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Gd-DISTANCE OF PRODUCT GRAPHS

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ABSTRACT. To study and to propose methods to classify the nature of problems in biological tissues, in [5], it is introduced the concept of $Gd-$distance of a graph and also computed the $Gd-$distance of some standard graphs and special graphs. In this paper, we determine the $Gd-$distance of the cartesian product of two graphs. Also we apply our results to compute the $Gd-$distance of nanotube, grid graph and nanotorus.

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a simple connected graph of order n. The distance between two vertices x and y in G, denoted by $d_G(x, y)$ or $d(x, y)$, is the length of the shortest $x - y$ path, also called a $x - y$ geodesic. The degree of a vertex $w \in V(G)$ is denoted by $d_G(w)$. Let P_n and C_n denote the path and the cycle on n vertices, respectively. The number of edges of G is denoted by $\varepsilon(G)$.

The Wiener index $W(G)$ is the first distance based topological index defined as

$$W(G) = \sum_{\{x,y\}\subseteq V(G)} d_G(x, y) = \frac{1}{2} \sum_{x,y\in V(G)} d_G(x, y)$$

with the summation going over all pairs of vertices of G.

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The degree distance was introduced by Dobrynin and Kochetova in [2]. The degree distance of G, denoted by $DD(G)$, is defined as

$$\begin{aligned} DD(G) &= \sum_{\{x,y\}\subseteq V(G)} d_G(x,y)[d_G(x)+d_G(y)] \\ &= \frac{1}{2}\sum_{x,y\in V(G)} d_G(x,y)[d_G(x)+d_G(y)]. \end{aligned}$$

$Dd-$distance in graphs was introduced by A. Anto Kinsley et.al. in [1]. $Dd-$length of a $x-y$ path is defined as $D^{Dd}(x,y) = D(x,y) + degx + degy$, where $D(x,y)$ is the length of the longest $x-y$ path.

V. Maheswari et.al. introduced the concept of $Gd-$distance between any two vertices in graphs and also the $Gd-$distance of a graph in [4, 5]. The $Gd-$distance of a $x-y$ path is defined as $d^{Gd}(x,y) = d(x,y) + degx + degy$. The $Gd-$distance of G, denoted by $d^{Gd}(G)$ is defined as

$$d^{Gd}(G) = \sum_{\{x,y\}\subseteq V(G)} [d(x,y) + degx + degy]$$

The Cartesian product $G_1 \square G_2$ of graphs G_1 and G_2 is a graph such that the vertex set of $G_1 \square G_2$ is the Cartesian product $V(G_1)\times V(G_2)$; two vertices (u,x) and (v,y) are adjacent in $G_1\square G_2$ if and only if either $u=v$ and $xy\in E(G_2)$ or $uv\in E(G_1)$ and $x=y$. The graphs $R = P_m\square C_n$, $T = P_m\square P_n$ and $S = C_m\square C_n$ are known as nanotube, grid and nanotorus respectively.

In this paper, we obtain the $Gd-$distance of the cartesian product of graphs.

Lemma 1.1. *[3] Let G_1 and G_2 be two connected graphs. Let $w_{ij} = (u_i, v_j)$ and $w_{pq} = (u_p, v_q)$ be in $V(G_1\square G_2)$. Then*

$$\begin{aligned} d_{G_1\square G_2}(w_{ij}, w_{pq}) &= d_{G_1}(u_i,u_p) + d_{G_2}(v_j,v_q) \\ d_{G_1\square G_2}(w_{ij}) &= d_{G_1}(u_i) + d_{G_2}(v_j)\,. \end{aligned}$$

Lemma 1.2. *[6] Let P_n and C_n denote the path and the cycle on n vertices, respectively.*

(1) *For $n\geq 2$, $W(P_n) = \frac{1}{6}n(n^2-1)$*

(2) *For $n\geq 3$,* $W(C_n) = \begin{cases} \frac{n^3}{8}, & if\ n\ is\ even, \\ \frac{n(n^2-1)}{8}, & if\ n\ is\ odd. \end{cases}$

Lemma 1.3. *[5] Let P_n and C_n denote the path and the cycle on n vertices, respectively.*

(1) *For* $n \geq 2, d^{Gd}(P_n) = \frac{1}{6}(n-1)(n^2+13n-12)$.

(2) *For* $n \geq 3$, $d^{Gd}(C_n) = \begin{cases} \frac{1}{8}n(n^2+16n-16), & if\ n\ is\ even, \\ \frac{1}{8}n(n-1)(n+17), & if\ n\ is\ odd. \end{cases}$

2. $Gd-$DISTANCE OF CARTESIAN PRODUCT OF GRAPHS

Theorem 2.1. *If G_1 and G_2 are two connected graphs with $|V(G_1)| = m$ and $|V(G_2)| = n$, where $m, n \geq 2$, then:*

$$d^{Gd}(G_1 \Box G_2) = 2m(m-1)\varepsilon(G_2) + 2n(n-1)\varepsilon(G_1) + n^2 d^{Gd}(G_1) + m^2 d^{Gd}(G_2)$$

where $\varepsilon(G)$ and $d^{Gd}(G)$ denote the number of edges and $Gd-$distance of G, respectively.

Proof. Let $G = G_1 \Box G_2$. Then

$$\begin{aligned} d^{Gd}(G) &= \frac{1}{2} \sum_{w_{ij}, w_{pq} \in V(G)} [d_G(w_{ij}, w_{pq}) + d_G(w_{ij}) + d_G(w_{pq})] \\ &= \frac{1}{2} \Bigg\{ \sum_{j=0}^{n-1} \sum_{\substack{i,p=0 \\ i \neq p}}^{m-1} [d_G(w_{ij}, w_{pj}) + d_G(w_{ij}) + d_G(w_{pj})] \\ &\quad + \sum_{i=0}^{m-1} \sum_{\substack{j,q=0 \\ j \neq q}}^{n-1} [d_G(w_{ij}, w_{iq}) + d_G(w_{ij}) + d_G(w_{iq})] \\ &\quad + \sum_{\substack{j,q=0 \\ j \neq q}}^{n-1} \sum_{\substack{i,p=0 \\ i \neq p}}^{m-1} [d_G(w_{ij}, w_{pq}) + d_G(w_{ij}) + d_G(w_{pq})] \Bigg\} \\ &= \frac{1}{2}(A_1 + A_2 + A_3) \end{aligned} \tag{2.1}$$

where A_1, A_2 and A_3 are the sums in the above term, in order as in (2.1). We calculate A_1, A_2 and A_3 of (2.1) separately.

First we compute

$$A_1 = \sum_{j=0}^{n-1} \sum_{\substack{i,p=0 \\ i \neq p}}^{m-1} [d_G(w_{ij}, w_{pj}) + d_G(w_{ij}) + d_G(w_{pj})]$$

For this, initially we calculate

$$\begin{aligned}
A_1' &= \sum_{\substack{i,p=0 \\ i \neq p}}^{m-1} [d_G(w_{ij}, w_{pj}) + d_G(w_{ij}) + d_G(w_{pj})] \\
&= \sum_{\substack{i,p=0 \\ i \neq p}}^{m-1} [d_{G_1}(u_i, u_p) + d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_p) + d_{G_2}(v_j)] \\
&= \sum_{\substack{i,p=0 \\ i \neq p}}^{m-1} [d_{G_1}(u_i, u_p) + d_{G_1}(u_i) + d_{G_1}(u_p) + 2d_{G_2}(v_j)] \\
&= \sum_{\substack{i,p=0 \\ i \neq p}}^{m-1} [d_{G_1}(u_i, u_p) + d_{G_1}(u_i) + d_{G_1}(u_p)] + \sum_{\substack{i,p=0 \\ i \neq p}}^{m-1} 2d_{G_2}(v_j) \\
&= 2d^{Gd}(G_1) + 2d_{G_2}(v_j) \sum_{\substack{i,p=0 \\ i \neq p}}^{m-1} 1 \\
&= 2d^{Gd}(G_1) + 2m(m-1)2d_{G_2}(v_j) \qquad (2.2)
\end{aligned}$$

Using (2.2), we obtain:

$$\begin{aligned}
A_1 &= \sum_{j=0}^{n-1} [2d^{Gd}(G_1) + 2m(m-1)2d_{G_2}(v_j)] \\
&= \sum_{j=0}^{n-1} 2d^{Gd}(G_1) + \sum_{j=0}^{n-1} 2m(m-1)2d_{G_2}(v_j) \\
&= 2d^{Gd}(G_1) \sum_{j=0}^{n-1} 1 + 2m(m-1) \sum_{j=0}^{n-1} 2d_{G_2}(v_j) \\
&= 2nd^{Gd}(G_1) + 4m(m-1)\varepsilon(G_2)
\end{aligned}$$

Therefore,

$$A_1 = 2nd^{Gd}(G_1) + 4m(m-1)\varepsilon(G_2) \qquad (2.3)$$

Next we compute

$$A_2 = \sum_{i=0}^{m-1} \sum_{\substack{j,q=0 \\ j \neq q}}^{n-1} [d_G(w_{ij}, w_{iq}) + d_G(w_{ij}) + d_G(w_{iq})]$$

For this, first we compute:

$$\begin{aligned}
A'_2 &= \sum_{\substack{j,q=0 \\ j \neq q}}^{n-1} [d_G(w_{ij}, w_{iq}) + d_G(w_{ij}) + d_G(w_{iq})] \\
&= \sum_{\substack{j,q=0 \\ j \neq q}}^{n-1} [d_{G_2}(v_j, v_q) + d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_i) + d_{G_2}(v_q)] \\
&= \sum_{\substack{j,q=0 \\ j \neq q}}^{n-1} [d_{G_2}(v_j, v_q) + d_{G_2}(v_j) + d_{G_2}(v_q) + 2d_{G_1}(u_i)] \\
&= \sum_{\substack{j,q=0 \\ j \neq q}}^{n-1} [d_{G_2}(v_j, v_q) + d_{G_2}(v_j) + d_{G_2}(v_q)] + \sum_{\substack{j,q=0 \\ j \neq q}}^{n-1} 2d_{G_1}(u_i) \\
&= 2d^{Gd}(G_2) + 2d_{G_1}(u_i) \sum_{\substack{j,q=0 \\ j \neq q}}^{n-1} 1
\end{aligned}$$

$$= 2d^{Gd}(G_2) + 2n(n-1)2d_{G_1}(u_i) \tag{2.4}$$

Using (2.4), we obtain

$$\begin{aligned}
A_2 &= \sum_{i=0}^{m-1} [2d^{Gd}(G_2) + 2n(n-1)2d_{G_1}(u_i)] \\
&= \sum_{i=0}^{m-1} 2d^{Gd}(G_2) + \sum_{i=0}^{m-1} 2n(n-1)2d_{G_1}(u_i) \\
&= 2d^{Gd}(G_2) \sum_{i=0}^{m-1} 1 + 2n(n-1) \sum_{i=0}^{m-1} d_{G_1}(u_i) \\
&= 2md^{Gd}(G_2) + 4n(n-1)\varepsilon(G_1)
\end{aligned}$$

Therefore,

$$A_2 = 2md^{Gd}(G_2) + 4n(n-1)\varepsilon(G_1) \tag{2.5}$$

Next we compute

$$A_3 = \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} \sum_{\substack{i,p=0\\ i\neq p}}^{m-1} [d_G(w_{ij}, w_{pq}) + d_G(w_{ij}) + d_G(w_{pq})]$$

For this, first we compute

$$\begin{aligned}
A'_3 &= \sum_{\substack{i,p=0\\ i\neq p}}^{m-1} [d_G(w_{ij}, w_{pq}) + d_G(w_{ij}) + d_G(w_{pq})] \\
&= \sum_{\substack{i,p=0\\ i\neq p}}^{m-1} [d_{G_1}(u_i, u_p) + d_{G_2}(v_j, v_q) + d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_p) + d_{G_2}(v_q)] \\
&= \sum_{\substack{i,p=0\\ i\neq p}}^{m-1} [d_{G_1}(u_i, u_p) + d_{G_1}(u_i) + d_{G_1}(u_p) + d_{G_2}(v_j, v_q) + d_{G_2}(v_j) + d_{G_2}(v_q)] \\
&= \sum_{\substack{i,p=0\\ i\neq p}}^{m-1} [d_{G_1}(u_i, u_p) + d_{G_1}(u_i) + d_{G_1}(u_p)] + \sum_{\substack{i,p=0\\ i\neq p}}^{m-1} [d_{G_2}(v_j, v_q) + d_{G_2}(v_j) + d_{G_2}(v_q)] \\
&= 2d^{Gd}(G_1) + [d_{G_2}(v_j, v_q) + d_{G_2}(v_j) + d_{G_2}(v_q)] \sum_{\substack{i,p=0\\ i\neq p}}^{m-1} 1 \\
&= 2d^{Gd}(G_1) + m(m-1)[d_{G_2}(v_j, v_q) + d_{G_2}(v_j) + d_{G_2}(v_q)]
\end{aligned}$$

(2.6)

Using (2.6), we obtain

$$\begin{aligned}
A_3 &= \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} [2d^{Gd}(G_1) + m(m-1)[d_{G_2}(v_j, v_q) + d_{G_2}(v_j) + d_{G_2}(v_q)]] \\
&= \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} 2d^{Gd}(G_1) + \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} m(m-1)[d_{G_2}(v_j, v_q) + d_{G_2}(v_j) + d_{G_2}(v_q)] \\
&= 2d^{Gd}(G_1) \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} 1 + m(m-1) \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} [d_{G_2}(v_j, v_q) + d_{G_2}(v_j) + d_{G_2}(v_q)] \\
&= 2n(n-1)d^{Gd}(G_1) + 2m(m-1)d^{Gd}(G_2)
\end{aligned}$$

Therefore,

$$A_3 = 2n(n-1)d^{Gd}(G_1) + 2m(m-1)d^{Gd}(G_2) \tag{2.7}$$

Using (2.3), (2.5) and (2.7) in (2.1), we get:

$$\begin{aligned}
d^{Gd}(G_1\Box G_2) &= \frac{1}{2}(A_1 + A_2 + A_3) \\
&= \frac{1}{2}\{2nd^{Gd}(G_1) + 4m(m-1)\varepsilon(G_2) + 2md^{Gd}(G_2) + 4n(n-1) \\
&\quad \varepsilon(G_1) + 2n(n-1)d^{Gd}(G_1) + 2m(m-1)d^{Gd}(G_2)\} \\
&= \frac{1}{2}\{4m(m-1)\varepsilon(G_2) + 4n(n-1)\varepsilon(G_1) + 2n^2d^{Gd}(G_1) \\
&\quad +2m^2d^{Gd}(G_2)\} \\
d^{Gd}(G_1\Box G_2) &= 2m(m-1)\varepsilon(G_2) + 2n(n-1)\varepsilon(G_1) + n^2d^{Gd}(G_1) + m^2d^{Gd}(G_2)
\end{aligned}$$

□

Theorem 2.2. *For a graph $G = P_m\Box C_n$,*

$$d^{Gd}(P_m\Box C_n) = \begin{cases} \frac{mn}{24}[3mn^2 + 4nm^2 + 96mn - 52n - 96 + \frac{48}{m}], & if\ n\ is\ even \\ \frac{mn}{24}[3mn^2 + 4nm^2 + 96mn - 52n - 3m - 96 + \frac{48}{m}], & if\ n\ is\ odd \end{cases}$$

where $m \geq 2$ and $n \geq 3$.

Proof. We know that

$$d^{Gd}(G_1\Box G_2) = 2m(m-1)\varepsilon(G_2) + 2n(n-1)\varepsilon(G_1) + n^2d^{Gd}(G_1) + m^2d^{Gd}(G_2)$$

Case (i): if n is even:

$$\begin{aligned}
d^{Gd}(P_m\Box C_n) &= 2m(m-1)\varepsilon(C_n)+2n(n-1)\varepsilon(P_m)+n^2d^{Gd}(P_m)+m^2d^{Gd}(C_n)\\
&= 2m(m-1)n+2n(n-1)(m-1)+\frac{1}{6}n^2(m-1)(m^2+13m-12)\\
&\quad +\frac{1}{8}m^2n(n^2+16n-16)\\
&= \frac{1}{24}\{48mn(m-1)+48n(n-1)(m-1)+4(m-1)n^2(m^2+13m-12)\\
&\quad +3m^2n(n^2+16n-16)\}\\
&= \frac{1}{24}\{3nm[mn^2+16mn-16]+4n(m-1)[nm^2+13mn-12]\}\\
&= \frac{1}{24}\{3m^2n^3+48m^2n^2-48mn+4n^2m^3+52n^2m^2-48mn-4n^2m^2\\
&\quad -52n^2m+48n\}\\
&= \frac{mn}{24}\{3mn^2+4nm^2+96mn-52n-96+\frac{48}{m}\}
\end{aligned}$$

Case (ii): if n is odd:

$$\begin{aligned}
d^{Gd}(P_m\Box C_n) &= 2m(m-1)\varepsilon(C_n)+2n(n-1)\varepsilon(P_m)+n^2d^{Gd}(P_m)+m^2d^{Gd}(C_n)\\
&= 2mn(m-1)+2n(n-1)(m-1)+\frac{1}{6}n^2(m-1)(m^2+13m-12)\\
&\quad +\frac{1}{8}m^2n(n-1)(n+17)\\
&= \frac{n}{24}\{48mn(m-1)+48n(n-1)(m-1)+4n(m-1)(m^2+13m-12)\\
&\quad +3m^2(n-1)(n+17)\}\\
&= \frac{n}{24}\{48m(m-1)+4(m-1)n(m^2+13m-12)+48(n-1)(m-1)\\
&\quad +3m^2(n-1)(n+17)\}\\
&= \frac{n}{24}\{4(m-1)[12m+n(m^2+13m-12)]+3(n-1)[16(m-1)\\
&\quad +m^2(n+17)]\}\\
&= \frac{n}{24}\{4nm^3+3m^2n^2+96m^2n-52mn-3m^2-96m+48\}\\
&= \frac{mn}{24}\{3mn^2+4nm^2+96mn-52n-3m-96+\frac{48}{m}\}
\end{aligned}$$

□

Theorem 2.3. *For a graph* $G=P_m\Box P_n$*,* $d^{Gd}(P_m\Box P_n)=\frac{1}{6}\{(m+n)(m^2n^2-13mn+12)+24m^2n^2-24mn\}$*, where* $m,n\geq 2$.

Proof. We have:

$$\begin{aligned}
d^{Gd}(G_1\Box G_2) &= 2m(m-1)\varepsilon(G_2)+2n(n-1)\varepsilon(G_1)+n^2d^{Gd}(G_1)+m^2d^{Gd}(G_2)\\
d^{Gd}(P_m\Box P_n) &= 2m(m-1)\varepsilon(P_n)+2n(n-1)\varepsilon(P_m)+n^2d^{Gd}(P_m)+m^2d^{Gd}(P_n)\\
&= 2m(m-1)(n-1)+2n(n-1)(m-1)+\frac{1}{6}n^2(m-1)(m^2+13m-12)\\
&\quad +\frac{1}{6}m^2(n-1)(m^2+13n-12)\\
&= 2m(m-1)(n-1)+\frac{1}{6}m^2(n-1)(n^2+13n-12)+2n(n-1)(m-1)\\
&\quad +\frac{1}{6}n^2(m-1)(m^2+13m-12)\\
&= \frac{1}{6}\{m(n-1)[12(m-1)+m(n^2+13n-12)]+n(m-1)[12(n-1)\\
&\quad +n(m^2+13m-12)]\}\\
&= \frac{1}{6}\{(mn-m)[mn^2+13mn-12]+(mn-n)[nm^2+13mn-12]\}\\
&= \frac{1}{6}\{m^2n^2(m+n)+26m^2n^2-24mn-2m^2n^2-13mn(m+n)+12(m+n)\}\\
&= \frac{1}{6}\{(m+n)(m^2n^2-13mn+12)+24m^2n^2-24mn\}.
\end{aligned}$$

□

Theorem 2.4. *For a graph $G=C_m\Box C_n$,*

$$d^{Gd}(C_m\Box C_n)=\begin{cases}\frac{mn}{8}[mn(m+n+32)-32] & \text{if } m \text{ is even and } n \text{ is even}\\ \frac{mn}{8}[(mn-1)(m+n+32)] & \text{if } m \text{ is odd and } n \text{ is odd}\\ \frac{mn}{8}[mn(m+n)+32(mn-1)-m] & \text{if } m \text{ is even and } n \text{ is odd}\\ \frac{mn}{8}[mn(m+n)+32(mn-1)-n] & \text{if } m \text{ is odd and } n \text{ is even}\end{cases}$$

where $m,n\geq 3$.

Proof. We have

$$d^{Gd}(G_1\Box G_2)=2m(m-1)\varepsilon(G_2)+2n(n-1)\varepsilon(G_1)+n^2d^{Gd}(G_1)+m^2d^{Gd}(G_2)$$

Case (i): both m and n are even:

$$\begin{aligned}
d^{Gd}(C_m\Box C_n) &= 2m(m-1)\varepsilon(C_n)+2n(n-1)\varepsilon(C_m)+n^2d^{Gd}(C_m)+m^2d^{Gd}(C_n)\\
&= 2m(m-1)n+2n(n-1)m+n^2\frac{1}{8}m(m^2+16m-16)\\
&\quad +m^2\frac{1}{8}n(n^2+16n-16)\\
&= \frac{mn}{8}[16(m-1)+16(n-1)+n(m^2+16m-16)\\
&\quad +m(n^2+16n-16)]\\
&= \frac{mn}{8}[nm^2+mn^2+32mn-32]\\
&= \frac{mn}{8}[mn(m+n)+32mn-32]\\
&= \frac{mn}{8}[mn(m+n+32)-32]
\end{aligned}$$

Thus, $d^{Gd}(C_m\Box C_n)=\frac{mn}{8}[mn(m+n+32)-32]$, when both m and n are even.

Case (ii): both m and n are odd:

$$\begin{aligned}
d^{Gd}(C_m\Box C_n) &= 2m(m-1)\varepsilon(C_n)+2n(n-1)\varepsilon(C_m)+n^2d^{Gd}(C_m)+m^2d^{Gd}(C_n)\\
&= 2m(m-1)n+2n(n-1)m+n^2\frac{1}{8}m(m-1)(m+17)\\
&\quad +m^2\frac{1}{8}n(n-1)(n+17)\\
&= \frac{mn}{8}[16(m-1)+16(n-1)+n(m-1)(m+17)+m(n-1)(n+17)]\\
&= \frac{mn}{8}[16(m-1)+n(m-1)(m+17)+16(n-1)+m(n-1)(n+17)]\\
&= \frac{mn}{8}[(m-1)[16+n(m+17)]+(n-1)[16+m(n+17)]]\\
&= \frac{mn}{8}[32mn+(m+n)mn-m-n-32]\\
&= \frac{mn}{8}[(m+n)(mn-1)+32(mn-1)]\\
&= \frac{mn}{8}[(mn-1)(m+n+32)]\,.
\end{aligned}$$

Thus, $d^{Gd}(C_m\Box C_n)=\frac{mn}{8}[(mn-1)(m+n+32)]$, when both m and n are odd.

Case (iii): if m is even and n is odd:

$$
\begin{aligned}
d^{Gd}(C_m\Box C_n) &= 2m(m-1)\varepsilon(C_n)+2n(n-1)\varepsilon(C_m)+n^2d^{Gd}(C_m)+m^2d^{Gd}(C_n)\\
&= 2m(m-1)n+2n(n-1)m+n^2\frac{1}{8}m(m^2+16m-16)\\
&\quad +m^2\frac{1}{8}n(n-1)(n+17)\\
&= \frac{mn}{8}[16(m-1)+16(n-1)+n(m^2+16m-16)+m(n-1)(n+17)]\\
&= \frac{mn}{8}[mn^2+nm^2+32mn-m-32]\\
&= \frac{mn}{8}[mn(m+n)+32(mn-1)-m]\,.
\end{aligned}
$$

Thus, $d^{Gd}(C_m\Box C_n)=\frac{mn}{8}[mn(m+n)+32(mn-1)-m]$, if m is even and n is odd.

Case (iv): if m is odd and n is even:

$$
\begin{aligned}
d^{Gd}(C_m\Box C_n) &= 2m(m-1)\varepsilon(C_n)+2n(n-1)\varepsilon(C_m)+n^2d^{Gd}(C_m)+m^2d^{Gd}(C_n)\\
&= 2m(m-1)n+2n(n-1)m+n^2\frac{1}{8}m(m-1)(m+17)\\
&\quad +m^2\frac{1}{8}n(n^2+16n-16)\\
&= \frac{mn}{8}[16(m-1)+16(n-1)+n(m-1)(m+17)+m(n^2+16n-16)]\\
&= \frac{mn}{8}[mn^2+nm^2+32mn-n-32]\\
&= \frac{mn}{8}[mn(m+n)+32(mn-1)-n].
\end{aligned}
$$

Thus, $d^{Gd}(C_m\Box C_n)=\frac{mn}{8}[mn(m+n)+32(mn-1)-n$, if m is odd and n is even. $\Box$

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