ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **9** (2020), no.5, 2607–2614
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.5.23

GREY SETS IN METRIC SPACE

J. SUBHASHINI[1] AND G. ROHINI VIJAYA LAXMI

ABSTRACT. The grey system theory was first initiated by Deng Julong in 1982. After then, Yingjie Yang proposed the grey sets in 2011. The French Mathematician Maurice Frechet initiated the study of Metric space in 1905. In this study we focus to define a metric in grey sets, thereby introduce the concept of grey sets in metric space and discuss its properties under the topic "Grey Sets in Metric Space".

1. INTRODUCTION

The grey system theory was first initiated by Deng Julong in 1982. As far as information is concerned, the system which lack information, such as structure message, operation mechanism and behaviour document are referred to as Grey Systems. For example, the human body, agriculture, economy, etc. are in the form of Grey Systems. In general, a system containing knowns and unknowns is called a grey system. It is a new methodology that focuses on the study of problems involving small samples and poor information. It deals with uncertain systems with partially known information.

Yingjie Yang proposed the grey sets on 2011. Grey sets apply the basic concept of grey numbers in grey systems, and consider the characteristic function values of a set as grey numbers. They provide an alternative approach for the representation of uncertainity in sets, [3,4].

[1]*corresponding author*

2010 *Mathematics Subject Classification.* 54E35.

Key words and phrases. Grey system, grey sets, metric, metric space.

In 1906 Maurice Frechet introduced metric spaces in his work "Sur quelques points du calcul fontionnel". However, the name is due to Felix Hausdorff, [2]. In Mathematics, a metric space is a set together with a metric on the set. A metric is a function that defines a concept of distance between any two members of the set. It is also called as distance function or simply distance.

In this study, we define two types of distance between grey sets and few properties of metric space have been studied. This paper consists of three sections. The first section is the preliminary section. In the preliminary section, the basic definitions needed for the study have been given. In the next section, we have defined the distance between two grey sets and also proved few results based on it. In the last section, we have defined another distance function and its properties were studied.

2. Preliminary

The basic definitions needed for this study is given in this section.

Definition 2.1. *[3] A grey number is a number with clear upper and lower boundaries but which has an unknown position within the boundaries. A grey number for the system is expressed mathematically as* $g^{\pm} \in [g^{-}, g^{+}] = \{g^{-} \leq t \leq g^{+}\}$*, where* $g^{\pm}$ *is a grey number,* t *is information,* g^{-} *and* g^{+} *are the upper and lower limits of the information.*

Definition 2.2. *[3] Let* $g^{\pm} \in R$ *be an unknown real number within a union set of closed or open intervals.* $g^{\pm} \in \bigcup_{i=1}^{n}[a_i^{-}, a_i^{+}],\ i = 1, 2, ..., n,$ n *is an integer and* $0 < n < \infty$, $a_i^{-}, a_i^{+} \in \mathbb{R}$ *and* $a_{+}^{i-1} \leq a_i^{-} \leq a_i^{+} \leq a_{i+1}^{-}$*. For any interval* $[a_i^{-}, a_i^{+}] \subseteq \bigcup_{i=1}^{n}[a_i^{-}, a_i^{+}]$, p_i *is the probability for* $g^{\pm} \in [a_i^{-}, a_i^{+}]$*. If the following condition holds*

(1) $p_i > 0$

(2) $\sum_{i=1}^{n} p_i = 1.$

then we call $g^{\pm}$ *a generalized grey number.* $g^{-} = inf_{a_i^{-} \in g^{\pm}} a_i^{-}$ *and* $g^{+} = sup_{a_i^{+} \in g^{\pm}} a_i^{+}$ *are called as the lower and upper limits of* $g^{\pm}$*.*

Note that the intervals involved in grey numbers do not need to be closed although our expression uses the closed representation. Definition 2.2 removes the limitation for open sets and discrete set to represent a grey number. A grey number could be represented as a set of intervals with gaps in between. For example, $g^{\pm} \in \{[5,6],[10,12]\}$ is a grey number.

Theorem 2.1. *[3] $g^{\pm}$ is a grey number defined by Definition 2.2. The following properties hold for $g^{\pm}$:*

(1) *$g^{\pm}$ is a continuous grey number $g^{\pm} \in [a_1^-, a_n^+]$ iff $a_i^- = a_{i-1}^+$ $(\forall i > 1)$ or $n = 1$.*
(2) *$g^{\pm}$ is a discrete grey number $g^{\pm} \in \{a_1, a_2, ..., a_n\}$ iff $a_i = a_i^- = a_i^+$.*
(3) *$g^{\pm}$ is a mixed grey number iff part of its intervals shrink to crisp numbers and others keep as intervals.*

Definition 2.3. *[3] Let U be the initial universal set. For a set $A \subseteq U$, if the characteristic function value of x with respect to A can be expressed with a grey number $g_A^{\pm}(x) \in \bigcup_{i=1}^{n}[a_i^-, a_i^+] \in D[0,1]^{\pm}$, i.e., $\chi : U \to D[0,1]^{\pm}$ then A is a grey set. Here, $D[0,1]^{\pm}$ refers to the set of all grey numbers within the interval $[0,1]$. A grey set is represented with its relevant elements and their associated grey number for characteristic function:*

$$A = g_A^{\pm}(x_1)/x_1 + g_A^{\pm}(x_2)/x_2 + \cdots + g_A^{\pm}(x_n)/x_n.$$

Definition 2.4. *[1] Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}$ is said to be a metric on X if*

(i) $d(x,y) \geq 0,\ \forall x, y \in X.$
(ii) $d(x,y) = 0 \Leftrightarrow x = y, \quad x, y \in X$
(iii) $d(x,y) = d(y,x),\ \forall x, y \in X$
(iv) $d(x,z) \leq d(x,y) + d(y,z),\ \forall x, y \in X.$

The ordered pair (X,d) is called a metric space.

Definition 2.5. *[1] A sequence $\{x_n\}$ in is said to be convergent if there is a point $x \in X$ such that for each $\epsilon > 0$, $\exists$ a positive integer N such that $d(x_n, x) < \epsilon$, $\forall n \geq N$. We write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$ and say x_n approaches x or that x_n converges to x.*

Definition 2.6. *[1] A sequence $\{x_n\}$ in a metric space (X,d) is said to be a Cauchy sequence if for each $\epsilon > 0$, $\exists$ a positive integer N such that $d(x_n, x_m) < \epsilon$, $\forall m, n \geq N$.*

Definition 2.7. *[1] A metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.*

3. Distance between grey sets - approach I

Definition 3.1. *Let U be an initial universe and let $\mathcal{P}(U)$ denote the set of all grey sets of U. Let $A, B \in \mathcal{P}(U)$. The distance between the grey sets A and B, denoted by d is given by*

$$d(A, B) = sup_{x \in U}\ max\{|g_A^-(x) - g_B^-(x)|, |g_A^+(x) - g_B^+(x)|\}.$$

Theorem 3.1. *Let U be an initial universe and let $\mathcal{P}(U)$ denote the set of all grey sets of U. Let $A, B \in \mathcal{P}(U)$. Then the distance between the grey sets A and B, d given in definition 3.1 is a metric on $\mathcal{P}(U)$.*

Proof. Clearly d is a function from $\mathcal{P}(U) \times \mathcal{P}(U)$ to $\mathbb{R}$.
(i) Clearly, $d(A, B) \geq 0$.
(ii) Let $d(A, B) = 0$.
Then $\forall x \in U,\ max\{|g_A^-(x) - g_B^-(x)|, |g_A^+(x) - g_B^+(x)|\} = 0$.

$$\begin{aligned} &\Rightarrow\ |g_A^-(x) - g_B^-(x)| = 0 \text{ and } |g_A^+(x) - g_B^+(x)| = 0,\ \forall x \in U \\ &\Rightarrow\ g_A^-(x) = g_B^-(x) \text{ and } g_A^+(x) = g_B^+(x),\ \forall x \in U \\ &\Rightarrow\ A = B. \end{aligned}$$

Conversely, suppose $A = B$.
Then $g_A^-(x) = g_B^-(x)$ and $g_A^+(x) = g_B^+(x),\ \forall x \in U$.
$\Rightarrow d(A, B) = 0$.
Thus $d(A, B) = 0 \Leftrightarrow A = B$, where $A, B \in \mathcal{P}(U)$.
(iii) Consider

$$\begin{aligned} d(A, B) &= sup_{x \in U}\ max\{|g_A^-(x) - g_B^-(x)|, |g_A^+(x) - g_B^+(x)|\} \\ &= sup_{x \in U}\ max\{|g_B^-(x) - g_A^-(x)|, |g_B^+(x) - g_A^+(x)|\} \\ &= d(B, A) \end{aligned}$$

(iv) Consider

$$\begin{aligned} d(A,B) &= sup_{x\in U}\ max\{|g_A^-(x)-g_B^-(x)|,|g_A^+(x)-g_B^+(x)|\} \\ &\leq sup_{x\in U}\ max\{\left(|g_A^-(x)-g_C^-(x)|,|g_C^+(x)-g_B^+(x)|\right), \\ &\qquad \left(|g_A^-(x)-g_C^-(x)|,|g_C^+(x)-g_B^+(x)|\right)\} \end{aligned}$$

where $g_C^-(x)$ and $g_C^+(x)$ forms the lower and upper limits of $C\in\mathcal{P}(U)$.

$$\begin{aligned} &= sup_{x\in U}\ max\{|g_A^-(x)-g_C^-(x)|,|g_A^+(x)-g_C^+(x)|\} \\ &\quad +sup_{x\in U}\ max\{|g_B^-(x)-g_C^-(x)|,|g_B^+(x)-g_C^+(x)|\} \\ &= d(A,C)+d(C,B) \end{aligned}$$

Thus $d(A,B)\leq d(A,C)+d(C,B),\ \forall A,B,C\in\mathcal{P}(U)$.
Hence d is a metric on $\mathcal{P}(U)$. □

Definition 3.2. *The ordered pair $(\mathcal{P}(U),d)$ forms a metric space and is called as grey metric space.*

Definition 3.3. *For any sequence of grey sets $A_n\in\mathcal{P}(U)$ and for some $A\in\mathcal{P}(U)$, $d(A_n,A)\to 0$ as $n\to\infty$ iff $g_{A_n}^-(x)$ converges to $g_A^-(x)$ and $g_{A_n}^+(x)$ converges to $g_A^+(x)$.*

Theorem 3.2. *The grey metric space $(\mathcal{P}(U),d)$ is a complete metric space.*

Proof. Assume that $\{A_k\}$ is a Cauchy sequence in $\mathcal{P}(U)$.

Then $\forall\epsilon>0$, $\exists$ a positive integer N such that $d(A_n,A_m)<\epsilon,\ \forall m,n\geq N$.

$\therefore |g_{A_m}^-(x)-g_{A_n}^-(x)|\leq d(A_n,A_m)<\epsilon$ and $|g_{A_m}^+(x)-g_{A_n}^+(x)|\leq d(A_n,A_m)<\epsilon$, $\forall x\in U$.

$\Rightarrow \{g_{A_k}^-(x)\}$ and $\{g_{A_k}^+(x)\}$ are Cauchy sequences in $\mathbb{R}$.

Since $\mathbb{R}$ is complete, $\{g_{A_k}^-(x)\}$ and $\{g_{A_k}^+(x)\}$ converges.

Let $g_{A_k}^-(x)\to g_A^-(x)$ and $g_{A_k}^+(x)\to g_A^+(x)$.

$\Rightarrow d(A_k,A))\to 0$ as $k\to\infty$.

Claim: $A\in\mathcal{P}(U)$.

We have $g_{A_k}^-(x)\leq g_{A_k}^+(x)$

$$\Rightarrow g_A^-(x)\leq g_A^+(x)$$

Let $g_A^-(x)$ and $g_A^+(x)$ forms the lower and upper limit of $g_A^{\pm}(x)$. Then $g_A^{\pm}(x)$ is a grey number $\forall x\in U$.

$\therefore A\in\mathcal{P}(U)$.

Thus $\{A_k\} \to A$ in $\mathcal{P}(U)$.
Hence $(\mathcal{P}(U), d)$ is complete. □

4. Distance between grey sets - approach II

Definition 4.1. *The distance between the two grey sets A and B, denoted by d' is given by*

$$d'(A,B) = sup_{x\in U} \left\{ \frac{1}{2} \left[|g_A^-(x) - g_B^-(x)| + |g_A^+(x) - g_B^+(x)| \right] \right\}.$$

Theorem 4.1. *The distance between the grey sets A and B, d' given in definition 4.1 is a metric on $\mathcal{P}(U)$.*

Proof. Clearly d is a function from $\mathcal{P}(U) \times \mathcal{P}(U)$ to $\mathbb{R}$.
(i) Clearly, $d'(A,B) \geq 0$.
(ii) Let $d'(A,B) = 0$.
Then $\forall x \in U,\ m|g_A^-(x) - g_B^-(x)| + |g_A^+(x) - g_B^+(x) = 0$.

$$\begin{aligned} &\Rightarrow \quad |g_A^-(x) - g_B^-(x)| = 0 \text{ and } |g_A^+(x) - g_B^+(x)| = 0,\ \forall x \in U \\ &\Rightarrow \quad g_A^-(x) = g_B^-(x) \text{ and } g_A^+(x) = g_B^+(x),\ \forall x \in U \\ &\Rightarrow \quad A = B. \end{aligned}$$

Conversely, suppose $A = B$.
Then $g_A^-(x) = g_B^-(x)$ and $g_A^+(x) = g_B^+(x),\ \forall x \in U$.
$\Rightarrow d'(A,B) = 0$.
Thus $d'(A,B) = 0 \Leftrightarrow A = B$, where $A, B \in \mathcal{P}(U)$.
(iii) Consider

$$\begin{aligned} d'(A,B) &= sup_{x\in U} \left\{ \frac{1}{2} \left[|g_A^-(x) - g_B^-(x)| + |g_A^+(x) - g_B^+(x)| \right] \right\} \\ &= sup_{x\in U} \left\{ \frac{1}{2} \left[|g_B^-(x) - g_A^-(x)| + |g_B^+(x) - g_A^+(x)| \right] \right\} \\ &= d'(B,A) \end{aligned}$$

(iv) Consider

$$\begin{aligned} d'(A,B) &= sup_{x\in U}\left\{\frac{1}{2}\left[|g_A^-(x)-g_B^-(x)|+|g_A^+(x)-g_B^+(x)|\right]\right\} \\ &= sup_{x\in U}\left\{\frac{1}{2}\left[|g_A^-(x)-g_C^-(x)-g_B^-(x)+g_C^-(x)|\right.\right. \\ &\qquad \left.\left.+|g_A^+(x)-g_C^+(x)-g_B^+(x)+g_C^+(x)|\right]\right\}, \end{aligned}$$

where $g_C^-(x)$ and $g_C^+(x)$ forms the lower and upper limits of $C \in \mathcal{P}(U)$.

$$\begin{aligned} &\leq sup_{x\in U}\left\{\frac{1}{2}\left[|g_A^-(x)-g_C^-(x)|+|g_A^+(x)-g_C^+(x)|\right]\right\} \\ &\quad +sup_{x\in U}\left\{\frac{1}{2}\left[|g_B^-(x)+g_C^-(x)|+|g_B^+(x)+g_C^+(x)|\right]\right\}, \\ &= d'(A,C)+d'(C,B) \end{aligned}$$

Thus $d'(A,B) \leq d'(A,C)+d'(C,B),\ \forall A,B,C \in \mathcal{P}(U)$.
Hence d' is a metric on $\mathcal{P}(U)$. □

Definition 4.2. *The ordered pair $(\mathcal{P}(U), d')$ forms a metric space and is called the grey metric space.*

Theorem 4.2. *The grey metric space $(\mathcal{P}(U), d')$ is a complete metric space.*

Theorem 4.3. *The proof is similar to Theorem 3.2.*

5. Conclusion

In this paper we introduced the concept of grey set into the field of metric space by defining two type of distance function. We have also introduced grey metric space and proved related properties. This paper leads further research under the concept of compactness, connectedness, fixed point theorem, etc. for the new researchers.

References

[1] P. K. Jain, K. Ahmad: *Metric Spaces*, Narosa Publishing House, New Delhi, ISBN 81-85198-99-3.

[2] P. Grzegorzewski: *Distances between intuitionistic fuzzy sets and/or interval- valued fuzzy sets based on the Hausdorff metric*, Fuzzy Sets and Systems, **148** (2004), 319–328.

[3] Y. YANG, R. JOHN: *Grey Sets and Greyness*, Information Sciences, **185** (1) (2012), 249–264.

[4] Y. YANG, R. JOHN, S. LIU: *Some Extended Operations of Grey Sets*, Kybernetes, **41** (7/8) (2012), 860–873.

DEPARTMENT OF MATHEMATICS
ST. JOHN'S COLLEGE
PALAYAMKOTTAI

DEPARTMENT OF MATHEMATICS
ST. JOHN'S COLLEGE
PALAYAMKOTTAI