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SOME RESULTS ON DUAL DOMINATION IN GRAPHS

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ABSTRACT. Let $G = (V, E)$ be a simple graph. A set $S \subseteq V(G)$ is a dual dominating set of G (or bi-dominating set) if S is a dominating set of G and every vertex in S dominates exactly two vertices in $V - S$. The dual domination number $\gamma_{du}(G)$ (or bi-domination number $\gamma_{bi}(G)$) of a graph G is the minimum cardinality of the minimal dual dominating set (or bi-dominating set) of G. In this paper dual domination number of some Join of two graphs are determined.

1. INTRODUCTION

Let $G(V, E)$ be a simple, connected graph where $V(G)$ is its vertex set and $E(G)$ is its edge set. The degree of any vertex v in G is the number of edges incident with v and is denoted by $degv$. The minimum degree of a graph is denoted by $\delta(G)$ and the maximum degree of a graph G is denoted by $\Delta(G)$. A vertex of degree 0 is called an isolated vertex and a vertex of degree 1 is called a pendent vertex. In this paper, dual domination number of join of two graphs are determined. For graph theoretic notations, refer to [1] and [2].

Definition 1.1. *The join $G + H$ consists of $G \cup H$ and all edges joining a vertex of G and a vertex of H.*

Definition 1.2. *A set $S \subseteq V(G)$ is a dual dominating set if S is a dominating set of G and every vertex in S dominates exactly two vertices in $V - S$.*

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Remark 1.1. *The dual domination number $\gamma_{du}(G)$ of a graph G is the minimum cardinality of all minimal dual dominating sets. The maximum cardinality of a dual dominating set of G is called the upper dual domination number of G and it is denoted by $\Gamma_{du}(G)$.*

2. Main results

Theorem 2.1. *Let G be a connected graph with $m \geq 2$ full degree vertices. Let $|V(G)| = n$, $n \geq 3$, then $\gamma_{du}(G) = n - 2$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let the full degree vertices be $v_1, v_2, \ldots, v_m$.

(i) Suppose $m = 2$. $S = \{v_i : 3 \leq i \leq n\}$ is the unique dual dominating set of G. Hence $\gamma_{du}(G) = n - 2$.

(ii) Let $m \geq 3$. Let v_k and v_t be any two full degree vertices. Since every vertex of $T = \{v_i \colon 1 \leq i \leq n, i \neq k, t\}$ are dominates both v_k and v_t, hence T is a dual dominating set of G. If one vertex, say v_1, is removed from T then, every vertex of $V(G)$ dominates v_k, v_t and v_1. Hence T cannot be a dual dominating set of G. The case is similar if more than two elements are removed from T. Hence $\gamma_{du}(G) = n - 2$.

□

Theorem 2.2. *Let $G = P_n + P_m$, then G has a dual dominating set if and only if n or $m = 2, 3$.*

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(P_m) = \{u_1, u_2, \ldots, u_m\}$.

(1) Suppose $n = 2$. The vertices v_1 and v_2 are full degree vertices. Then by Theorem 2.1, $\{u_1, u_2, \ldots, u_n\}$ is the unique dual dominating set of $G, \gamma_{du}(G) = m$.

(2) Suppose $n = 3$. Then by Theorem 2.1, $\{v_2, u_j : 1 \leq j \leq n\}$ is the dual dominating set of G. Thus $\gamma_{du}(G) = m + 1$.

Conversely, let $n, m \neq 2$. Suppose S is a dual dominating set of G.

(1) Suppose v_1 belongs to S. v_1 dominates v_2 and u_j, $1 \leq j \leq n$.

(1a) Let u_1 and u_m do not belong to S. If v_i, $2 \leq i \leq n$ belong to S then u_2 dominates only vertex u_1 in $V - S$, a contradiction. Hence one of the vertex v_i, $2 \leq i \leq n$ does not belong to S. Let the vertex be v_k. Let v_t be the vertex adjacent with v_k. v_t dominates with v_k, u_1 and u_m, a contradiction. Hence v_1 does not belongs to S.

(1b) Let v_2 and $u_j, 1 \leq j \leq n$ do not belong to S. Let u_k be the vertex not adjacent with u_j. Hence one of the vertices, say v_t , $1 \leq t \leq n$ and $t \neq 1, 2$ does not belong to S. The vertex which is adjacent to u_j dominates three vertices u_j, v_2, v_t, a contradiction. Hence v_1 does not belongs to S.

(2) The proof is similar to (1) if any v_j, $2 \leq j \leq n$ belongs to S. Hence $v_1, v_2, \ldots, v_n$ do not belong to S. Therefore any element of S dominates more than two vertices of $V - S$, a contradiction.

The proof is similar if any u_j, $2 \leq j \leq m$ belong to S, a contradiction.
According to all the above cases, dual dominating set of G does not exists. □

Theorem 2.3. *Let $G = C_n + C_m$, then G has a dual dominating set if and only if n or m is 3 or 4.*

Proof. Let $G_1 = C_n$ and $G_2 = C_m$. Let $G = G_1 + G_2$, $V(G_1) = v_1, v_2, \ldots, v_n$ and $V(G_2) = \{u_1, u_2, \ldots, u_m\}$. Let $E(G_1) = \{v_i, v_{i+1} \colon 1 \leq i \leq n-1\} \cup \{v_n v_1\}$ and $E(G_2) = \{u_j, u_{j+1} \colon 1 \leq j \leq n-1\} \cup \{u_n u_1\}$.

(1) Let $m = 3$. v_1, v_2 and v_3 are full degree vertices of G. Hence by Theorem 2.1, $\gamma_{du}(G) = n + 1$.

(2) Let $m = 4$. Let $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{v_i u_j \colon 1 \leq i \leq 4, 1 \leq j \leq n\}$. Let $S_1 = \{v_2, v_4, u_j \colon 1 \leq j \leq n\}$ and Let $S_2 = \{v_1, v_3, u_j \colon 1 \leq j \leq n\}$ are the dual dominating set of G. $|S_i| = n + 2$, $i = 1, 2$. It is verified that no set with less than $n + 2$ elements is a dual dominating set of G. Therefore, $\gamma_{,du}(G) = n + 2$.

Conversely, let $m, n \geq 5$. Suppose S is a dual dominating set of G. Suppose v_1 belongs to S. (The proof is similar if any v_i belongs to S or any u_j belongs to S, $1 \leq i \leq n$, $1 \leq j \leq m$). v_1 dominates v_2, v_n and all u_j, $1 \leq j \leq m$. Since S is a dual dominating set of G, all neighbourhood of v_1 except any two must belong to S. Let v_2 and v_n do not belong to S. Therefore the vertices $v_3, v_4, \ldots, v_{n-1}$ belongs to S. v_3 and v_{n-1} dominates exactly only one vertex of $V - S$ and other v_k, $4 \leq k \leq n-2$ dominate no vertex $V - S$, a contradiction. The proof is similar any two u_j do not belongs to S or one v_i and one u_j do not belongs to S.
Hence a dual dominating set does not exists. □

Proposition 2.1. *Suppose G_1 and G_2 have atleast one full degree vertex. Then $G = G_1 + G_2$ has γ_{du} - set, $\gamma_{du}(G) = |V(G_1)| + |V(G_2)| - 2$.*

Proposition 2.2. *Let $G = G_1 + G_2$. Either G_1 or G_2 has two full degree vertex, then $\gamma_{du}(G) = |V(G_1)| + |V(G_2)| - 2$.*

Proposition 2.3. *Let G_1 be any simple graph with n vertices. Let $G_2 = \bar{K}_2$. Let $G = G_1 + G_2$, $V(G_1)$ is the unique dual dominating set of G. Hence $\gamma_{du}(G) = n$.*

Theorem 2.4. *Let G_1 be a connected graph with no full degree vertices. Let $|V(G_1)| = n$, $n \geq 4$. Let $G_2 = P_3$. Let $G = G_1 + G_2$. Then $\gamma_{du}(G) = n + 1$.*

Proof. Let $V(G_1) = \{v_1, v_2, \dots, v_n\}$ and $V(G_2) = \{u_1, u_2, u_3\}$. Let $E(G_2) = \{u_1u_2, u_2u_3\}$. Let $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{v_iu_j : 1 \leq i \leq n, 1 \leq j \leq 3\}$. $S = \{v_i, u_2 : 1 \leq i \leq n\}$ is the unique dual dominating set of G. Therefore $\gamma_{du}(G) = n + 1$. □

Theorem 2.5. *Let G_1 be a connected graph with no full degree vertices. Let $|V(G_1)| = n$. Let $G_2 = C_4$. Let $G = G_1 + G_2$. Then $\gamma_{du}(G) = n + 2$.*

Proof. Let $V(G_1) = \{v_1, v_2, \dots, v_n\}$ and $V(G_2) = \{u_1, u_2, u_3, u_4\}$. Let $E(G_2) = \{u_1u_2, u_2u_3, u_3u_4, u_4u_1\}$. Let $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{v_iu_j : 1 \leq i \leq n, 1 \leq j \leq 4\}$.$S_1 = \{v_i, u_1, u_3 : 1 \leq i \leq n\}$ and $S_2 = \{v_i, u_2, u_4 : 1 \leq i \leq n\}$ are the dual dominating set of G. Further it is verified that no γ_{du} - set of G with less than $n + 2$ elements exists. Hence $\gamma_{du}(G) = n + 2$. □

Theorem 2.6. *Let $G = C_m + P_n$, then G has a dual dominating set if and only if $n \leq 3$ or $m \leq 4$.*

Proof. Let $V(C_m) = \{v_1, v_2, \dots, v_m\}$ and $V(P_n) = \{u_1, u_2, \dots, u_n\}$.

(1) Let $m \leq 4$.
(1a) Suppose $m = 3$. By Proposition 2.1, $\gamma_{du}(G) = n + 1$.
(1b) Suppose $m = 4$. By Theorem 2.5, $\gamma_{du}(G) = n + 2$.
(2) Let $n \leq 3$.
(2a) Suppose $n = 2$. By Proposition 2.2, $\gamma_{du}(G) = m$.
(2b) Suppose $n = 3$. By Theorem 2.4, $\gamma_{du}(G) = m + 1$.

Conversely, let $m \geq 5$ and $n \geq 4$. Let S be a dual dominating set of G. Suppose v_i belong to S. $N(v_i) = \{v_{i-1}, v_i, v_{i+1}, u_j : 1 \leq j \leq n\}$.

(1) Let u_j, $1 \leq j \leq n$ belongs to S. If v_{i+2} belongs to S then v_{i+3} does not belongs to S. But all u_j, $1 \leq j \leq n$ dominates three vertices v_{i-1}, v_{i+1} and v_{i+3}, a contradiction. Hence v_{i+2} does not belongs to S. Now also u_j, $1 \leq j \leq n$ dominates three vertices v_{i-1}, v_{i+1} and v_{i+2}, a contradiction. The proof is similar for any v_k, $k \neq i$.

(2) Let $v_{i-1}, v_{i+1}, u_j \colon 1 \leq j \leq n$, $j \neq k, t$. If any v_s does not belong to S then v_{s+1} dominates three vertices v_s, u_k and u_t, a contradiction. Hence all v_i belong to S. The vertices which are not adjacent to both u_k and u_t has one element or no element in $V - S$, a contradiction. This case does not exists.

(3) Let v_{i-1} (or v_{i+1}), u_j, $1 \leq j \leq n$, $j \neq k$. If v_{i+2} belongs to S and v_{i+3} belongs to S, v_{i+4} does not belongs to S. But u_{k+1} dominates three vertices v_{i-1}, v_{i+4} and u_k, a contradiction. If v_{i+2} belongs to S and v_{i+3} does not belongs to S then also u_{k+1} dominates three vertices v_{i+1}, v_{i+3} and u_k, a contradiction. This case does not exist.

From all the above cases, it is observed that no $v_i, 1 \leq i \leq m$ belong to S. Hence all u_j, $1 \leq j \leq n$ dominates more than two vertices of $V - S$, a contradiction. Hence dual dominating set of G does not exists. □

Theorem 2.7. *Let $G = B_{r,s} + K_1$, $r, s \geq 2$ then $\gamma_{du}(G) = r + s$.*

Proof. Let $V(B_{r,s}) = \{u, v, u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_s\}, E(B_{r,s}) = \{uv, uu_i, vv_j \colon 1 \leq i \leq r, 1 \leq j \leq s\}$ and $V(K_1) = w$. Let $V(G) = V(B_{r,s}) \cup V(K_1)$ and $E(G) = E(B_{r,s}) \cup \{uw, wv, wv_i, wv_j \colon 1 \leq i \leq r, 1 \leq j \leq s\}$. $S = \{u_i, v_j \colon 1 \leq i \leq r, 1 \leq j \leq s\}$ is the unique dual dominating set of G. Therefore $\gamma_{du}(G) = r + s$. □

Theorem 2.8. *Let G_1 be any connected graph without a full degree vertex of order m, $G = G_1 + K_{2,n}$. Then $\gamma_{du}(G) = m + n$.*

Proof. Let $V(G_1) = \{v_1, v_2, \ldots, v_m\}$ and $V(K_{2,n}) = \{u, v, u_1, u_2, \ldots, u_n\}$. Let $V(G) = V(G_1) \cup V(K_{2,n})$ and $E(G) = E(G_1) \cup E(K_{2,n}) \cup \{uv_i, wv_i, v_iu_j \colon 1 \leq i \leq m, 1 \leq j \leq n\}$. $S = \{u_i, v_j \colon 1 \leq i \leq m, 1 \leq j \leq n\}$ is the unique dual dominating set of G. Therefore $\gamma_{du}(G) = m + n$. □

Theorem 2.9. *Let $G = K_{m,n} + K_{r,s}$, $m, n, r, s \geq 3$. It has no dual dominating set.*

Proof. Let $V(K_{m,n}) = \{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n \colon 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(K_{m,n}) = \{v_iu_j, 1 \leq i \leq m, 1 \leq j \leq n\}$.

Let $V(K_{r,s}) = \{w_1, w_2, \ldots, w_r, x_1, x_2, \ldots, x_s \colon 1 \leq k \leq r, 1 \leq l \leq s\}$ and $E(K_{r,s}) = \{w_k x_s 1 \leq k \leq r, 1 \leq l \leq s\}$. Let $V(G) = V(K_{m,n}) \cup V(K_{2,n})$ and $E(G) = E(K_{m,n}) \cup E(K_{m,n}) \cup \{v_i w_k, v_i x_l, u_j w_k, u_j x_l \colon 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq r, 1 \leq l \leq s\}$. Let S be a dual dominating set of G.

(1): Suppose w_k(or x_l) belongs to S.
$N(w_k) = \{v_i, u_j, x_l \colon 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq s\}$.

(1a) Let x_l and x_{1+2} does not belongs to S. Let x_l, $3 \leq l \leq s$, v_i and u_j must belongs to S. The vertices x_l, $3 \leq l \leq s$ dominates more than two vertices of $V - S$, a contradiction. This case does not exists.

(1b) Let v_i and v_j does not belongs to S. Let v_i, $3 \leq i \leq n$. u_j and x_l must belongs to S. This case is similar to (1a).

(2) Suppose v_i (or u_j) belongs to S.
$N(v_i) = \{u_j, w_k, x_l \colon 1 \leq j \leq n, 1 \leq k \leq r, 1 \leq l \leq s\}$. This case is similar to (1).

From all the above cases x_l , $3 \leq l \leq s$ or u_j, $3 \leq j \leq m$ or v_i, $3 \leq i \leq m$ or w_k, $3 \leq k \leq r$ dominates more than two vertices in $V - S$. Hence dual dominating set does not exists. □

Theorem 2.10. *Let $G = W_n + C_4$, $n \geq 4$, then $\gamma_{du}(G) = n + 2$.*

Proof. Let $V(W_n) = \{v, v_1, v_2, \ldots, v_{n-1}\}$ and $E(W_n) = \{vv_i, v_1v_2, v_2v_3, \ldots, v_{n-2}v_{n-1} \colon 1 \leq i \leq n-1\}$. Let $V(C_4) = \{u_1, u_2, u_3, u_4\}$ and $E(C_4) = \{u_1u_2, u_2u_3, u_3u_4, u_4u_1\}$. Let $V(G) = V(W_n) \cup V(C_4)$ and $E(G) = E(W_n) \cup E(C_4) \cup \{vu_j, v_iu_j \colon 1 \leq i \leq n-1, 1 \leq j \leq 4\}$. $S_1 = \{v, v_i, u_1, u_3 \colon 1 \leq i \leq n-1\}$ and $S_2 = \{v_i, u_2, u_4 \colon 1 \leq i \leq n-1\}$ are the dual dominating set of G.
Further it is verified that no γ_{du} - set of G with less than $n+2$ elements exists.
Hence $\gamma_{du}(G) = n + 2$. □

Theorem 2.11. *Let G_1 be a simple connected graph. Let $H = G_1 \odot (\bar{K_n})$ and $G = H + K_1$, then $\gamma_{du}(G) = mn$.*

Proof. Let $V(G_1) = \{v_1, v_2, \ldots, v_m\}$, $V(K_1) = \{v\}$, $V(H) = V(G_1) \cup \{u_{i1}, u_{i2}, \ldots, u_{in} \colon 1 \leq i \leq m\}$ and $E(H) = E(G) \cup \{v_i u_{ij} \colon 1 \leq i \leq m, 1 \leq j \leq n\}$. Let $V(G) = V(H) \cup \{v\}$ and $E(G) = E(H) \cup \{v_i v, vu_{ij} \colon 1 \leq i \leq m, 1 \leq j \leq n\}$.
$S = \{u_{ij} \colon 1 \leq i \leq m, 1 \leq j \leq n\}$ is the unique dual dominating set of H.
Therefore $\gamma_{du}(H) = mn$. □

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