

RESULTS CONNECTING DOMINATION, STEINER AND STEINER DOMINATION NUMBER OF GRAPHS

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ABSTRACT. In this paper, relation between domination number, steiner number and steiner domination number is studied in detail.

1. INTRODUCTION

The concept of domination in graphs was introduced by Ore and Berge [4]. Let $G = (V, E)$ denotes a finite undirected simple graph with vertex set V and edge set E . A subset D of $V(G)$ is a dominating set of G if every vertex in $V - D$ is adjacent to at least one vertex in D . The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$.

The concept of Steiner number of a graph was introduced by G. Chatrand and P. Zhang [1]. For a nonempty set W of vertices in a connected graph G , the Steiner distance $d(W)$ of W is the minimum size of a connected subgraph of G containing W . Necessarily each such subgraph is a tree and is called a Steiner tree with respect to W or a Steiner W -tree. The set of all vertices of G that lie in some Steiner W -tree is denoted by $S(W)$. If $S(W) = V$, then W is called a Steiner set for G . A Steiner set with minimum cardinality is the Steiner number of G and is denoted by $s(G)$.

The concept of Steiner domination number of a graph was introduced by J. John et al., [3]. For a connected graph G , a set of vertices W in G is called

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a Steiner dominating set if W is both a Steiner set and a dominating set. The minimum cardinality of a Steiner dominating set of G is its Steiner domination number and is denoted by $\gamma_s(G)$. A steiner dominating set of cardinality $\gamma_s(G)$ is said to be a γ_s -set.

Theorem 1.1. [2] *The domination number of some standard graphs are given as follows:*

- (1) $\gamma(P_p) = \lceil \frac{p}{3} \rceil$, $p \geq 3$.
- (2) $\gamma(C_p) = \lceil \frac{p}{3} \rceil$, $p \geq 3$.
- (3) $\gamma(K_p) = \gamma(W_p) = \gamma(K_{1,n}) = 1$.
- (4) $\gamma(K_{m,n}) = 2$ if $m, n \geq 2$.

Theorem 1.2. [5]

$$\gamma_s(P_n) = \begin{cases} \lceil \frac{n-4}{3} \rceil + 2 & \text{if } n \geq 5; \\ 2 & \text{if } n = 2, 3 \text{ or } 4. \end{cases}$$

Observation 1.1. [3] *If G is a connected graph of order p , then*

$$2 \leq \max\{s(G), \gamma(G)\} \leq \gamma_s(G) \leq p.$$

Theorem 1.3. [1] *Every end vertex of G belongs to every steiner set of G .*

Theorem 1.4. [3] *Every extreme vertex of G belongs to every steiner dominating set of G .*

2. MAIN RESULTS

In this paper, we find graphs for which all the three parameters: domination number, steiner number and steiner domination number are equal. The relation between the three parameters is discussed in detail. For any two positive integers a and b , relation between the above three parameters is discussed in detail when $a < b$ and $a > b$. In these cases, existence of such a graph is explained and proved. The case in which no graph exists is also discussed and the reason is given. Also, we prove for any given set of three positive integers a, b and c with $a, b \geq 2$, and $\max\{a, b\} \leq c \leq a + b$, there exists a graph G such that $\gamma(G) = a$, $s(G) = b$ and $\gamma_s(G) = c$.

Lemma 2.1. *Given a positive integer $k \geq 2$, there exists a graph G with $\gamma(G) = s(G) = \gamma_s(G) = k$.*

Proof. Consider the path P_4 . Name the initial and end vertices of P_4 as u_0 and u_3 respectively. Attach $k - 2$ $u_0 - u_3$ paths of length four and name the middle vertices of the $k - 2$ paths as $w_1, w_2, \dots, w_{k-2}$. Then, we get the following graph.

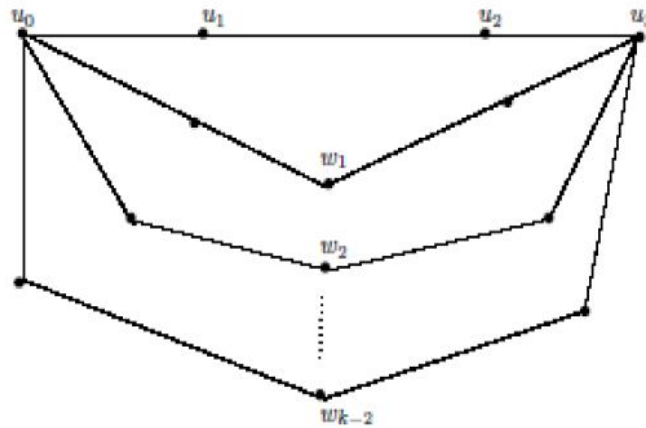


FIGURE 1

Clearly, $\{u_0, w_1, w_2, \dots, w_{k-2}, u_3\}$ is one of the minimum dominating sets of G . It is also a minimum steiner set and a minimum steiner dominating set of G and so $\gamma(G) = s(G) = \gamma_s(G) = k$. $\square$

Lemma 2.2. *Given two positive integers a and b with $a < b$, there exists a connected graph G such that $\gamma(G) = a$ and $\gamma_s(G) = s(G) = b$.*

Proof.

- (i) Let $a = 1$ and $a < b$. The star graph $K_{1,b}$ satisfies the required condition.
- (ii) Let $a = 2$ and $a < b$. Let s and t be two positive integers such that $s + t = b$. Consider the edge uv . Join s pendant vertices $v_1, v_2, \dots, v_s$ to u and also t pendant vertices $w_1, w_2, \dots, w_t$ to v . The resulting graph is given in Figure 2 which is a bistar.

Here, $\{u, v\}$ is a minimum dominating set of G and so $\gamma(G) = 2$. Further, $\{v_1, v_2, \dots, v_s, w_1, w_2, \dots, w_t\}$ is the unique minimum steiner set and

unique minimum steiner dominating set of G . Hence, $\gamma_s(G) = s(G) = s + t = b$.



FIGURE 2

- (iii) Let $a > 2$ and $a < b$. Let $P_a : (u_1, u_2, \dots, u_a)$. Choose positive integers $s_1, s_2, \dots, s_a$ such that $s_1 + s_2 + \dots + s_a = b$. Join $s_1, s_2, \dots, s_a$ pendant vertices respectively to the vertices $u_1, u_2, \dots, u_a$ of P_a . The resulting graph appears as in Figure 3.

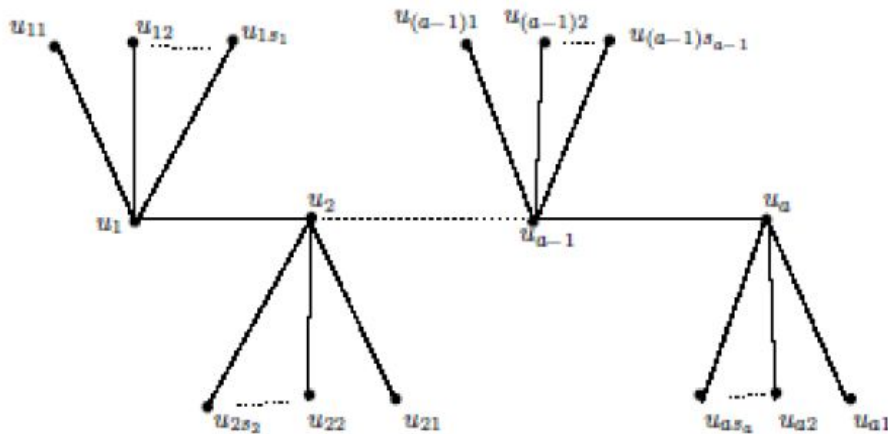


FIGURE 3

Clearly, $u_1, u_2, \dots, u_a$ is a minimum dominating set of G and so $\gamma(G) = a$. Also, $\{u_{11}, u_{12}, \dots, u_{1s_1}, u_{21}, \dots, u_{2s_2}, \dots, u_{a1}, u_{a2}, \dots, u_{as_a}\}$ is the unique minimum steiner set and the unique minimum steiner dominating set of G . Hence, $\gamma_s(G) = s(G) = b$.

□

Remark 2.1. From Observation 1.1, we observe the following result. Given two positive integers a, b with $b > 2$ and $a > b$, there exists no graph with $\gamma(G) = a$ and $\gamma_s(G) = s(G) = b$.

Lemma 2.3. Given two positive integers a, b with $2 \leq b < a$, there exists a graph with $s(G) = b$ and $\gamma_s(G) = \gamma(G) = a$.

Proof.

Case (i): $b = 2$. P_{3a-2} satisfies the required condition since:

- (1) The set of end vertices of P_{3a-2} is the unique minimum steiner set of P_{3a-2} and so $s(P_{3a-2}) = 2 = b$.
- (2) By Theorem 1.1, $\gamma(P_{3a-2}) = \lceil \frac{3a-2}{3} \rceil = a$.
- (3) Further, $3a - 2 > 5$ as $a > 2$. Therefore:

$$\begin{aligned} \gamma(P_{3a-2}) &= \lceil \frac{(3a-2)-4}{3} \rceil + 2 \\ &= \lceil \frac{3a-6}{3} \rceil + 2 \\ &= a - 2 + 2 \\ &= a. \end{aligned}$$

Case (ii): $b > 2$. Let $P_{3a-2b+1} = (v_1, v_2, \dots, v_{3a-2b+1})$. As $a > b$, $3a - 2b + 1 > b$. Let G be a graph in Figure 4 which is obtained by attaching $b - 1$ pendant vertices $w_1, w_2, \dots, w_{b-1}$ respectively to $v_1, v_2, \dots, v_{b-1}$ of the path $P_{3a-2b+1}$. Therefore, $\{w_1, w_2, \dots, w_{b-1}, v_{3a-2b+1}\}$ is a minimum steiner set of G and so $s(G) = b$. The set $S = \{v_1, v_2, \dots, v_{b-1}\}$ together with any minimum dominating set of the path $P' = \{v_{b+1}, v_{b+2}, \dots, v_{3a-2b+1}\}$ dominating set of G .

Number of vertices is

$$\begin{aligned} P' &= 3a - 2b + 1 - b \\ &= 3(a - b) + 1. \end{aligned}$$

Therefore,

$$\begin{aligned}\gamma(G) &= b - 1 + \gamma(P') \\ &= b - 1 + \left\lceil \frac{3(a - b) + 1}{3} \right\rceil \\ &= b - 1 + a - b + 1 = a.\end{aligned}$$

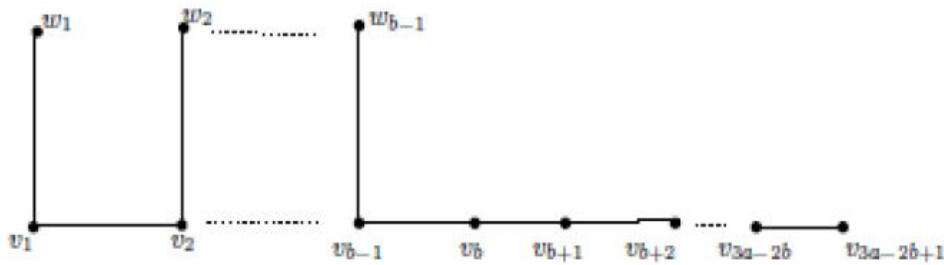


FIGURE 4

Further, $\{w_1, w_2, \dots, w_{b-1}, v_{3a-2b+1}\}$ along with any minimum steiner dominating set of path $P'' = \{v_b, v_{b+1}, \dots, v_{3a-2b-1}\}$ forms a minimum steiner dominating set of G .

Number of vertices in $P'' = 3a - 2b - 1 - (b - 1) = 3(a - b)$. Therefore,

$$\begin{aligned}\gamma_s(G) &= b + \gamma_s(P'') \\ &= b + \left\lceil \frac{3(a - b)}{3} \right\rceil \\ &= b + (a - b) = a.\end{aligned}$$

□

Remark 2.2. As in Remark 2.1, for any two positive integers a, b with $a < b$, there exists no connected graph G with $s(G) = b$ and $\gamma(G) = \gamma_s(G) = a$.

Theorem 2.1. For three positive integers a, b and c with $a, b \geq 2$ and $\max\{a, b\} \leq c \leq a + b$, there exists a graph G with $\gamma(G) = a$, $s(G) = b$ and $\gamma_s(G) = c$.

Proof. Let a, b and c be three positive integers satisfying the given condition.

Case 1: $c = \max\{a, b\}$.

If $a = b$, then the result follows from Lemma 2.1.

If $a < b$, then the result follows from Lemma 2.2.

If $a > b$, then the result follows from Lemma 2.3.

Case 2: $c > \max\{a, b\}$.

Subcase 2a: Suppose $a = b$. Then, $a + 1 \leq c \leq a + b \leq 2a$. Let $r = c - a$. Then, $1 \leq r \leq a$.

Suppose $1 \leq r \leq a$. Consider the graph G in Figure 5:

$$\begin{aligned} V(G) &= \{v_1, v_2, \dots, v_a, w_1, w_2, \dots, w_a, x_{i1}, x_{i2} : 1 \leq i \leq r\} \text{ and} \\ E(G) &= \{v_i w_i : 1 \leq i \leq a\} \cup \{x_{i1} x_{i2} : 1 \leq i \leq r\} \cup \{w_i x_{i1} : 1 \leq i \leq r\} \\ &\quad \cup \{x_{i2} w_{i+1} : 1 \leq i \leq r\} \cup \{w_i w_{i+1} : r+1 \leq i \leq a-1\}. \end{aligned}$$

It is easy to observe that $\{w_1, w_2, \dots, w_a\}$ and $\{v_1, v_2, \dots, v_a\}$ are the unique minimum dominating set and unique minimum steiner set of G , respectively.

Therefore, $\gamma(G) = s(G) = a$.

Further, $\{v_1, v_2, \dots, v_a, x_{11}, x_{21}, \dots, x_{r1}\}$ is one of the minimum steiner dominating set of G and so $\gamma_s(G) = a + r = c$.

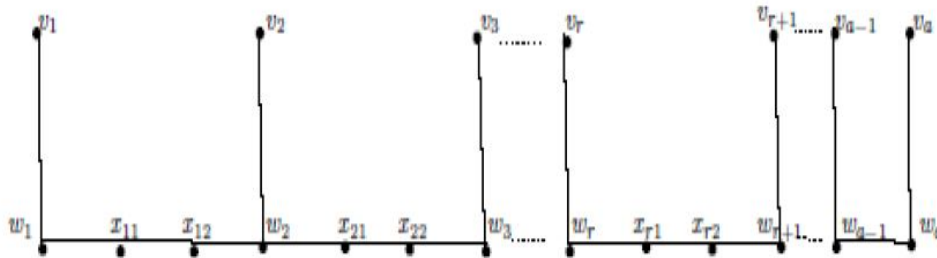


FIGURE 5

Suppose $r = a$.

Considering the graph as in Figure 6:

$$\begin{aligned} V(G) &= \{v_1, v_2, \dots, v_a, w_1, w_2, \dots, w_a, x_{i1}, x_{i2}, y_{i1}, y_{i2} : 1 \leq i \leq a-1\} \text{ and} \\ E(G) &= \{v_i w_i : 1 \leq i \leq a\} \cup \{x_{i1} x_{i2}, y_{i1} y_{i2} : 1 \leq i \leq a-1\} \cup \\ &\quad \{w_i x_{i1}, w_i y_{i1} : 1 \leq i \leq a-1\} \cup \{x_{i2} w_{i+1}, y_{i2} w_{i+1} : 1 \leq i \leq a-1\}. \end{aligned}$$

As, $\{w_1, w_2, \dots, w_a\}$ and $\{v_1, v_2, \dots, v_a\}$ are the unique minimum dominating set and unique minimum steiner set of G , $\gamma(G) = s(G) = a$.

Further, $\{v_1, v_2, \dots, v_a, w_1, w_2, \dots, w_a\}$ is the unique minimum steiner dominating set of G .

Therefore, $\gamma_s(G) = 2a = a + a = a + r = c$, as $r = c - a$.

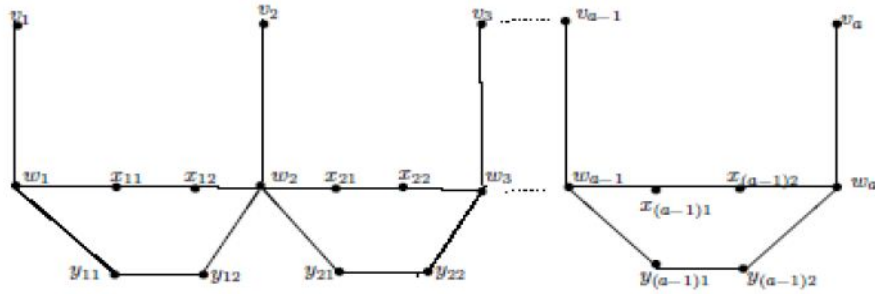


FIGURE 6

Subcase 2b: Suppose $a \neq b$. Then, a, b and c are distinct. Let $r = a + b - c$. As, $c \leq a + b$, $r \geq 0$.

Suppose $r > 0$. Let P be a path $(u_0, u_1, u_{3(a-r)})$. Let s and t be positive integers such that $s + t = b - (r - 1)$.

Add s and t pendant vertices $v_1, v_2, \dots, v_s$ and $w_1, w_2, \dots, w_t$ respectively to the vertices u_0 and $u_{3(a-r)}$ of the path and add $r - 1$ paths of length 4 from u_0 to u_2 , say

$$(u_0, x_1, x_2, x_3 u_2), (u_0, x_4, x_5, x_6, u_2), (u_0, x_{3(r-2)+1}, x_{3(r-2)+2}, x_{3(r-2)}, u_2)$$

The resulting graph is given in Figure 7.

Let D_1 be any minimum dominating set of P containing the end vertices u_0 and $u_{3(a-r)}$ and let D be any minimum dominating set of P . The number of vertices of the path $P = 3(a - r) + 1 \equiv 1 \pmod{3}$. Hence, $|D_1| = |D| = \gamma(P) = \lceil \frac{3(a-r)+1}{3} \rceil = a - r + 1$.

It is easy to see that $D_1 \cap S_1$ where $S_1 = \{x_1, x_5, \dots, x_{3(r-2)+2}\}$ forms a minimum dominating set of G . Therefore,

$$\begin{aligned} \gamma(G) &= a - r + 1 + |S| \\ &= a - r + 1 + r - 1 \\ &= a. \end{aligned}$$

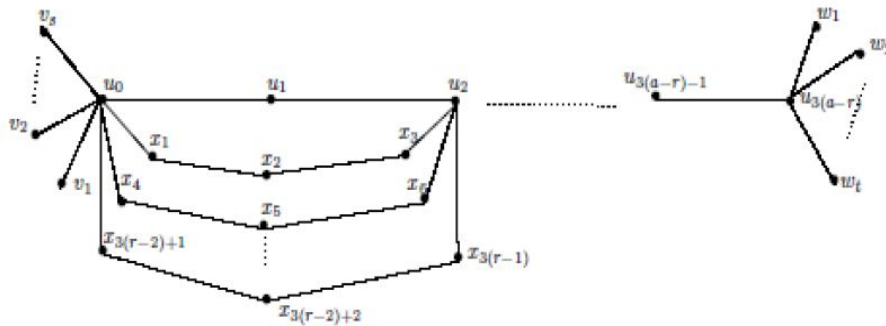


FIGURE 7

Further, $S_2 = S_1 \cup \{v_1, v_2, \dots, v_s, w_1, w_2, \dots, w_t\}$ is a minimum steiner set of G and so $s(G) = s + t + r - 1 = b - (r - 1) + (r - 1) = b$.

Also, S_2 together with any minimum dominating set of the path $(u_1, u_2, \dots, u_{3(a-r)-1})$ forms a minimum steiner dominating set of G . Therefore,

$$\begin{aligned} \gamma_s(G) &= b + \left\lceil \frac{3(a-r)-1}{3} \right\rceil \\ &= b + (a-r) = c, \text{ as } a + b - c = r. \end{aligned}$$

Let $r = 0$. Then, $c = a + b$. Considering the graph G in Figure 8:

$$V(G) = \{v_1, v_2, \dots, v_{3(a-1)+1}\} \cup \{x_1, x_2, y_1, y_2\} \cup \{u_1, u_2, \dots, u_s, w_1, w_2, \dots, w_t\}$$

and

$$\begin{aligned} E(G) &= \{v_1 u_i : 1 \leq i \leq s\} \cup \{v_{3(a-1)+1} w_i : 1 \leq i \leq t\} \cup \\ &\quad \{v_i v_{i+1} : 1 \leq i \leq 3(a-1)\} \cup \{v_1 x_1, x_1 x_2, x_2 v_4\} \cup \\ &\quad \{v_{3(a-1)-2} y_1, y_1 y_2, y_2 v_{3(a-1)+1}\}, \text{ where } s + t = b. \end{aligned}$$

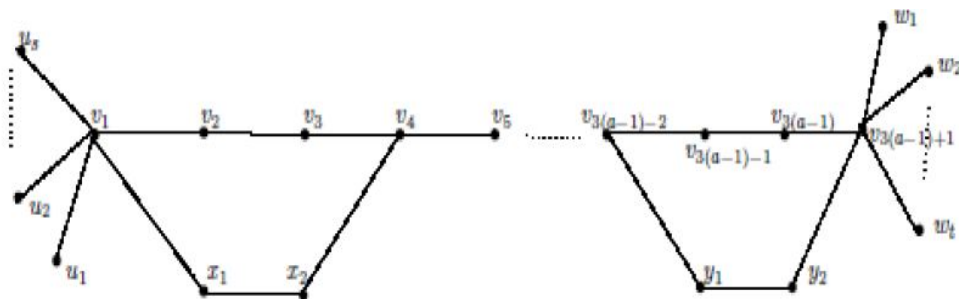


FIGURE 8

Let S be a minimum dominating set of the path $(v_1, v_2, \dots, v_{3(a-1)}, v_{3(a-1)+1})$ containing the end vertices. S is also a minimum dominating set of G and so $\gamma(G) = |S|$. Further, $3(a-1) + 1 \equiv 1 \pmod{3}$. Therefore,

$$\begin{aligned}\gamma(G) &= |S| \\ &= \left\lceil \frac{3(a-1) + 1}{3} \right\rceil \\ &= a - 1 + 1 = a.\end{aligned}$$

The set of end vertices of G forms a minimum steiner set of G and so $s(G) = s + t = b$.

As the set of all end vertices together with $\{v_1, v_4, v_7, v_{3(a-1)+1}\}$ forms a minimum steiner dominating set of G , $\gamma_s(G) = b + a = c$.

Hence, it is proved that for three positive integers a , b and c with $a, b \geq 2$ and $\max\{a, b\} \leq c \leq a + b$, there exists a graph G with $\gamma(G) = a$, $s(G) = b$ and $\gamma_s(G) = c$.

□

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