected regular dominating set, if it satisfies the following conditions, (i) D is dominating set (ii) D is connected and (iii) D is regular. The minimum cardinality of connected regular dominating set is called connected regular domination number of G and is denoted by $\gamma_{cr}(G)$.*

2. CONNECTED REGULAR DOMINATION NUMBER OF ZERO-DIVISOR GRAPHS

In this section we evaluate the connected regular domination number of some product and line graph of $\Gamma(\mathbb{Z}_n)$.

Theorem 2.1. *For any graph $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$, where $p > 2$ is any prime number, holds $\gamma_{cr}(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)) = 1$.*

Proof. The vertex set of $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$ is defined as $V = \{(0,1), (0,2), (0,3, ..., (0, p-1), (1,0)\}=\{v_1, v_2, v_3, ..., v_{(p-1)}, v_p\}$.

The total number of vertices in $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$ is p. $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$ is a star graph.

It is clear that, v_p is adjacent to every other vertices in V. Thus, $\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p))$ is one. $\{v_p\}$ is itself connected and regular. Therefore, $\gamma_{cr}(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)) = 1$. $\square$

Theorem 2.2. *For any graph $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{np})$, where n is a non-zero positive integer and $p > 2$ is any prime number, holds $\gamma_{cr}(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{np})) = 1$.*

Proof. The vertex set of $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{np})$ is defined as $V = \{(0,1),(0,2),(0,3),...,(0,np-1),(1,0)\}=\{v_1,v_2,v_3,...,v_{(np-1)},v_{np}\}$.

The total number of vertices in $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{np})$ is np. $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{np})$ is a star graph.

It is clear that, v_{np} is adjacent to every other vertices in V. Thus, $\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{np}))$ is one. $\{v_{np}\}$ is itself connected and regular.

Therefore, $\gamma_{cr}(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{np}))) = 1$. □

Theorem 2.3. *For any graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$, where $q > p$ and p, q are odd prime numbers, holds $\gamma_{cr}(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)) = 2$.*

Proof. The vertex set of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ is defined as $V = \{(0,1),(0,2),...,(0,q-1),(1,0),(2,0),...,(p-1,0)\}$. The total number of vertices in $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ is $p+q-2$.

It is clear that, $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ is a complete bi-partite graph. So, we can split the vertex set into two set of vertices V_1 and V_2.

Let $V_1 = \{(0,1),(0,2),...,(0,q-1)\}=\{v_{11},v_{12},...,v_{1(q-1)}\}$ and $V_2 = \{(1,0),(2,0),...,(p-1,0)\}=\{v_{21},v_{22},...,v_{2(p-1)}\}$.

Clearly, any vertex from the vertex set V_1 is adjacent to all vertices in V_2. Similarly, any vertex from the vertex set V_2 is adjacent to all vertices in V_1.

Therefore, any one vertex from V_1 and one vertex from V_2 form the dominating set. Thus, $\gamma(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q))$ is two.

The induced subgraph formed from the dominating set is both connected as well as regular. Therefore, $\gamma_{cr}(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)) = 2$. □

Theorem 2.4. *For any graph $\Gamma(\mathbb{Z}_{p^3})$, where p is any prime number, holds $\gamma_{cr}(\Gamma(\mathbb{Z}_{p^3})) = 1$.*

Proof. The vertex set of the graph $\Gamma(\mathbb{Z}_{p^3})$ is $\{p,2p,3p,...,(p^2-1)p,p^2,2p^2,...,(p-1)p^2\}$. Decompose the vertex set into the following disjoint sets as multiples of p but not p^2 and multiples of p^2 respectively.

- $V_1 = \{p,2p,3p,...,(p^2-1)p\}$
- $V_2 = \{p^2,2p^2,...,(p-1)p^2\}$

Clearly, the vertices in V_2 are adjacent to each other and all vertices in V_1. Any vertex from V_2 form the dominating set with minimal cardinality.

The induced subgraph formed from the dominating set is both connected as well as regular. Therefore, $\gamma_{cr}(\Gamma(\mathbb{Z}_{p^3})) = 1$. □

Theorem 2.5. *For any graph $\Gamma(\mathbb{Z}_{pq^2})$, where $q > p$ and p, q are distinct prime numbers, holds $\gamma_{cr}(\Gamma(\mathbb{Z}_{pq^2})) = 2$.*

Proof. The vertex set of the graph $\Gamma(\mathbb{Z}_{pq^2})$ is $\{p, 2p, 3p, ..., (q-1)p, pq, (q+1)p, ..., (q^2-1)p, q, 2q, ..., (p-1)q, (p+1)q, ..., ((q-1)p-1)q, ((q-1)p+1)q, ..., (pq-1)q, 2pq, ..., (q-1)pq, q^2, 2q^2, ..., (p-1)q^2\}$.

Decompose the vertex set into the following sets as multiples of p, q, pq and q^2 respectively.

- $V_1 = \{p, 2p, (q-1)p, (q+1)p, ..., (q^2-1)p\}$
- $V_2 = \{q, 2q, ..., (p-1)q, (p+1)q, ..., ((q-1)p-1)q, ((q-1)p+1)q, ..., (pq-1)q\}$
- $V_3 = \{pq, 2pq, ..., (q-1)pq\}$
- $V_4 = \{q^2, 2q^2, ..., (p-1)q^2\}$

Clearly, the vertices in V_3 are adjacent to all vertices in V_2 and V_4 and also V_1 and V_4 forms a complete bipartite graph.

$\{pq, q^2\}$ forms the dominating set with minimal cardinality. The induced subgraph formed from the dominating set is both connected as well as regular.

Therefore, $\gamma_{cr}(\Gamma(\mathbb{Z}_{pq^2})) = 2$. $\square$

Theorem 2.6. *Let $L(\Gamma(\mathbb{Z}_n))$ be a line graph of $\Gamma(\mathbb{Z}_n)$. If $n = 2p$ where $p > 2$ is any prime number then $\gamma_{cr}(L(\Gamma(\mathbb{Z}_n)))$ is 1.*

Proof. If $n = 2p$, then $\Gamma(\mathbb{Z}_n)$ is a star graph. So there is a common vertex which is adjacent to all other vertices and that vertex is also called the centre of the graph.

We draw the line graph of $\Gamma(\mathbb{Z}_n)$, for $n = 2p$. Let v_1 be the common vertex of $\Gamma(\mathbb{Z}_n)$, which is end point of every edge of $\Gamma(\mathbb{Z}_n)$. Then v_1 appears in every vertex of the line graph.

$(v_1, u_i) \in V(L(\Gamma(\mathbb{Z}_n)))$, where $\{u_i = 2, 2.2, 2.3, ..., 2(p-1), p = v_1\}$.

The line graph of forms a complete graph. Therefore, $D = \{(v_1, u_i)\}$ is the minimum dominating set with cardinality 1. So, $\gamma_{cr}(L(\Gamma(\mathbb{Z}_n)))$ is 1. $\square$

Theorem 2.7. *For any graph $L(\Gamma(\mathbb{Z}_{3p}))$, where $p \geq 3$ is an odd prime number, $\gamma_{cr}(L(\Gamma(\mathbb{Z}_{3p})))$ is 2.*

Proof. If $n = 3p$ then $\Gamma(\mathbb{Z}_n)$ is a complete bipartite graph. The vertex set of $\Gamma(\mathbb{Z}_n)$ is $\{3, 6, 9, ..., 3(p-1), p, 2p\}$. Decompose the vertex set into the following disjoint subsets:

- the set S of non-zero multiples of 3 which are less than n.
- the set M of multiples of prime p which are less than n.

In $\Gamma(\mathbb{Z}_n)$, every vertex of M is adjacent to all the vertices of S, but the distinct vertices of S are not adjacent to each other. These two vertices of $\Gamma(\mathbb{Z}_n)$ appears in line graph as the end points of edges.

Let (p, v_1) and $(2p, v_2) \in V(L(V))$, they are not adjacent to each other but (p, v_1) is adjacent to $(p, v_i) \in V(L(\Gamma(\mathbb{Z}_n)))$ where v_i is multiple of 3 and $2 \leq i \leq p-1$.

Similarly, $(2p, v_1)$ is adjacent to $(2p, v_i) \in V(L(\Gamma(\mathbb{Z}_n)))$.

Then (p, v_1) and $(2p, v_2)$ are two vertices adjacent to remaining all vertices of line graph $\Gamma(\mathbb{Z}_n)$.

Therefore, $D = \{(p, v_i), (2p, v_i)\}$ is a minimum dominating set. The induced subgraph formed from the dominating set is both connected as well as regular. Therefore, $\gamma_{cr}(L(\Gamma(\mathbb{Z}_n))) = 2$. □

Theorem 2.8. *For any graph $L(\Gamma(\mathbb{Z}_{pq}))$, where p, q are odd prime numbers and $2 < p < q$, it holds $\gamma_{cr}(L(\Gamma(\mathbb{Z}_n))) = p - 1$.*

Proof. Let $\Gamma(\mathbb{Z}_n)$ be a line graph of $\Gamma(\mathbb{Z}_n)$. If $n = pq$ where p, q are odd prime numbers and $2 < p < q$. Then the vertex set of $L(\Gamma(\mathbb{Z}_n))$ is
$V = \{(q,p), (q,2p), (q,3p), ..., (q, p(q-1)), (2q,p), (2q,2p), ..., (2q, p(q-1)),$
$(3q,p), (3q,2p), ..., (3q, p(q-1)), (4q,p), (4q,2p), ..., (4q, p(q-1)), ..., ((p-1)q, p),$
$((p-1)q, 2p), ..., ((p-1)q, p(q-1))\}$.

Decompose the vertex set of line graph of $\Gamma(\mathbb{Z}_n)$ into the following disjoint subsets.

The set $P = \{(q,p), (q,2p), (q,3p), ..., (q, p(q-1))\}$.
The set $Q = \{(2q,p), (2q,2p), (2q,3p), ..., (2q, p(q-1))\}$.
The set $R = \{(3q,p), (3q,2p), (3q,3p), ..., (3q, p(q-1))\}$.

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The set $Y = \{((p-1)q, p), ((p-1)q, 2p), ((p-1)q, 3p), ..., ((p-1)q, (q-1))\}$.
The edge set $E(G)$ of line graph of $\Gamma(\mathbb{Z}_n)$ is defined by

$$E(G) = \{((u_1, v_1), (u_2, v_2))/u_1 = u_2 \text{ or } v_1 = v_2\}$$

i.e., either first ordinate is common or second ordinate is common.

Let us consider a vertex $(q,p) \in P$. So (q,p) is adjacent to all vertices of P because q is the common and (q,p) is adjacent to only $(2q,p) \in Q, (3q,p) \in R, ..., ((p-1)q,p) \in Y$ but (q,p) is not adjacent to the remaining vertices. So that $D = \{(q,p)\}$ is not a dominating set.

Let us consider another vertex $(2q,p) \in Q$. It is adjacent to all the vertices of Q. Since $2q$ is the common and $(2q,p)$ is also adjacent to $(q,p) \in P, (3q,p) \in R, ..., ((p-1)q,p) \in Y$. So that $D = \{(q,p),(2q,p)\}$ is not a dominating set, since some of the vertices of $R, S, ..., Y$ are not adjacent to D.

Let us consider another vertex $(3q,p) \in R$. $(3q,p)$ is adjacent to all the vertices of R because $3q$ is the common in R and also $(3q,p)$ is adjacent to $(q,p) \in P, (2q,p) \in Q, ..., ((p-1)q,p) \in Y$. The remaining vertices of $S, T, ..., Y$ are not adjacent. Therefore, $D = \{(q,p),(2q,p),(3q,p)\}$ is not a dominating set.

Continuing like this let us consider $((p-1)q,p) \in Y$. $((p-1)q,p)$ is adjacent to all the vertices of Y and it is adjacent to only a single vertex in each set. i.e.,$(q,p) \in P, (2q,p) \in Q, (3q,p) \in R, ...$

Therefore, $D = \{(q,p),(2q,p),(3q,p), ..., ((p-1)q,p)\}$ is a minimum dominating set with cardinality $p-1$. The induced subgraph formed from the dominating set is both connected as well as regular. Therefore, $\gamma_{cr}(L(\Gamma(\mathbb{Z}_n))) = p-1$. □

Theorem 2.9. *For any graph $L(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{np}))$, where n is a non-zero positive integer and $p > 2$ is any prime number, holds $\gamma_{cr}(L(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{np}))) = 1$.*

Proof. The vertex set of $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{np})$ is defined as $V = \{(0,1),(0,2),(0,3), ..., (0,np-1),(1,0)\} = \{v_1, v_2, v_3, ..., v_{np-1}, v_{np}\}$.

The total number of vertices in $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{np})$ is np. $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{np})$ is a star graph.

Clearly, the line graph of $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{np})$ is a complete graph.

Therefore, $\gamma_{cr}(L(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{np}))) = 1$. □

Theorem 2.10. *For any graph $L(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q))$, where $q > p$ and p, q are odd prime numbers, it holds $\gamma_{cr}(L(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q))) = p-1$.*

Proof. The vertex set of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ is defined as $V = \{(0,1),(0,2), ..., (0,q-1),(1,0),(2,0), ..., (p-1,0)\}$. The total number of vertices in $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ is $p+q-2$.

It is clear that, $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ is a complete bi-partite graph. So, we can split the vertex set into two set of vertices V_1 and V_2.

Let $V_1 = \{(0,1),(0,2),...,(0,q-1)\}=\{v_{11},v_{12},...,v_{1(q-1)}\}$ and $V_2 = \{(1,0),(2,0),...,(p-1,0)\}=\{v_{21},v_{22},...,v_{2(p-1)}\}$.

Clearly, any vertex from the vertex set V_1 is adjacent to all vertices in V_2. Similarly, any vertex from the vertex set V_2 is adjacent to all vertices in V_1.

Now, the vertex set of $L(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q))$ is $\{(v_{11},v_{21}),(v_{11},v_{22}),...,(v_{11},v_{2(p-1)}),...,(v_{1(q-1)},v_{21}),(v_{1(q-1)},v_{22}),...,(v_{1(q-1)},v_{2(p-1)})\}$. The edge set $E(G)$ of line graph of $\Gamma(\mathbb{Z}_n)$ is defined by

$$E(G) = \{((u_1,v_1),(u_2,v_2))/u_1 = u_2 \ or \ v_1 = v_2\}.$$

Let us consider a vertex $(v_{11},v_{21}) \in V(L(\Gamma(\mathbb{Z}_n)))$. (v_{11},v_{21}) is adjacent to all (v_{11},v_{2i}) and (v_{1j},v_{21}) but (v_{11},v_{21}) is not adjacent to $(v_{1k},v_{22}),(v_{1k},v_{23}),...,(v_{1k},v_{2(p-1)})$ where $2 \leq k \leq q-1$.

Similarly, (v_{12},v_{22}) is adjacent to all (v_{12},v_{2i}) and (v_{1j},v_{22}) but (v_{12},v_{22}) is not adjacent to $(v_{1k},v_{21}),(v_{1k},v_{23}),(v_{1k},v_{24}),...,(v_{1k},v_{2(p-1)})$ where $k = 1$ and $3 \leq k \leq q-1$.

Continuing like this we get the dominating set as $\{(v_{11},v_{21}),(v_{11},v_{22}),...,(v_{11},v_{2(p-1)})\}$. The induced subgraph formed from the dominating set is both connected as well as regular. Therefore, $\gamma_{cr}(L(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q))) = p-1$. □

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PG AND RESEARCH DEPARTMENT OF MATHEMATICS
A.P.C. MAHALAXMI COLLEGE FOR WOMEN, THOOTHUKUDI
E-mail address: maheswari@apcmcollege.ac.in

PG AND RESEARCH DEPARTMENT OF MATHEMATICS
A.P.C. MAHALAXMI COLLEGE FOR WOMEN, THOOTHUKUDI
E-mail address: kayathiri906@gmail.com

PG AND RESEARCH DEPARTMENT OF MATHEMATICS
A.P.C. MAHALAXMI COLLEGE FOR WOMEN, THOOTHUKUDI

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A.P.C. MAHALAXMI COLLEGE FOR WOMEN, THOOTHUKUDI