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FUZZY t-DERIVATIONS B*-IDEALS IN BCI-ALGEBRAS

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ABSTRACT. In this paper, we introduce the notion of B*-Ideals in BCI-Algebra. In an associative BCI-Algebra, we prove that every B*-Ideal is a BCI-Ideal (a-ideal, pideal and H-ideal) and converse of the statement is also true. We apply the definition of B*-Ideals into left (right) t-derivations in BCI-Algebra and prove that every t derivations BCI-Ideal is a t-derivations subalgebra in BCI-Algebra. Furthermore, we extend the definition of t-derivations B*-Ideal into Fuzzy left (right) t-derivations B*-Ideal. Also, we prove that every fuzzy t-derivations B*-Ideal of a BCI-Algebra preserves the order. We give the definition of fuzzy closed t-derivations B*-Ideal and prove that every fuzzy t-regular derivation B*-Ideal is a fuzzy closed t-derivation B*-Ideal of an associative BCI-Algebra. Finally, we define the fuzzy t-derivations subalgebra and prove that every fuzzy closed t-derivation B*-Ideal is a fuzzy t derivations subalgebra of a BCI-Algebra.

1. INTRODUCTION

In 1966, Y. Imai and K. Iseki [3] introduced the two classes of logical algebras, namely, BCK-Algebras and BCI-Algebras. Their names were derived from Combinators B (cut), C (exchange), I (identity) and K (weakening) in combinatory logic. It is known that the class of BCI-algebra is a superclass of a class of BCK-Algebra and subclass of Groupoids. K. Iseki proposed the concept ideals in

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BCK-algebras. H. E. Bell and G. Mason [2] presented derivations in near-rings and near-fields in 1987. Jun and Xin [5] had applied the notion of derivations in ring and near-ring theory to BCI-algebras. In 2012, G. Muhiuddin and M. Al-roqi [6] given the t-derivations in BCI-Algebra/p-semisimple BCI-Algebra.

The theory of fuzzy sets was first inspired by Zadeh [9] in 1965. Rosenfeld framed the concept of fuzzy subgroups, latterly these concepts have been applied to the algebraic structures like groups, rings, nearrings and ideals in 1977. O. G. Xi extended the concept of fuzzy sets to BCK-Algebras. In 1994, Y. B. Jun and E. H. Roh applied the concept of ideals to BCK-Algebras. In 2015, Mostafa and Hassan given the new idea of Fuzzy derivations BCC-Ideals on BCC-Algebras. Recently, Tamadhur and Ahmad [7] introduced the view of fuzzy left (right)-derivations BCI-Ideals, p-Ideals and H-Ideals on BCI-Algebra in 2018.

In the scope of this paper, first we define the concept of B*-Ideals in BCI-Algebra and we give the connection between B*-Ideal and other ideals like BCI-Ideal, a-Ideal, p-Ideal and H-Ideal in BCI-Algebra. We apply the concept of BCI-Ideal and B*-Ideal into left (right) t-derivations in BCI-Algebra. Further, we define the fuzzy t-derivations B*-Ideals in BCI-Algebras and study some of its relevant properties.

2. Preliminaries

We initiate with some following definitions and results that will be essential for our results.

Definition 2.1. *[4,8] An algebra $(X;*,0)$ is called a BCI-Algebra if for all $x,y,z \in X$ it satisfies the following conditions:*

(BCI-1): $((x*y)*(x*z))*(z*y)=0$;
(BCI-2): $(x*(x*y))*y=0$;
(BCI-3): $x*x=0$;
(BCI-4): $x*y=0$ *and* $y*x=0 \Rightarrow x=y$.

Here, the binary operation '$*$' on X is called a '$*$' multiplication on X, and constant 0 of X is the zero element of X.

Suppose that $(X;*,0)$ is a BCI-Algebra. Define a binary relation $\leq$ on X with $x \leq y$ if and only if $x*y=0$ for any $x,y \in X$. Then $(X;\leq)$ is a partially ordered set with 0 as a minimal element in the meaning that $x \leq 0$ implies $x=0$ for any $x \in X$. Let $x,\ y,\ z$ be any elements in a BCI-Algebra X. Then the following properties hold:

(p1) : $(x * y) * z = (x * z) * y$

(p2) : $x * 0 = x$

(p3) : $(x * y) * z = x * (y * z)$ for all $x, y, z \in X$, i.e. A BCI-Algebra X is an associative algebra

Let X be a BCI-Algebra, we denote $X_+ = \{x \in X | 0 \leq x\}$ the BCK-part of X and by $G(x) = \{x \in X | 0 * x = x\}$ the BCI G-part of X. If $X^+ = \{0\}$, then X is called a p-semisimple BCI-Algebra.

If X is a p-semisimple BCI-Algebra, then for all $x, y, z \in X$, the following properties hold:

(p4) : $x * (x * y) = y$

(p5) : $x * (0 * y) = y * (0 * x)$

(p6) : $(x * z) * (y * z) = x * y$.

Definition 2.2. *[8] A nonempty subset S of a BCI-Algebra X is called subalgebra of X if $x * y \in S$ for every $x, y \in S$.*

Definition 2.3. *[1, 7, 8] Let X be a BCI-Algebra.*

(1) *A subset $I(6 = \phi)$ of X is called a BCI-Ideal (Ideal) of X if it satisfies:*
  (i) $0 \in I$ and
  (ii) $x * y \in I$ and $y \in I \Rightarrow x \in I$, for every $x, y \in X$.

(2) *An ideal I of X is called a closed ideal, if $0 * x \in$I implies $x \in I$, for every $x \in X$.*

(3) *A subset $I(6 = \phi)$ of X is called p-Ideal of X if it satisfies:*
  (i) $0 \in I$ and
  (ii) $(x * z) * (y * z) \in I$ and $y \in I \Rightarrow x \in I$, for every $x, y, z \in X$.

(4) *A subset $I(6 = \phi)$ of X is called H-Ideal or q-Ideal of X if it satisfies:*
  (i) $0 \in I$ and
  (ii) $x * (y * z) \in I,\ y \in I \Rightarrow x * z \in I$, for every $x, y, z \in X$.

(5) *A subset $I(6 = \phi)$ of X is called an a-ideal of X if it satisfies:*
  (i) $0 \in I$ and
  (ii) $(x*z)*(0*y) \in I$ and $z \in I \Rightarrow y*x \in I$, for every $x, y, z \in X$.

Theorem 2.1. *[1] A BCI-algebra X is associative if and only if $0 * x = x$ for all $x \in X$.*

Theorem 2.2. *[1] A BCI-algebra X is associative if and only if it is both quasi-associative and p-semisimple.*

Definition 2.4. *[6] If X is a BCI-Algebra, we define a self-map $d_t : X \to X$ for any $t \in X$, by $d_t(x) = x * t$ for every $x \in X$.*

Let X be a BCI-Algebra, we express $x \wedge y = y * (y * x)$ for every $x, y \in X$.

Definition 2.5. *[6] If a self map $d_t : X \to X$ on a BCI-Algebra X satisfies the identity $d_t(x * y) = (d_t(x) * y) \wedge (x * d_t(y))$ for every $x, y \in X$, then d_t is said to be a left-right $t-$derivation or $(l, r) - t$-derivation of X.*

If a self map $d_t : X \to X$ satisfies the identity $d_t(x * y) = (x * d_t(y)) \wedge (d_t(x) * y)$ for all $x, y \in X$, then d_t is said to be a right-left t-derivation or (r, l)-t-derivation of X. Moreover, if d_t is both a (l, r)- and a $(r, l) - t$-derivation on X, we say that d_t is a t-derivation on X.

Definition 2.6. *[6] A self map d_t of a BCI-Algebra X is said to be t-regular if $d_t(0) = 0$.*

A t-derivation d_t is a t-regular in a BCI-Algebra [6] .

Theorem 2.3. *[6] Let d_t be a t-regular $(r, l) - t$-derivation of a BCI-Algebra X. Then, the following hold:*

(i) $d_t(x) * y \leq x * d_t(y)$, *for all* $x, y \in X$.
(ii) $d_t(x * y) = d_t(x) * y \leq d_t(x) * d_t(y)$, *for all* $x, y \in X$.

Definition 2.7. *[9] In a BCI-Algebra X, a function $\alpha : X \to [0, 1]$ is called a fuzzy set of X.*

3. B*-IDEAL

In this section, we define the notion of B*-Ideals in BCI-Algebras. Also, we give the relationship between B*-Ideals and other ideals in an associative BCI-Algebra.

Definition 3.1. *A subset $I(6 = \phi)$ of a BCI-Algebra $(X, *, 0)$ is called a B*-Ideal of X if it obeys the below conditions:*

$(B_1^*) : 0 \in I$;
$(B_2^*) : (a * b) * c \in I$ *and* $a \in I \Rightarrow b * c \in I$, *for every* $a, b, c \in X$.

Example 1. *Let $X = \{0, 1, 2\}$ be a BCI-Algebra with the following Cayley Table.*

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Let us take $I = \{0, 1, 2\}$, then it can be easily verified that I is a B*-Ideal of X.

Theorem 3.1. *Let X be an associative BCI-Algebra. Every B*-Ideal I of X is a subalgebra of X.*

Proof. Let X be an associative BCI-Algebra and I be a B*-Ideal of X.

Let $a, b \in X$. Put $b = a * b$ and $c = 0$ in (B_2^*), we get $(a * (a * b)) * 0 \in I$ and $a \in I$ implies $(a * b) * 0 \in I$.

By the properties (p_2) and (p_4), we get $b \in I$ and $a \in I$ implies $a * b \in I$. Hence I is a subalgebra of X. □

Theorem 3.2. *Let X be an associative BCI-Algebra. Every B*-Ideal I of X is a BCI Ideal (ideal) of X.*

Proof. Let X be an associative BCI-Algebra and I be a B*-Ideal of X.

Let $a, b \in X$. Put $a = b * a$ in (B_2^*), we get $((b * a) * b) * c \in I$ and $b * a \in I \Rightarrow b * c \in I$. Replace a by b, b by a and Put $c = 0$, $((a * b) * a) * 0 \in I$ and $a * b \in I \Rightarrow a * 0 \in I$.

By the properties (p_2) and (p_1), $(a * a) * b \in I$ and $a * b \in I \Rightarrow a \in I$.

$0 * b \in I$ and $a * b \in I \Rightarrow a \in I$. $b \in I$ and $a * b \in I \Rightarrow a \in I$. $a * b \in I$ and $b \in I \Rightarrow a \in I$. Hence I is a BCI-Ideal of X. □

Corollary 3.1. *Every B*-Ideal I of an associative BCI-Algebra X is a p-Ideal of X.*

Theorem 3.3. *In an associative BCI-Algebra X, every B*-Ideal I of X is a q-Ideal (H-Ideal) of X.*

Proof. Let X be an associative BCI-Algebra and I be a B*-Ideal of X.

Let $a, b, c \in X$. Replace a by b and b by a in (B_2^*), we have $(b * a) * c \in I$ and $b \in I \Rightarrow a * c \in I$, for all $a, b, c \in X$.

By the property (p_2), $((b * 0) * a) * c \in I$ and $b \in I \Rightarrow a * c \in I$, for all $a, b, c \in X$.

By the property (p_3), $(b * (0 * a)) * c \in I$ and $b \in I \Rightarrow a * c \in I$, for all $a, b, c \in X$.

By the property (p_5), $(a*(0*b))*c \in I$ and $b \in I \Rightarrow a*c \in I$, for all $a,b,c \in X$. $(a*b)*c \in I$ and $b \in I \Rightarrow a*c \in I$, for all $a,b,c \in X$.

Again by the property (p_3), $a*(b*c) \in I$ and $b \in I \Rightarrow a*c \in I$, for all $a,b,c \in X$. Hence I is a q-Ideal of X. □

Theorem 3.4. *Every B*-Ideal I of an associative BCI-Algebra X, is an a-Ideal of X.*

Proof. Let X be an associative BCI-Algebra and I be a B*-Ideal of X.

Let $a,b,c \in X$. Replace a by c and c by a in (B_2^*), $(c*b)*a \in I$ and $c \in I \Rightarrow b*a \in I$, for all $a,b,c \in X$.

By the property (p_1), $(c*a)*b \in I$ and $c \in I \Rightarrow b*a \in I$, for all $a,b,c \in X$.

By Theorem 2.1, $(c*(0*a))*b \in I$ and $c \in I \Rightarrow b*a \in I$, for all $a,b,c \in X$.

By the property (p_5), $(a*(0*c))*b \in I$ and $c \in I \Rightarrow b*a \in I$, for all $a,b,c \in X$.

Again by Theorem 2.1, we get $(a*c)*(0*b) \in I$ and $c \in I \Rightarrow b*a \in I$, for all $a,b,c \in X$. Hence I is an a-Ideal of X. □

Note 1. *If X is an associative BCI-Algebra and I is a B*-Ideal of X, then I is a BCI-Ideal of X. Similarly, the B*-Ideal of X is also a p-Ideal, q-Ideal and a-Ideal of X.*

The converse of the above theorem is true. i.e., Every BCI-Ideal is a B*-Ideal of an associative BCI-Algebra X. Similarly, p-Ideal, q-Ideal and a-Ideal are also an B*-Ideal of an associative BCI-Algebra X. See Figure 1.

Suppose if X is not an associative BCI-Algebra, then the above results are not true.

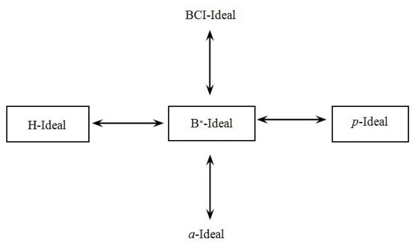

FIGURE 1. Relation between B*-Ideals and other Ideals in Associative BCI-Algebra

4. t-DERIVATIONS B*-IDEALS

Definition 4.1. *Let d_t be a self-map from X to itself in X. A subset $I(6=\phi)$ of X is called left t-derivations BCI-Ideal or (briefly, $l-d_t$ -BCI) of X if it obeys the below conditions:*

($l-d_t$ -BCI$_1$) : $0\in I$;

$(l-d_t-BCI_2)$: $d_t(a)*b\in I$ *and* $d_t(b)\in I\Rightarrow d_t(a)\in I$ *for all* $a,b\in I$.

Definition 4.2. *Let d_t be a self-map from X to itself in X. A subset $I(6=\phi)$ of X is called right t-derivations BCI-Ideal or (briefly, $r-d_t$ -BCI) of X if it obeys the below conditions:*

$(r-d_t-BCI_1):0\in I$;

$(r-d_t-BCI_2)$: $a*d_t(b)\in I$ *and* $d_t(b)\in I\Rightarrow d_t(a)\in I$ *for all* $a,b\in I$.

Definition 4.3. *Let d_t be a self-map from X to itself in X. A subset $I(6=\phi)$ of X is called t-derivations BCI-Ideal or (briefly, d_t -BCI) of X if it obeys the below conditions:*

$(d_t-BCI_1):0\in I$;

(d_t-BCI_2) : $d_t(a*b)\in I$ *and* $d_t(b)\in I\Rightarrow d_t(a)\in I$ *for all* $a,b\in I$.

Example 2. *Let $X=\{0,1,2\}$ be a BCI-Algebra (refer Example 1.) For any $t\in X$, define a self-map $d_t:X\to X$ by*

$$d_t(x)=x*t=\begin{cases}0 & if\ x=0,1\\ 2 & otherwise.\end{cases}$$

Take $I=\{0,1,2\}$, then it can be easily verified that I is a t-derivations BCI-Ideal of X.

Definition 4.4. *Let d_t be a self-map from X to itself in X. A subset $I(6=\phi)$ of X is called left t-derivations B*-Ideal ($l-d_t$ -B*) of X if it obeys the below conditions:*

$(l-d_t-B_1^*):0\in I$;

$(l-d_t-B_2^*)$: $d_t(a*b)*c\in I$ *and* $d_t(a)\in I\Rightarrow d_t(b*c)\in I$ *for all* $a,b,c\in I$.

Definition 4.5. *Let d_t be a self-map from X to itself in X. A subset $I(6=\phi)$ of X is called right t-derivations B*-Ideal ($r-d_t$ -B*) of X if it obeys the below conditions:*

$(r-d_t-B_1^*):0\in I$;

$(r-d_t-B_2^*)$: $(a*b)*d_t(c)\in I$ *and* $d_t(a)\in I\Rightarrow d_t(b*c)\in I$ *for all* $a,b,c\in I$.

Definition 4.6. *Let d_t be a self-map from X to itself in X. A subset $I(6 = \phi)$ of X is called t-derivations B*-Ideal $(d_t$ -B*$)$ of X if it obeys the below conditions:*

$(d_t - B_1^*) : 0 \in I$;

$(d_t - B_2^*) : d_t((a * b) * c) \in I$ *and* $d_t(a) \in I \Rightarrow d_t(b * c) \in I$ *for all* $a, b, c \in I$.

Example 3. *Let $X = \{0, 1, 2\}$ be a BCI-Algebra (refer Example 1.) For any $t \in X$, define a self-map $d_t : X \to X$ by*

$$d_t(x) = x * t = \begin{cases} 0 & if\ x = 0, 1 \\ 2 & otherwise. \end{cases}$$

Take $I = \{0, 1, 2\}$, then it can be easily verified that I is a t-derivations B-Ideal of X.*

Definition 4.7. *Let X be a BCI-Algebra and d_t be a self-map. A non empty subset S of X is called t-derivations subalgebra of X if $d_t(a), d_t(b) \in S$ implies $d_t(a * b) \in S$ for every $a, b \in X$.*

Theorem 4.1. *Let X be an associative BCI-Algebra and I be a non empty subset of X. I is a t-derivations BCI-Ideal of X if and only if I is a t-derivations subalgebra of X.*

Proof. Let X be an associative BCI-Algebra. Assume that I is a t-derivations BCI-Ideal of X.

Let $a, b \in X$. Put $a = a * b$ in $(d_t - BCI_2) \Rightarrow d_t((a * b) * b) \in I$ and $d_t(b) \in I$ implies $d_t(a * b) \in I$.

By the property (p_3), $d_t(a * (b * b)) \in I$ and $d_t(b) \in I$ implies $d_t(a * b) \in I$.

By definition (BCI-3), $d_t(a * 0) \in I$ and $d_t(b) \in I$ implies $d_t(a * b) \in I$.

By the property (p_2), $d_t(a) \in I$ and $d_t(b) \in I$ implies $d_t(a * b) \in I$. Hence I is a t-derivations subalgebra of X.

Conversely, suppose that I is a t-derivations subalgebra of X.

1) Since I is a non empty subset of X, $0 \in I$.

2) Let $a, b \in X$.

Put $a = a * b$ in the definition of t-derivations subalgebra $\Rightarrow d_t(a * b) \in I$ and $d_t(b) \in I$ implies $d_t((a * b) * b) \in I$.

By the property (p_3), $d_t(a * b) \in I$ and $d_t(b) \in I$ implies $d_t(a * (b * b)) \in I$.

By (BCI-3), $d_t(a * b) \in I$ and $d_t(b) \in I$ implies $d_t(a * 0) \in I$.

By the property (p_2), $d_t(a * b) \in I$ and $d_t(b) \in I$ implies $d_t(a) \in I$. Hence I is a t-derivations BCI-Ideal of X. □

5. Fuzzy t-Derivations B*-Ideals

Definition 5.1. *Let d_t be a self-map on X. A fuzzy set $\alpha : X \to [0,1]$ in X is called fuzzy left t-deivations B*-Ideal of X or $(F_l - d_t - B^*)$ if it satisfies the following conditions:*

$(F_l - d_t - B_1^*) : \alpha(0) \geq \alpha(a)$ *for all* $a \in X$,

$(F_l - d_t - B_2^*) : \alpha(d_t(b * c)) \geq min\{\alpha((d_t(a * b)) * c), \alpha(d_t(a))\}$ *for all* $a, b, c \in X$.

Definition 5.2. *Let d_t be a self-map on X. A fuzzy set $\alpha : X \to [0,1]$ in X is called fuzzy right t-deivations B*-Ideal of X or $(F_r - d_t - B^*)$ if it satisfies the following conditions:*

$(F_r - d_t - B_1^*) : \alpha(0) \geq \alpha(a)$ *for all* $a \in X$,

$(F_r - d_t - B_2^*) : \alpha(d_t(b * c)) \geq min\{\alpha((d_t(a * b)) * (d_t(c))), \alpha(d_t(a))\}$ *for all* $a, b, c \in X$.

Definition 5.3. *Let d_t be a self-map on X. A fuzzy set $\alpha : X \to [0,1]$ in X is called fuzzy t-deivations B*-Ideal of X or $(F - d_t - B^*)$ if it satisfies the following conditions:*

$(F - d_t - B_1^*) : \alpha(0) \geq \alpha(d_t(a))$ *for all* $a \in X$,

$(F - d_t - B_2^*) : \alpha(d_t(b * c)) \geq min\{\alpha((d_t(a * b)) * c), \alpha(d_t(a))\}$ *for all* $a, b, c \in X$.

Example 4. *Let $X = \{0, 1, 2, 3\}$ be a BCI-Algebra with the following Cayley Table.*

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

For any $t \in X$, Define a self-map $d_t : X \to X$ by

$$d_t(x) = x * t = \begin{cases} 0 & if\ x = 0 \\ 2 & if\ x = 1, 2, 3 \end{cases}$$

and define a fuzzy set $\alpha : X \to [0,1]$ by $\alpha(0) = 0.5$, $\alpha(1) = 0.2$, $\alpha(2) = 0.03$ and $\alpha(3) = 0.01$.

It can be easily verified that α is not a fuzzy t-derivations B-Ideal of X.*

Example 5. *Let $X = \{0, 1, 2\}$ be a BCI-Algebra with the following Cayley Table.*

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	1	0

For any $t \in X$, Define a self-map $d_t : X \to X$ by

$$d_t(x) = x * t = \begin{cases} 0 & if\ x = 0, 1 \\ 2 & if\ x = 2 \end{cases}$$

and define a fuzzy set $\alpha : X \to [0, 1]$ by $\alpha(0) = 0.8$, $\alpha(1) = 0.3$ and $\alpha(2) = 0.05$.

It can be easily verified that α is a fuzzy t-derivations B*-Ideal of X.

Lemma 5.1. (i) *Let α be a fuzzy left t-derivations B*-Ideal of X and d_t be t-regular self-map on X. If $d_t(a) \leq b$ then $\alpha(d_t(a)) \leq \alpha(d_t(b))$ for all $a, b \in X$.*

(ii) *Let α be a fuzzy right t-derivations B*-Ideal of X and d_t be t-regular self-map on X. If $a \leq d_t(b)$ then $\alpha(d_t(a)) \leq \alpha(d_t(b))$ for all $a, b \in X$.*

Proof. (i) Let α be a fuzzy left t-derivations B*-Ideal of X and let $a, b \in X$. Assume that $d_t(a) \leq b$ and d_t is a t-regular $\Rightarrow d_t(a) * b = 0$ and $d_t(0) = 0$.

By the property, $a * b \geq d_t(a) * b$ and $d_t(a) * b = 0 \Rightarrow a * b \geq d_t(a) * b = 0$. $\Rightarrow a * b \geq 0$.

Now, consider

$$\begin{aligned} \alpha(d_t(b)) = \alpha(d_t(b * 0)) &\geq min\{\alpha((d_t(a * b)) * 0), \alpha(d_t(a))\} \\ &\geq min\{\alpha(d_t(0) * 0), \alpha(d_t(a))\} \\ &= min\{\alpha(d_t(0)), \alpha(d_t(a))\} \\ &= min\{\alpha(0), \alpha(d_t(a))\} \\ &= \alpha(d_t(a)) \end{aligned}$$

Therefore, $\alpha(d_t(b)) \geq \alpha(d_t(a))$.

(ii) Let α be a fuzzy left t-derivations B*-Ideal of X and let $a, b \in X$. Assume that $a \leq d_t(b)$ and d_t is a t-regular. By the property, $a \leq d_t(b)$ and $d_t(b) \leq b \Rightarrow a \leq b \Rightarrow a * b = 0$.

Now, consider

$$\begin{aligned}\alpha(d_t(b)) &= \alpha(d_t(b*0)) \\ &\geq min\{\alpha((a*b)*d_t(0)), \alpha(d_t(a))\} \\ &\geq min\{\alpha(0*d_t(0)), \alpha(d_t(a))\} \\ &= min\{\alpha(0), \alpha(d_t(a))\} \\ &= \alpha(d_t(a))\end{aligned}$$

Therefore, $\alpha(d_t(b)) \geq \alpha(d_t(a))$.

□

Theorem 5.1. *Every fuzzy t-derivations B*-Ideal α of X preserves the order, i.e.* $d_t(a) \leq d_t(b) \Rightarrow \alpha(d_t(a)) \leq \alpha(d_t(b))$.

Proof. Let α be a t-derivations B*-Ideal of X and $a, b \in X$. Given $d_t(a) \leq d_t(b) \Rightarrow d_t(a) * d_t(b) = 0$. By the property, $d_t(a*b) = d_t(a) * b \leq d_t(a) * d_t(b) \Rightarrow d_t(a*b) \leq d_t(a) * d_t(b) \Rightarrow d_t(a*b) \leq d_t(a) * d_t(b)$ and $d_t(a) * d_t(b) = 0 \Rightarrow d_t(a*b) \leq 0$.

Since $d_t(a*b) \not< 0$. Therefore, $d_t(a*b) = 0$. Consider,

$$\begin{aligned}\alpha(d_t(b)) &= \alpha(d_t(b*0)) \\ &\geq min\{\alpha(d_t((a*b)*0)), \alpha(d_t(a))\} \\ &= min\{\alpha(d_t(a*b)), \alpha(d_t(a))\} \\ &= min\{\alpha(0), \alpha(d_t(a))\} \geq \alpha(d_t(a))\end{aligned}$$

Therefore, $\alpha(d_t(b)) \geq \alpha(d_t(a))$. □

Theorem 5.2. *Let α be a fuzzy t-derivations B*-Ideal of X satisfying the inequality $d_t(a*b) \leq d_t(c)$, then $\alpha(d_t(c)) \geq min\{\alpha(d_t(a)), \alpha(d_t(b))\}$ for all $a, b, c \in X$.*

Proof. Let $a, b, c \in X$. Assume that $d_t(a*b) \leq d_t(c) \Rightarrow d_t(a*b) * d_t(c) = 0$. By the property,

$$d_t(a*b) \leq d_t(a) * d_t(b). \tag{5.1}$$

Then

$$d_t((a*b)*c) \leq d_t(a*b) * d_t(c) \tag{5.2}$$

Comparing equations (5.1) and (5.2) we have:

$$d_t((a*b)*c) \leq 0 \Rightarrow d_t((a*b)*c) = 0\,.$$

Consider,

$$\begin{aligned}\alpha(d_t(c)) &= \alpha(d_t(c*0))\\ &\geq min\{\alpha(d_t((b*c)*0)),\alpha(d_t(b))\} \geq min\{\alpha(d_t(b*c)),\alpha(d_t(b))\}\\ &\geq min\{min\{\alpha(d_t((a*b)*c)),\alpha(d_t(a))\},\alpha(d_t(b))\}\\ &= min\{min\{\alpha(0),\alpha(d_t(a))\},\alpha(d_t(b))\}\\ &= min\{\alpha(d_t(a)),\alpha(d_t(b))\}.\end{aligned}$$

Therefore, $\alpha(d_t(c)) \geq min\{\alpha(d_t(a)),\alpha(d_t(b))\}$ □

Definition 5.4. *A nonempty fuzzy subset α in X is said to be a fuzzy closed t-derivation B*-Ideal of X if it satisfies the following conditions:*
$(F_C - d_t(i))$: $\alpha(d_t(0*a)) \geq \alpha(d_t(a))$, for all $a,b \in X$.
$(FC - d_t(ii))$: $\alpha(d_t(b*c)) \geq min\{\alpha(d_t((a*b)*c)),\alpha(d_t(a))\}$, for all $a,b,c \in X$.

Example 6. *Let $X = \{0,1,2\}$ be a BCI-Algebra with the following Cayley Table.*

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	1	0

For any $t \in X$, define a self-map $d_t : X \to X$ by

$$d_t(x) = x*t = \begin{cases} 0 & if\ x = 0,1 \\ 2 & if\ x = 2 \end{cases}$$

and define a fuzzy set $\alpha : X \to [0,1]$ by $\alpha(0) = 0.8$, $\alpha(1) = 0.3$ and $\alpha(2) = 0.05$.
It can be easily verified that α is a fuzzy t-derivations B-Ideal of X.*

Example 7. *Let $X = \{0,1,2,3\}$ be a BCI-Algebra with the following Cayley Table.*

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

For any $t \in X$, Define a self-map $d_t : X \to X$ by

$$d_t(x) = x*t = \begin{cases} 0 & if\ x = 0 \\ 2 & if\ x = 1,2,3 \end{cases}$$

and define a fuzzy set $\alpha : X \to [0,1]$ *by* $\alpha(0) = 0.5$, $\alpha(1) = 0.2$, $\alpha(2) = 0.03$ *and* $\alpha(3) = 0.01$.

It can be easily verified that α *is not a fuzzy* t*-derivations B***-Ideal of* X.

Theorem 5.3. *Every fuzzy* t*-regular derivation B***-Ideal of an associative BCI-Algebra* X *is a fuzzy closed* t*-derivation B***-Ideal of* X.

Proof. Let α be a fuzzy t-regular derivation B*-Ideal of an associative BCI-Algebra X and $d_t(0) = 0$.

Let $a, b, c \in X$. Put $c = a * b$ in $(F - d_t - B_2^*)$ we get

$$\begin{aligned} \alpha(d_t(b * (a * b)) &\geq min\{\alpha(d_t((a * b) * (a * b))), \alpha(d_t(a))\} \\ &\geq min\{\alpha(d_t(0)), \alpha(d_t(a))\} \\ &\geq min\{\alpha(d_t(0)), \alpha(d_t(a))\} \\ &\geq min\{\alpha(0), \alpha(d_t(a))\} \end{aligned}$$

Therefore, $\alpha(d_t(b * (a * b)) \geq \alpha(d_t(a))$. By associativity, we have that $\alpha(d_t((b * a) * b)) \geq \alpha(d_t(a))$. By the property (p_1), $\alpha(d_t((b * b) * a)) \geq \alpha(d_t(a))$ $\Rightarrow \alpha(d_t(0 * a)) \geq \alpha(d_t(a))$ for all $a \in X$. $(F_C - d_t(ii))$ is obviously from $(F - d_t - B_2^*)$. Hence α is a fuzzy closed t-derivation B*-Ideal of X. □

Definition 5.5. *Let* X *be a BCI-Algebra. A fuzzy subset* α *in* X *is said to be a fuzzy* t*-derivation subalgebra of* X *if it satisfies the following condition:*

$(F_S - d_t)\alpha(d_t(a * b)) \geq min\{\alpha(d_t(a)), \alpha(d_t(b))\}$, *for all* $a, b \in X$.

Theorem 5.4. *Every fuzzy closed* t*-derivation B***-Ideal* α *of* X *is a fuzzy* t*-derivations Subalgebra of* X.

Proof. Let α be a fuzzy closed t-derivation B*-Ideal of X.

Let $a, b \in X$. Now,

$$\begin{aligned} \alpha(d_t(a * b)) &\geq min\{\alpha(d_t((a * a) * b)), \alpha(d_t(a))\} \\ &= min\{\alpha(d_t(0 * b)), \alpha(d_t(a))\} \\ &\geq min\{\alpha(d_t(b)), \alpha(d_t(a))\} \\ \alpha(d_t(a * b)) &\geq min\{\alpha(d_t(a)), \alpha(d_t(b))\} \end{aligned}$$

Hence α is a fuzzy t-derivations subalgebra of X. □

Definition 5.6. *Let* α *be a fuzzy* t*-derivations B***-Ideal of* X *and* $s \in [0,1]$. *The subset* α_s *of* X *defined by* $\alpha_s = \{a \in X | \alpha(d_t(a)) \geq s\}$ *is called* α *-level set of* X.

Theorem 5.5. *Let α be a fuzzy set in X, then α is a fuzzy t-derivations B*-Ideal of X iff it satisfies for all $s \in [0,1]$, α_s is non-empty implies α_s is a t-derivations B*-Ideal of BCK-Algebra.*

Proof. Assume that α is a fuzzy t-derivations B*-Ideal of X.

Let $s \in [0,1]$ be such that $\alpha_s 6 = \phi$ and $a, b \in X$ such that $a \in \alpha_s$ then $\alpha(d_t(a)) \geq s$.

Let $a, b, c \in X$. Now,

$$\begin{aligned}
\alpha(d_t(0)) &= \alpha(d_t(0 * c)) \\
&\geq min\{\alpha(d_t((a * 0) * c)), \alpha(d_t(a))\} \\
&= min\{\alpha(d_t(a * (0 * c))), \alpha(d_t(a))\} \\
&= min\{\alpha(d_t(a * 0)), \alpha(d_t(a))\} \\
&= min\{\alpha(d_t(a)), \alpha(d_t(a))\} \\
&= \alpha(d_t(a)) \geq s \\
\alpha(d_t(0)) &\geq s
\end{aligned}$$

Therefore, $0 \in \alpha_s$.

Let $d_t((a*b)*c) \in \alpha_s$ and $d_t(a) \in \alpha_s$, then $\alpha(d_t((a*b)*c)) \geq s$ and $\alpha(d_t(a)) \geq s$.

Now,

$$\begin{aligned}
\alpha(d_t(b * c)) &\geq min\{\alpha(d_t((a * b) * c)), \alpha(d_t(a))\} \\
&\geq min\{s, s\} = s \\
\alpha(d_t(b * c)) &\geq s.
\end{aligned}$$

Hence α_s is a t-derivations B*-Ideal of X.

Conversely, for $s \in [0,1]$, α_s is a t-derivations B*-Ideal of X. Suppose that α satisfies fuzzy t-derivations B*-Ideal of X. Let $a, b, c \in X$ be such that $\alpha(d_t(b * c)) < min\{\alpha(d_t(a * b) * c)), \alpha(d_t(a))\}$.

Put $s' = \dfrac{1}{2}\{\alpha(d_t(b' * c')) + min\{\alpha(d_t((a' * b') * c')), \alpha(d_t(a'))\}\}$.

We have $s \in [0,1]$ and

$$\alpha(d_t(b * c)) < s \tag{5.3}$$

Then, $0 \leq s' < min\{\alpha(d_t((a' * b') * c')), \alpha(d_t(a'))\} \leq 1$.

Therefore, $min\{\alpha(d_t((a' * b') * c')), \alpha(d_t(a'))\} > s' \Rightarrow \alpha(d_t((a' * b') * c')) > s'$ and $\alpha(d_t(a')) > s' \Rightarrow d_t((a' * b') * c') \in \alpha'_s$ and $d_t(a') \in \alpha'_s$. Since α is a fuzzy t-derivations B*-Ideal of X. Therefore, we have $d_t(b' * c') \in \alpha'_s \Rightarrow \alpha(d_t(b' * c')) > s'$, which is a contradiction to (5.3).

Hence α is a fuzzy t-derivations B*-Ideal of X. □

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