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STRONG EFFICIENT EDGE BONDAGE NUMBER OF SOME GRAPHS

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ABSTRACT. Let $G = (V, E)$ be a simple graph. A subset S of $E(G)$ is a strong (weak) efficient edge dominating set of G if $|Ns[e] \in S| = 1$ for all $e \in E(G)$ ($|Nw[e] \in S| = 1$ for all $e \in E(G)$) where $Ns(e) = \{f/f \in E(G)\ and\ degf \geq dege\}$ $(Nw(e) = \{f/f \in E(G)\ and\ degf \leq dege\})$ and $Ns[e] = Ns(e) \cup \{e\}(Nw[e] = Nw(e) \cup \{e\})$. The minimum cardinality of a strong efficient edge dominating set of G (weak efficient edge dominating set of G) is called a strong efficient edge domination number of G and is denoted by $\gamma'_{se}(G)(\gamma'_{we}(G))$. In this paper, the strong efficient edge bondage number of some graphs is determined.

1. INTRODUCTION

Throughout this paper, only finite, undirected and simple graphs are considered. Two volumes on domination have been published by Haynes, Hedetniemi and Slater in [9, 10]. Edge dominating sets were studied by Mitchell and Hedetniemi in [12]. A set F of edges in a graph G is called an edge dominating set of G if every edge in $E - S$ is adjacent to at least one edge in F. The edge domination number $\gamma'(G)$ of a graph G is the minimum cardinality of an edge dominating set of G. The degree of an edge was introduced by Kulli in [8]. The concept of efficient domination was introduced by Bange et al. in [4, 5]. The concept of strong domination graphs was introduced by Kumar and Pushpalatha

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in [13] and efficient edge domination were studied by C. L. Lu et al. in [11]. A subset D of $E(G)$ is called an efficient edge dominating set if every edge in $E(G)$ is dominated by exactly one edge in D. The cardinality of the minimum efficient edge dominating set is called the efficient edge domination number of G. The strong efficient edge domination was introduced by M. Annapoopathi and N. Meena, [1]. The edge bondage number $b'(G)$ of a graph G was introduced by V.R.Kulli in [7]. In this paper, the strong efficient edge bondage number of some graphs is determined.
The following results are from [1–3, 6].

Definition 1.1. *Let $G = (V, E)$ be a simple graph. A subset S of $E(G)$ is a strong (weak) efficient edge dominating set of G if $Ns[e] \cap S = 1$ for all $e \in E(G)$ $[Nw[e] \cap S = 1$ for all $e \in E(G)]$ where $N_s(e) = \{ f/f \in E(G) \ and \ degf \geq dege\}$ $(N_w(e) = \{f/f \in E(G) \ and \ degf \leq dege\})$ and $N_s[e] = N_s e \cup \{e\} Nw[e] = N_w e \cup \{e\}$.*

The minimum cardinality of a strong efficient edge dominating set of G (weak efficient edge dominating set of G) is called a strong(weak) efficient edge domination number of G and is denoted by $\gamma'_{se}(G)(\gamma'_{we}(G))$.

Definition 1.2. *The helm H_n is the graph obtained from the cycle C_n with n spokes by adding a pendant edge at each vertex on the cycle's rim.*

Definition 1.3. *A flower F_n is constructed from a helm H_n by joining each vertex of degree one to the centre.*

Definition 1.4. *A fan S_n is constructed from the path P_{n+1} by adding a pendant vertex and by joining each vertices of the path P_{n+1} by new edges.*

Observation 1. $\gamma'_s(G) \leq \gamma'_{se}(G)$.

Definition 1.5. *Let $G = (V, E)$ be a simple graph. Let $E(G) = \{e_1, \ e_2, \ e_3, e_4, \ldots e_n \}$. An edge e_i is said to be full degree edge if and only if deg $e_i = n - 1$.*

Observation 2. $\gamma'_{se}(G) = 1$ *if and only if G has a full degree edge.*

Observation 3. $\gamma'_{se}(K_{1,n}) = 1, \ \ n \geq 1$ *and* $\gamma'_{se}(D_{r,s}) = 1, \ \ n \geq 1$.

Observation 4. $\gamma'_{se}(C_{3n}) = n, \ \ \forall\, n \in N$.

Observation 5. *C_{3n+1}, C_{3n+2} do not have efficient edge dominating sets, they do not have strong efficient edge dominating sets.*

Observation 6. *For any path P_m, $\gamma'_{se}(\ P_m\ ) = \begin{cases} n\ if m = 3n+1, n \geq 1 \\ n+1,\ if m = 3n, n \geq 2 \\ n+1,\ if m = 3n+2, n \geq 1 \end{cases}$.*

Observation 7. *Let W_m be a wheel graph. Then W_m has a strong efficient edge dominating set if and only if $m = 3n$ and $\gamma'_{se}(W_{3n}) = n,\ n \geq 2$.*

Observation 8. *Let $H_n = W_n \circ K_1$ be the helm graph. Then $\gamma'_{se}(Hm) = 2n$, $n \geq 2, m = 3n$.*

Observation 9. *A flower graph F_m has a strong efficient edge dominating set if and only if $m = 3n, n \geq 1$. Then $\gamma'_{se}(F_{3n}) = 2n, n \geq 2$.*

2. STRONG EFFICIENT EDGE BONDAGE NUMBER OF SOME GRAPHS

Definition 2.1. *The strong efficient edge bondage number $b_{see}(G)$ of a graph without isolated vertices is the minimum cardinality among all sets of edges $X \subseteq E$ such that $G - X$ has no isolated vertices $\gamma'_{se}(G - X) > \gamma'_{se}(G)$.*

Example 1. *Consider the following graph G The set $\{e_4, e_7\}$ is the unique strong*

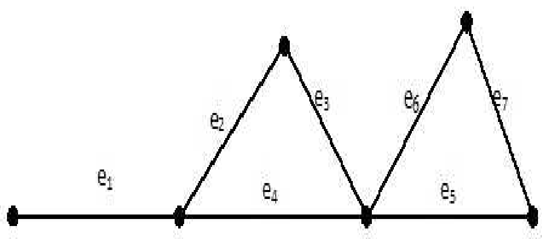

FIGURE 1

efficient edge dominating set of G. Therefore $\gamma'_{se}(G) = 2$.
Let $X = \{e_4, e_5\}$. Then $G - X = P_6$ and $\gamma'_{se}(G - X) = \gamma'_{se}(P_6) = 3$. Therefore $\gamma'_{se}(G - X) > \gamma'_{se}(G)$. Hence $b_{see}(G) \leq 2$. Suppose any edge is removed from G, it is verified that $\gamma'_{se}(G - e_i) \leq \gamma'_{se}(G)$. Hence $b_{see}(G) \geq 2$. Therefore $b_{see}(G) = 2$.

Theorem 2.1. *Let $G = P_m$. Then $b_{see}(G) = 1, m \geq 4$.*

Proof. **Case 1:** Let $G = P_{3n}, n \geq 4$. Let $V(G) = \{v_i/1 \leq i \leq 3n\}$. Let $e_i = v_i v_{i+1}, 1 \leq i \leq 3n - 1$. Then $E(G) = \{e_i/1 \leq i \leq 3n - 1\}$ and $|E(G)| = 3n - 1, n \geq 4$. $Let X = \{e_6\}$. Then $G - X = P_6 \cup P_{3n-6}$. Therefore $\gamma'_{se}(G - X) = \gamma'_{se}(P_6) + \gamma'_{se}(P_{3n-6}) = \gamma'_{se}(P_6) + \gamma'_{se}(P_{3(n-2)}) = 3 + n - 1 = n + 2, n \geq 4$. Since $\gamma'_{se}(P_{3n}) = n + 1, \gamma'_{se}(G - X) > \gamma'_{se}(G)$. Hence $b_{see}(G) = 1, n \geq 4$.
Case 2: Let $G = P_{3n+1}, n \geq 1$. Let $V(G) = \{v_i/1 \leq i \leq 3n + 1\}$. Let $e_i = v_i v_{i+1}, 1 \leq i \leq 3n$. Then $E(G) = \{e_i/1 \leq i \leq 3n\}$ and $|E(G)| = 3n, n \geq 1$. Let $X = \{e_2\}$. Then $G - X = P_2 \cup P_{3n-1}$. Therefore $\gamma'_{se}(G - X) = \gamma'_{se}(P_2) + \gamma'_{se}(P_{3n-1}) = \gamma'_{se}(P_2) + \gamma'_{se}(P_{3(n-1)+2}) = 1 + n - 1 + 1 = n + 1, n \geq 1$. Since $\gamma'_{se}(P_{3n+1}) = n, \gamma'_{se}(G - X) > \gamma'_{se}(G)$. Hence $b_{see}(G) = 1, n \geq 1$.
Case 3: Let $G = P_{3n+2}, n \geq 2$. Let $V(G) = \{v_i/1 \leq i \leq 3n + 2\}$. Let $e_i = v_i v_{i+1}, 1 \leq i \leq 3n + 1$. Then $E(G) = \{e_i/1 \leq i \leq 3n + 1\}$ and $|E(G)| = 3n + 1, n \geq 1$. Let $X = \{e_2\}$. Then $G - X = P_2 \cup P_{3n}$. Therefore $\gamma'_{se}(G - X) = \gamma'_{se}(P_2) + \gamma'_{se}(P_{3n}) = n + 2, n \geq 2$. Since $\gamma'_{se}(P_{3n+2}) = n + 1, \gamma'_{se}(G - X) > \gamma'_{se}(G)$. Hence $b_{see}(G) = 1, n \geq 2$. □

Remark 2.1. *An edge in P_2 or P_3 is removed, isolated vertex exists. Hence $b_{see}(P_2)$ and $b_{see}(P_3)$ does not exist.*

Remark 2.2. *Let $G = P_5$. Let $V(G) = \{v_i/1 \leq i \leq 5\}$. Let $e_i = v_i v_{i+1}, 1 \leq i \leq 4$. Then $E(G) = \{e_i/1 \leq i \leq 4\}$ and Let $X = \{e_2\}$. Then $G - X = P_2 \cup P_3$. Therefore $\gamma'_{se}(G - X) = \gamma'_{se}(P_2) + \gamma'_{se}(P_3) = 1 + 1 = 2$. Since $\gamma'_{se}(P_5) = 2, \gamma'_{se}(G - X) = \gamma'_{se}(G)$. Hence $b_{see}(G)$ does not exists.*

Remark 2.3. *Let $G = P_6$. Let $V(G) = \{v_i/1 \leq i \leq 6\}$. Let $e_i = v_i v_{i+1}, 1 \leq i \leq 5$. Then $E(G) = \{e_i/1 \leq i \leq 5\}$ and Let $X = \{e_2, e_4\}$. Then $G - X = 3P_2$. Therefore $\gamma'_{se}(G - X) = 3\gamma'_{se}(P_2) = 3$. Since $\gamma'_{se}(P_6) = 3, \gamma'_{se}(G - X) = \gamma'_{se}(G)$. Hence $b_{see}(G)$ does not exists.*

Remark 2.4. *Let $G = P_9$. Let $V(G) = \{v_i/1 \leq i \leq 9\}$. Let $e_i = v_i v_{i+1}, 1 \leq i \leq 8$. Then $E(G) = \{e_i/1 \leq i \leq 8\}$ and Let $X = \{e_6\}$. Then $G - X = P_6 \cup P_3$. Therefore $\gamma'_{se}(G - X) = 3\gamma'_{se}(P_2) + \gamma'_{se}(P_3) = 3 + 1 = 4$. Since $\gamma'_{se}(P_9) = 4, \gamma'_{se}(G - X) = \gamma'_{se}(G)$. Hence $b_{see}(G)$ does not exists.*

Theorem 2.2. *Let $G = C_{3n}, n \geq 2$. Then $b_{see}(G) = 1, n \geq 2$.*

Proof. Let $G = C_{3n}, n \geq 2$. Let $V(G) = \{v_i/1 \leq i \leq 3n\}$. Let $e_i = v_iv_{i+1}, 1 \leq i \leq 3n-1, e_{3n} = v_{3n}v_1$. Then $E(G) = \{e_i/1 \leq i \leq 3n\}$ and $|E(G)| = 3n, n \geq 2$. Let $X = \{e_i\}, 1 \leq i \leq 3n$. Then $G - X = C_{3n} - X = P_{3n}$. Therefore $\gamma'_{se}(G - X) = \gamma'_{se}(P_{3n}) = n + 1, n \geq 2$. Since $\gamma'_{se}(C_{3n}) = n, \gamma'_{se}(G - X) > \gamma'_{se}(G)$. Hence $b_{see}(G) = 1, n \geq 2$. □

Remark 2.5. *Let $G = C_3$. Let $V(G) = \{v_i/1 \leq i \leq 3\}$. Then $E(G) = \{e_1, e_2, e_3\}$ and Let $X = \{e_1\}$. Then $G - X = P_3$. Therefore $\gamma'_{se}(G - X) = \gamma'_{se}(P_3) = 1$ Since $\gamma'_{se}(C_3) = 1, \gamma'_{se}(G - X) = \gamma'_{se}(G)$. Hence $b_{see}(G)$ does not exists.*

Theorem 2.3. *Let $G = W_{3n}, n \geq 2$. Then $b_{see}(G) = 1, n \geq 2$.*

Proof. Let $G = W_{3n}, n \geq 2$. Let $V(G) = \{v, v_i/1 \leq i \leq 3n - 1\}$. Let $e_i = vv_i, 1 \leq i \leq 3n - 1, f_i = v_iv_{i+1}, 1 \leq i \leq 3n - 2, f_{3n-1} = v_{3n-1}v_1$. Then $E(G) = \{e_i, f_i/1 \leq i \leq 3n - 1\}$ and $|E(G)| = 6n - 2, n \geq 2$. Let $X = \{f_1\}$. In $G - X, Dege_1 = dege_2 = 3n - 1, dege_i = 3n, 3 \leq i \leq 3n - 1, degf_2 = degf_{3n-1} = 3, degf_i = 4, 3 \leq i \leq 3n - 2$. The edges $e_i, 3 \leq i \leq 3n - 1$ have maximum degree. To dominate them, any one e_i is considered. Without loss of generality, let it be the edge e_4. Then it strongly efficiently dominates all e_i, f_3, f_4. Also the edges $f_2, f_6, f_9, \ldots, f_{3n-3}, f_{3n-1}$ belong to strong efficient edge dominating set. Therefore total number of elements in the strong efficient edge dominating set is $n+1, n \geq 2$. Hence $\gamma'_{se}(G-X) \leq n+1, n \geq 2$. Also no set with less than $n+1$ edge is a strong efficient edge dominating set of G. Therefore $\gamma'_{se}(G-X) \geq n+1, n \geq 2$. Hence $\gamma'_{se}(G - X) = n + 1, n \geq 2$. Since $\gamma'_{se}(W_{3n}) = n, \gamma'_{se}(G - X) > \gamma'_{se}(G)$. Hence $b_{see}(G) = 1, n \geq 2$. □

Theorem 2.4. *Let $G = D_{r,s}, r, s \geq 1$. Then $b_{see}(G) = 1, r, s \geq 1$.*

Proof. Let $G = D_{r,s}, r, s \geq 1$. $V(G) = \{u, v, u_i, v_j/1 \leq i \leq r, 1 \leq j \leq s\}, E(G) = \{uv, uu_i, vv_j/1 \leq i \leq r, 1 \leq j \leq s\}$. Let $X = \{uv\}$. Then $G - X = K_1, r \cup K_1, s$. Therefore $\gamma'_{se}(G - X) = \gamma'_{se}(K_{1,r}) + \gamma'_{se}(K_{1,s}) = 1 + 1 = 2, r, s \geq 1$. Since $\gamma'_{se}(D_{r,s}) = 1, \gamma'_{se}(G - X) > \gamma'_{se}(G)$. Hence $b_{see}(G) = 1, r, s \geq 1$. □

Theorem 2.5. *Let H_m be the helm graph and let $G = H_m, m = 3n, n \geq 2$. Then $b_{see}(G) = 3, n \geq 2$.*

Proof. Let $G = H_{3n}, n \geq 2$. Let $V(G) = \{v, v_i, u_i/1 \leq i \leq 3n-1\}$. Let $e_i = vv_i, 1 \leq i \leq 3n - 1, f_i = v_iv_{i+1}, 1 \leq i \leq 3n - 2, f_{3n-1} = v_{3n-1}v_1, g_i = u_iv_i, 1 \leq i \leq 3n - 1$. Then $E(G) = \{e_i, f_i, g_i/1 \leq i \leq 3n - 1\}$ and $|E(G)| = 9n - 3, n \geq 2$. Let

$S_i = \{e_i, f_{i+2}, f_{i+5}, \ldots, f_{i+3n-4}, g_{i+1}, g_{i+4}, \ldots, g_{i+3n-2}, 1 \leq i \leq 3n-1$ is the strong efficient edge dominating set of G and $|S| = 2n, n \geq 2$. Hence $\gamma'_{se}(H_{3n}) = 2n, n \geq 2$. Let $X = \{f_1, f_2, f_3\}$. In $G - X, Dege_2 = dege_3 = 3n-1, dege_1 = dege_4 = 3n, dege_i = 3n+1$, $5 \leq i \leq 3n-1, degf_4 = degf_{3n-1} = 5, degf_i = 6, 5 \leq i \leq 3n-2, degg_2 = degg_3 = 1, degg_1 = degg_4 = 2, degg_i = 3, 5 \leq i \leq 3n-1$. All $e_i's$ are adjacent with each other. The edges $e_i, 5 \leq i \leq 3n-1$ have maximum degree. To dominate them, any one e_i is considered. Without loss of generality, let it be the edge e_5. Then it strongly efficiently dominates all e_i and the edges f_4, f_5. Also the edges $f_7, f_{10}, f_{13}, \ldots, f_{3n-2}$ and $g_1, g_2, g_3, g_4, g_6, g_9, \ldots, g_{3n-3}$ belongs to strong efficient edge dominating set. Therefore total number of elements in the strong efficient edge dominating set is $2n+1, n \geq 2$. Hence $\gamma'_{se}(G - X) \leq 2n+1, n \geq 2$. Therefore $b_{see}(G) \leq 3, n \geq 2$.

Case(i): Let one edge be removed from G. Let $X = \{e_i\}, 1 \leq i \leq 3n-1$ or $X = \{f_i\}, 1 \leq i \leq 3n-1$. Then $G - X$ has strong efficient edge dominating sets $S_j, 1 \leq j \leq 3n-1, i \neq j$. Hence $\gamma'_{se}(G - X) = \gamma'_{se}(G)$.

Case (ii): Two edges are removed from G. Let $X = \{e_i, e_j\}, 1 \leq i, j \leq 3n-1$ or $X = \{e_i, f_j\}, 1 \leq i, j \leq 3n-1$ or $X = \{f_i, f_j\}, 1 \leq i, j \leq 3n-1$ or $X = \{e_i, f_i\}, 1 \leq i, j \leq 3n-1$ or $X = \{e_i, f_{i-1}, 1 \leq i, j \leq 3n-1$. Then $G - X$ has strong efficient edge dominating sets $S_k, 1 \leq k \leq 3n-1, k \neq i, j$. Hence $\gamma'_{se}(G - X) = \gamma'_{se}(G)$. From the cases (i) and (ii) $b_{see}(G) \geq 3$. Hence $b_{see}(G) = 3, n \geq 2$. □

Theorem 2.6. *Let F_n be a flower graph and let $G = F_{3n}, n \geq 2$. Then $b_{see}(G) = 3, n \geq 2$.*

Proof. Let $G = F_{3n}, n \geq 2$. Let $V(G) = \{v, v_i, u_i/1 \leq i \leq 3n-1\}$. Let $e_i = vv_i, 1 \leq i \leq 3n-1, f_i = v_iv_{i+1}, 1 \leq i \leq 3n-2, f_{3n-1} = v_{3n-1}v_1, g_i = u_iv_i, h_i = vu_i, 1 \leq i \leq 3n-1$. Then $E(G) = \{e_i, f_i, g_i, h_i/1 \leq i \leq 3n-1\}$ and $|E(G)| = 12n-4, n \geq 2$. Let $X = \{f_1, f_2, f_3\}$. In $G - X, Dege_2 = dege_3 = 6n-2, dege_1 = dege_4 = 6n-1, dege_i = 6n, 5 \leq i \leq 3n-1, degf_4 = degf_{3n-1} = 5, degf_i = 6, 5 \leq i \leq 3n-2, degg_2 = degg_3 = 2, degg_1 = degg_4 = 3, degg_i = 4, 5 \leq i \leq 3n-1$. All $e_i's$ are adjacent with each other. The edges $e_i, 5 \leq i \leq 3n-1$ have maximum degree. To dominate them, any one e_i is considered. Without loss of generality, let it be the edge e_5. Then it strongly efficiently dominates all $e_i \& all h_i$ and the edges f_4, f_5. Also the edges $f_7, f_{10}, f_{13}, \ldots, f_{3n-2}$ and $g_1, g_2, g_3, g_4, g_6, g_9, \ldots, g_{3n-3}$ belongs to strong efficient edge dominating set. Therefore total number of elements in the strong efficient edge dominating set is $2n+1, n \geq 2$. Hence $\gamma'_{se}(G-X) \leq 2n+1, n \geq 2$. Therefore $b_{see}(G) \leq 3, n \geq 2$.

Case(i): Let one edge be removed from G. Let $X = \{e_i\}, 1 \leq i \leq 3n-1$ or $X = \{f_i\}, 1 \leq i \leq 3n-1$. Then $G - X$ has strong efficient edge dominating sets $S_j, 1 \leq j \leq 3n-1, i \neq j$. Hence $\gamma'_{se}(G-X) = \gamma'_{se}(G)$.

Case (ii): Two edges are removed from G. Let $X = \{e_i, e_j\}, 1 \leq i, j \leq 3n-1$ or $X = \{e_i, f_j\}, 1 \leq i, j \leq 3n-1$ or $X = \{f_i, f_j\}$, $1 \leq i, j \leq 3n-1$ or $X = \{e_i, f_i\}, 1 \leq i, j \leq 3n-1$ or $X = \{e_i, f_{i-1}, 1 \leq i, j \leq 3n-1$. Then $G - X$ has strong efficient edge dominating sets $S_k, 1 \leq k \leq 3n-1, k \neq i, j$. Hence $\gamma'_{se}(G-X) = \gamma'_{se}(G)$. From the cases (i) and (ii) $b_{see}(G) \geq 3$. Hence $b_{see}(G) = 3, n \geq 2$. □

Theorem 2.7. *Let $G = F_m$ be the fan graph. Then*

$$\gamma'_{se}(\ F_m\ ) = \begin{cases} n+1, & if m = 3n, n \geq 1 \\ n+2, & if m = 3n+1, n \geq 1 \\ n+1, & if m = 3n+2, n \geq 1 \end{cases}$$

Proof. **case (1):** Let $m = 3n$ and $G = F_{3n} = P_{3n+1} + K_1, n \geq 1$. Let $V(G) = \{u, v_i / 1 \leq i \leq 3n+1\}$. Let $e_i = uv_i, 1 \leq i \leq 3n+1, f_j = v_j v_{j+1}, 1 \leq j \leq 3n, E(G) = \{e_i, f_j / 1 \leq i \leq 3n+1, 1 \leq j \leq 3n\}$. $|V(G)| = 3n+2, |E(G)| = 6n+1, n \geq 1$. $Degu = 3n+1, degv_1 = degv_{3n+1} = 2, degv_i = 3, 2 \leq i \leq 3n, dege_1 = dege_{3n+1} = 3n+1, dege_i = 3n+2, 2 \leq i \leq 3n, degf_1 = degf_{3n} = 3, degf_j = 4, 2 \leq j \leq 3n-1$. Then $S_1 = \{e_2, f_4, f_7, \ldots, f_{3n}\}, S_2 = \{e_3, f_1, f_5, \ldots, f_{3n-1}\}, S_3 = \{e_5, f_2, f_7, \ldots, f_{3n}\}, S_4 = \{e_{3n}, f_1, f_3, f_6, \ldots, f_{3n-3}\}$ are some strong efficient edge dominating sets of G and $|S_i| = n+1, 1 \leq i \leq 4$. Therefore $\gamma'_{se}(G) \leq n+1, n \geq 1$. Also no set with less than $n+1$ edge is a strong efficient edge dominating set of G. Therefore $\gamma'_{se}(G) \geq n+1, n \geq 1$. Hence $\gamma'_{se}(G) = n+1, n \geq 1$.

case (2): Let $m = 3n+1$ and $G = F_{3n+1} = P_{3n+2} + K_1, n \geq 1$. Let $V(G) = \{u, v_i / 1 \leq i \leq 3n+2\}$. Let $e_i = uv_i, 1 \leq i \leq 3n+2, f_j = v_j v_{j+1}, 1 \leq j \leq 3n+1, E(G) = \{e_i, f_j / 1 \leq i \leq 3n+2, 1 \leq j \leq 3n+1\}$. $V(G) = 3n+3, E(G) = 6n+2, n \geq 1$. $Degu = 3n+2, degv_1 = degv_{3n+2} = 2, degv_i = 3, 2 \leq i \leq 3n+1, dege_1 = dege_{3n+2} = 3n+2, dege_i = 3n+3, 2 \leq i \leq 3n+1, degf_1 = degf_{3n+1} = 3, degf_j = 4, 2 \leq j \leq 3n$. Therefore $S = \{e_3, f_1, f_5, f_8, \ldots, f_{3n+1}\}$ is the unique strong efficient edge dominating sets of G and $|S| = n+2, n \geq 1$. Hence $\gamma'_{se}(G) = n+2, n \geq 1$.

case (3): Let $m = 3n+2$ and $G = F_{3n+2} = P_{3n+3} + K_1, n \geq 1$. Let $V(G) =$

$\{u, v_i/1 \leq i \leq 3n+3\}$. Let $e_i = uv_i, 1 \leq i \leq 3n+3, f_j = v_jv_{j+1}, 1 \leq j \leq 3n+2, E(G) = \{e_i, f_j/1 \leq i \leq 3n+3, 1 \leq j \leq 3n+2\}$. $V(G) = 3n+4, E(G) = 6n+3, n \geq 1$. $Degu = 3n+3, degv_1 = degv_{3n+3} = 2, degv_i = 3, 2 \leq i \leq 3n+2, dege_1 = dege_{3n+3} = 3n+3, dege_i = 3n+4, 2 \leq i \leq 3n+2, degf_1 = degf_{3n+2} = 3, degf_j = 4, 2 \leq j \leq 3n+1$. Therefore $S = \{e_2, f_4, f_7, f_{10}, \ldots, f_{3n+1}\}$ is the unique strong efficient edge dominating sets of G and $|S| = n+1, n \geq 1$. Hence $\gamma'_{se}(G) = n+1, n \geq 1$. □

Theorem 2.8. *Let $G = F_m$ be the fan graph. Then*

$$b_{see}(G) = \begin{cases} 1, m = 3n, n \geq 2 \qquad m = 3n+2, n \geq 1 \\ 3, m = 3n+1, n \geq 3 \end{cases}.$$

Proof. **Case (1):** Let $m = 3n$ and $G = F_{3n}, n \geq 2$. Let $V(G) = \{v, v_i/1 \leq i \leq 3n+1\}$. Let $e_i = vv_i, 1 \leq i \leq 3n+1, f_j = v_jv_{j+1}, 1 \leq j \leq 3n, E(G) = \{e_i, f_j/1 \leq i \leq 3n+1, 1 \leq j \leq 3n\}$ and $|E(G)| = 6n+1, n \geq 2$. Let $X = \{f_2\}$. In $G-X, degv = 3n+1, degv_1 = degv_2 = degv_3 = degv_{3n+1} = 2, degv_i = 3, 4 \leq i \leq 3n, dege_1 = dege_2 = dege_3 = dege_{3n+1} = 3n+1, dege_i = 3n+2, 4 \leq i \leq 3n, degf_1 = 2, degf_3 = degf_{3n+1} = 3, degf_i = 4, 4 \leq i \leq 3n$. The edges $e_i, 4 \leq i \leq 3n$ have maximum degree. To dominate them, any one e_i is considered. Without loss of generality, let it be the edge e_5. Then it strongly efficiently dominates all e_i and f_4, f_5. The sub graph induced by the remaining edges $= 2P_2 \cup P_{3n-4} = 2P_2 \cup P_3(n-2)+2 = G'$ and $\gamma'_{se}(G') = n+1$. Therefore these $n+1$ edges together with e_5 form a strong efficient edge dominating set for $G-X$. Hence $\gamma'_{se}(G-X) \leq n+2, n \geq 2$. Also no set with less than $n+2$ edge is a strong efficient edge dominating set of G. Therefore $\gamma'_{se}(G-X) \geq n+2, n \geq 2$. Hence $\gamma'_{se}(G-X) = n+2, n \geq 2$. Since $\gamma'_{se}(F_{3n}) = n+1, \gamma'_{se}(G-X) > \gamma'_{se}(G)$. Hence $b_{see}(G) = 1, n \geq 2$.

Case (2): Let $m = 3n+1$ and $G = F_{3n+1}, n \geq 3$. Let $V(G) = \{v, v_i/1 \leq i \leq 3n+2\}$. Let $e_i = vv_i, 1 \leq i \leq 3n+2, f_j = v_jv_{j+1}, 1 \leq j \leq 3n+1, E(G) = \{e_i, f_j/1 \leq i \leq 3n+2, 1 \leq j \leq 3n+1\}$ and $|E(G)| = 6n+3, n \geq 3$. Let $X = \{f_2, f_4, f_6\}$. In $G-X, dege_i = 3n+2, 1 \leq i \leq 7, dege_{3n+2} = 3n+2, dege_i = 3n+3, 8 \leq i \leq 3n+1, degf_1 = degf_3 = degf_5 = 2, degf_7 = degf_{3n+1} = 3, degf_i = 4, 8 \leq i \leq 3n$. The edges $e_i, 8 \leq i \leq 3n$ have maximum degree. To dominate them, any one e_i is considered. Without loss of generality, let it be the edge e_9. Then it strongly efficiently dominates all e_i and f_8, f_9. The sub graph induced by the remaining edges $= 4P_2 \cup P_{3n-9} = 2P2 \cup P_{3(n-3)} = G'$ and $\gamma'_{se}(G') = n+2$. Therefore these $n+2$ edges together with e_9 form a strong efficient edge dominating set for $G-X$. Hence $\gamma'_{se}(G-X) \leq n+3, n \geq 3$. Also no set with less than $n+3$ edges is a strong efficient edge dominating set of G. Therefore $\gamma'_{se}(G-X) \geq n+3, n \geq 3$. Hence $\gamma'_{se}(G-X) = n+3, n \geq 3$. Since $\gamma'_{se}(F_{3n+1}) = n+2, \gamma'_{se}(G-X) > \gamma'_{se}(G)$. Hence $b_{see}(G) = 3, n \geq 3$.

Case (3): Let $m = 3n+2$ and $G = F_{3n+2}, n \geq 1$. Let $V(G) = \{v, v_i/1 \leq i \leq 3n+3\}$. Let $e_i = vv_i, 1 \leq i \leq 3n+3, f_j = v_jv_{j+1}, 1 \leq j \leq 3n+2, E(G) = \{e_i, f_j/1 \leq i \leq 3n+3, 1 \leq j \leq 3n+2\}$ and $|E(G)| = 6n+5, n \geq 1$. Let $X = \{f_2\}$. In $G-X, degv = 3n+3, degv_1 = degv_2 = degv_3 = degv_{3n+3} = 2, degv_i = 3, 4 \leq i \leq 3n+2, dege_1 = dege_2 = dege_3 = dege_{3n+3} = 3n+3, dege_i = 3n+4, 4 \leq i \leq 3n+2, degf_1 = 2, degf_3 = degf_{3n+3} = 3, degf_i = 4, 4 \leq i \leq 3n+2$. The edges $e_i, 4 \leq i \leq 3n+2$ have maximum degree. To dominate them, any one e_i is considered. Without loss of generality, let it be the edge e_4. Then it strongly efficiently dominates all e_i and f_3, f_4. The sub graph induced by the remaining edges $= P_2 \cup P_{3n-3} = P_2 \cup P_{3(n-1)} = G'$ and $\gamma'_{se}(G') = n+1$. Therefore these $n+1$ edges together with e_4 form a strong efficient edge dominating set for $G-X$. Hence $\gamma'_{se}(G-X) \leq n+2, n \geq 1$. Also no set with less than $n+2$ edges is a_4 strong efficient edge dominating set of G. Therefore $\gamma'_{se}(G-X) \geq n+2, n \geq 1$. Hence $\gamma'_{se}(G-X) = n+2, n \geq 2$. Since $\gamma'_{se}(F_{3n+2}) = n+1$, $\gamma'_{se}(G-X) > \gamma'_{se}(G)$. Hence $b_{see}(G) = 1, n \geq 1$. □

3. CONCLUSION

In this paper, the strong efficient edge bondage number of some standard graphs is determined.

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