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PRIME PAIR LABELING OF GRAPHS AND DIGRAPHS

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ABSTRACT. K. Palani et.al introduced the concept of prime pair labeling of digraphs. Let $D(p, q)$ be a digraph. An injective function $f : V \to \{1, 2, 3, \ldots, p + q\}$ is said to be a prime pair labeling of D if it is both an in and outdegree prime pair labeling of D. In this paper, we introduce the concept of prime pair labeling of graphs and investigate the existence of prime pair labeling in some standard and special graphs. Also, we investigate the existence of prime pair labeling in some digraphs.

1. INTRODUCTION

The notion of prime labeling for graphs originated with Roger Entringer and was introduced in the paper by Tout et.al [8] in the early 1980's and since then it is an active field of research for many scholars. S. K. Patel and N. P. Shrimali [7] introduced one of the variation of prime labeling which is known as neighbourhood-prime labeling of a graph. Definitions of comb, crown and dragon graph are from [2]. For the definitions of varieties of comb, crown and dragon digraphs we refer [5, 6]. In and Outdegree prime pair labeling in digraphs and concept of prime pair labeling of directed graphs were introduced by K. Palani et.al [3–5]. In this paper, we introduces the concept of prime pair labeling of graphs and investigate the existence of prime pair labeling in some

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standard and special graphs. Also, we investigate the existence of prime pair labeling in some digraphs. The following facts are from [6]:

Theorem 1.1. *The path P_n is a neighbourhood-prime graph for every n.*

Theorem 1.2. *The cycle C_n is neighbourhood-prime if $n \not\equiv 2(mod4)$.*

Theorem 1.3. *[1] Bertrand-Chebyshev theorem: $\pi(x) - \pi(\frac{x}{2}) \geq 1$, for all $x \geq 2$ where $\pi(x)$ is the prime counting function (number of primes less than or equal to x.)*

2. Main results

Definition 2.1. *Let $G = (p, q)$ be a graph. An injective function $f : V \to \{1, 2, 3, \ldots, p + q\}$ is said to be a prime pair labeling, if for every vertex $v \in V(G)$ with $d(v) > 1, gcd\{f(x), f(y)\} = 1 \forall x, y \in N(v)$. A graph which admits prime pair labeling is called a prime pair graph.*

Observation 1.

(1) *A graph G in which every vertex is of degree at most 1, is prime pair vacuously.*
(2) *A neighbourhood prime graph G in which every vertex is of degree at most 2 is prime pair.*

Remark 2.1. *By Theorem 1.1 and Observation 1 (2), the path P_n is prime pair.*

Theorem 2.1. *The cycle $C_n, n \geq 3$ is prime pair.*

Proof.

Case (i): $n \not\equiv 2(mod4)$
The result follows from Theorem 1.2 and Observation 1 (2).

Case (ii): $n \equiv 2(mod4)$
Let $V(C_n) = \{u_i | 1 \leq i \leq n\}$ be the vertex set. The graph has n vertices and n edges. Let p_r be the first prime number between n and $2n$. Such a prime exists by Theorem 1.3.

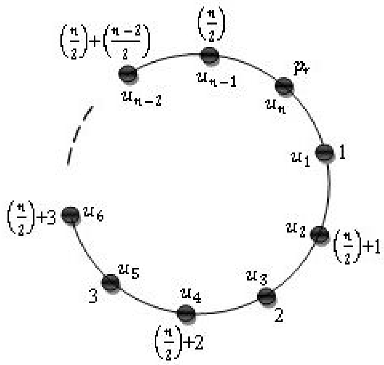

FIGURE 1

Define an injective function $f: V \to \{1, 2, 3, \ldots, 2n\}$ by

$$f(u_i) = \begin{cases} \{\frac{i+1}{2}\} \; if \; i \; isodd \\ (\frac{n}{2}) + (\frac{i}{2}) \; if \; i \; iseven \end{cases}$$

for $1 \leq i \leq n-1$ and $f(u_n) = p_r$.
Now, $N(u_1) = \{u_n, u_2\}$. Since $f(u_n) = p_r$ it implies $gcd[f(u_n), f(u_2)] = 1$. Therefore,

$$gcd[f(x), f(y)] = 1 \forall x, y \in N(u_1) \tag{2.1}$$

Let $2 \leq i \leq n-2$. Then $N(u_i) = \{u_{i-1}, u_{i+1}\}$. Since (u_{i-1}) and $f(u_{i+1})$ are consecutive integers, gcd of every label pair in $N(u_i)$ is equal to 1 where $2 \leq i \leq n-2$. $N(u_{n-1}) = \{u_{n-2}, u_n\}$. Since $f(u_n) = p_r$ it implies $gcd[f(u_{n-2}), f(u_n)] = 1$. Therefore

$$gcd[f(x), f(y)] = 1 \forall x, y \in N(u_{n-1}) \tag{2.2}$$

Further, $N(u_n) = \{u_{n-1}, u_1\}$. Since $f(u_1) = 1$ implies $gcd[f(u_{n-1}), f(u_1)] = 1$. Therefore

$$gcd[f(x), f(y)] = 1 \forall x, y \in N(u_n) \tag{2.3}$$

From (2.1), (2.2) and (2.3), f is a prime pair labeling.

By Case (i) and Case (ii), C_n admits prime pair labeling. Hence $C_n, n \geq 3$ is prime pair. □

Theorem 2.2. *Let $G = (p, q)$ be a graph with $p + q \geq (p-1)^{th}$ prime, then G is prime pair.*

Proof. Suppose $G = (p, q)$ be a graph with $p + q \geq (p-1)^{th}$ prime. Let $V(G) = \{v_1, v_2, v_3, \ldots, v_p\}$ be the vertex set. Define an injective function $f : V \to \{1, 2, 3, \ldots, p+q\}$ by $f(v_1) = 1$ and $f(v_i) = (i-1)^{th}$ prime for $2 \leq i \leq p$. Then, gcd of every label pair in $N(v)$ is equal to 1 for all $v \in V(G)$ with $deg(v) > 1$. Therefore f is a prime pair labeling of G. Hence G is prime pair. □

Theorem 2.3. *The star graph $K_{1,p-1}$ (where $p \geq 2$) is prime pair if $p+q \geq (p-2)^{th}$ prime.*

Proof. Suppose $G \cong K_{1,p-1}$ (where $p \geq 2$) and $p + q \geq (p-2)^{th}$ prime. Let $V(G) = \{u, v_1, v_2, v_3, \ldots, v_{p-1}\}$ be the vertex set where u represent the root vertex and v_i's represent the set of end vertices.

(i) If $p = 2$, then $G \cong K_{1,1}$ and is prime pair vacuously.

(ii) Let $p > 2$. Then $p + q \geq 5$.
Define an injective function $f : V \to \{1, 2, 3, \ldots, p+q\}$ by $f(u) = 4$; $f(v_1) = 1$; $f(v_i) = (i-1)^{th}$ prime for $2 \leq i \leq p-1$. Then, gcd of every label pair in $N(u)$ is equal to 1. Also, $|N(v_i)| = 1$ for $1 \leq i \leq p-1$. Therefore, gcd of every label pair in $N(v)$ is equal to 1 for all $v \in V(G)$ with $deg(v) > 1$. So, f is a prime pair labeling of G. Hence G is prime pair.

□

Theorem 2.4. *Let $G \cong K_{m,n}$ with $m, n > 1$. Then $p = m + n$ and $q = mn$. Suppose p_k (where $k \leq min\{m, n\}$) is the greatest k^{th} prime such that $p_k^2 \leq p + q$ and $p + q \geq (p - (k+1))^{th}$ prime, then G is prime pair.*

Proof. Let $V(G) = V_1 \cup V_2 = \{u_1, u_2, u_3, \ldots, u_m\} \cup \{v_1, v_2, v_3, \ldots, v_n\}$ be the vertex set. Without loss of generality we assume that $m \geq n$. Let p_k (where $k \leq min\{m, n\}$) is the greatest k^{th} prime such that $p_k^2 \leq p+q$ and $p+q \geq (p-(k+1))^{th}$ prime. Define an injective function $f : V \to \{1, 2, 3, \ldots, p+q\}$ by $f(u_1) = 1$; $f(u_i) = (i-1)^{th}$ prime for $2 \leq i \leq m$; $f(v_i) = p_i^2$ for $1 \leq i \leq k$ and $f(v_i) = (m+i-(k+1))^{th}$ prime for $k+1 \leq i \leq n$. Then, gcd of every label pair in $N(v)$ is equal to 1 for all $v \in V(G)$. Therefore f is a prime pair labeling of G. Hence G is prime pair. □

Theorem 2.5. *The comb graph $P_n \bigodot K_1$ is prime pair.*

Proof. Let $V(P_n \bigodot K_1) = \{u_i, v_i | 1 \leq i \leq n\}$ be the vertex set where $u_i's$ and $v_i's$ represent the i^{th} vertex of the path and the i^{th} copy of K_1 respectively. The graph has $2n$ vertices and $2n-1$ edges.

Case (i): n is odd. Define an injective function $f : V \to \{1, 2, 3, \ldots, 4n-1\}$ by:

$$f(u_i) = \begin{cases} i \ if \ i \ is \ odd \\ n+i \ if \ i \ is \ even \end{cases} \quad \text{and } f(v_i) = \begin{cases} n+i \ if \ i \ is \ odd \\ i \ if \ i \ is \ even \end{cases} \quad \text{for } 1 \leq i \leq n.$$

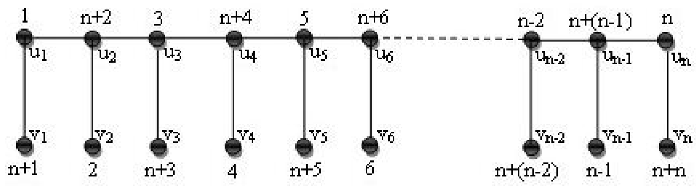

FIGURE 2

Case (ii): n is even. Define an injective function $f : V \to \{1, 2, 3, \ldots, 4n-1\}$ by:

$$f(u_i) = \begin{cases} iif \ i \ is \ odd \\ 2n-(i-1)if \ i \ is \ even \end{cases} \quad \text{and } f(v_i) = \begin{cases} 2n-(i-1) \ if \ i \ is \ odd \\ i \ if \ i \ is \ even \end{cases} \quad \text{for}$$

$1 \leq i \leq n$.

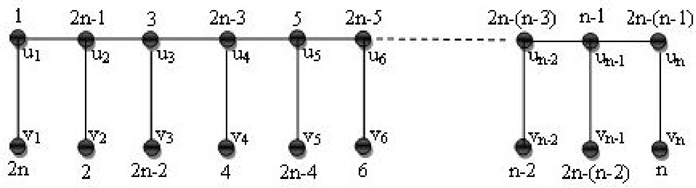

FIGURE 3

In both cases, $N(u_1) = \{v_1, u_2\}$. Since $f(v_1)$ and $f(u_2)$ are consecutive integers, gcd of every label pair in $N(u_1)$ is equal to 1. For $2 \leq i \leq n-1, N(u_i) = \{u_{i-1}, v_i, u_{i+1}\}$. Since $f(u_{i-1})$, $f(v_i)$ and $f(u_{i+1})$ are three consecutive integers starting from an odd number, gcd of every label pair in $N(u_i)$ is equal to 1 where

$2 \leq i \leq n-1$. Also $N(u_n) = \{u_{n-1}\ , v_n\}$. Since $f(u_{n-1})$ and $f(v_n)$ are consecutive integers, gcd of every label pair in $N(u_n)$ is equal to 1. Further, for $1 \leq i \leq n$, $N(v_i) = \{u_i\}$ and so $|N(v_i)| = 1$. Thus, gcd of every label pair in $N(v)$ is equal to 1 for all $v \in V(G)$ with $deg(v) > 1$. Therefore, f is a prime pair labeling of $P_n \bigodot K_1$. Hence $P_n \bigodot K_1$ is prime pair. $\square$

Theorem 2.6. *The crown graph $C_n \bigodot K_1$ is prime pair.*

Proof. Let $V(C_n \bigodot K_1) = \{u_i, v_i | 1 \leq i \leq n\}$ be the vertex set where $u_i's$ and $v_i's$ represent the i^{th} vertex of the cycle and the i^{th} copy of K_1 respectively. The graph has $2n$ vertices and $2n$ edges.

Case (i): n is odd. Define an injective function $f : V \to \{1, 2, 3, \ldots, 4n\}$ by:

$$f(u_i) = \begin{cases} i\ if\ i\ is\ odd \\ n + iif\ i\ is\ even \end{cases} \quad \text{and } f(v_i) = \begin{cases} n + i\ if\ i\ is\ odd \\ i\ if\ i\ is\ even \end{cases} \quad \text{for } 1 \leq i \leq n.$$

Now, $N(u_1) = \{u_n, v_1, u_2\}$. Also $gcd[f(u_n), f(v_1)] = gcd[n, n+1] = 1$; $gcd[f(u_n), f(u_2)] = gcd[n, n+2] = 1$ and $gcd[f(v_1), f(u_2)] = gcd[n+1, n+2] = 1$. Therefore,

$$gcd[f(x)\ , f(y)] = 1 \forall x, y \in N(u_1)\,. \tag{2.4}$$

Let $2 \leq i \leq n-1$. Then $N(u_i) = \{u_{i-1}, v_i, u_{i+1}\}$. Since $f(u_{i-1})$, $f(v_i)$ and $f(u_{i+1})$ are three consecutive integers starting from an odd number, gcd of every label pair in $N(u_i)$ is equal to 1 where $2 \leq i \leq n-1$. Further, $N(u_n) = \{u_{n-1}, v_n, u_1\}$. Also $gcd[f(u_{n-1}),\ f(v_n)] = gcd[n+n-1, n+n] = gcd[2n-1, 2n] = 1$; $gcd[f(v_n), f(u_1)] = gcd[n+n, 1] = 1$ and $gcd[f(u_{n-1}), f(u_1)] = gcd[n+n-1, 1] = 1$. Therefore,

$$gcd[f(x)\ , f(y)] = 1, \forall x, y \in N(u_n) \tag{2.5}$$

Further for $1 \leq i \leq n, N(v_i) = \{u_i\}$ and so

$$|N(v_i)| = 1 \tag{2.6}$$

From (2.4), (2.5) and (2.6), we conclude f is a prime pair labeling.

Case (ii): n is even. Let p_r be the first prime number between $2n$ and $4n$. Such a prime exists by Theorem 1.3. Define an injective function $f : V \to \{1, 2, 3, \ldots, 4n\}$ by:

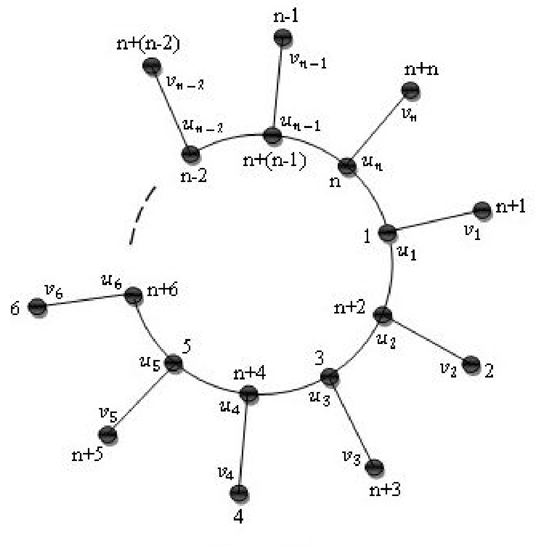

FIGURE 4

$f(u_i) = \begin{cases} i \ if \ i \ is \ odd \\ n+(i+1) \ if \ i \ is \ even \end{cases}$ and $f(v_i) = \begin{cases} n+(i+1) \ if \ i \ is \ odd \\ i \ if \ i \ is \ even \end{cases}$ for $1 \leq i \leq n-1$; $f(u_n) = p_r$ and $f(v_n) = n$.

Now, $N(u_1) = \{u_n, v_1, u_2\}$. Also $gcd[f(u_n), f(v_1)] = gcd[p_r, n+2] = 1$; $gcd[f(u_n), f(u_2)] = gcd[p_r, n+3] = 1$ and $gcd[f(v_1), f(u_2)] = gcd[n+2, n+3] = 1$. Therefore

(2.7) $$gcd[f(x), f(y)] = 1, \forall x, y \in N(u_1)$$

Let $2 \leq i \leq n-2$. Then $N(u_i) = \{u_{i-1}, v_i, u_{i+1}\}$. Since $f(u_{i-1})$,$f(v_i)$ and $f(u_{i+1})$ are three consecutive integers starting from an odd number, gcd of every label pair in

(2.8) $$N(u_i) = 1, 2 \leq i \leq n-2$$

Further, $N(u_{n-1}) = \{u_{n-2}, v_{n-1}, u_n\}$. Also $gcd[f(u_{n-2}), f(v_{n-1})] = gcd[n+n-1, n+n] = 1$; $gcd[f(v_{n-1}), f(u_n)] = gcd[n+n, p_r] = 1$ and $gcd[f(u_{n-2}), f(u_n)] = gcd[n+n-1, p_r] = 1$. Therefore

(2.9) $$gcd[f(x), f(y)] = 1 \forall x, y \in N(u_{n-1})$$

Further, $N(u_n) = \{u_{n-1}, v_n, u_1\}$. Also $gcd[f(u_{n-1}), f(v_n)] = gcd[n-1, n] = 1$; $gcd[f(v_n), f(u_1)] = gcd[n, 1] = 1$ and $gcd[f(u_{n-1}), f(u_1)] = gcd[n-1, 1] = 1$. Therefore,

$$gcd[f(x), f(y)] = 1, \forall x, y \in N(u_n) \tag{2.10}$$

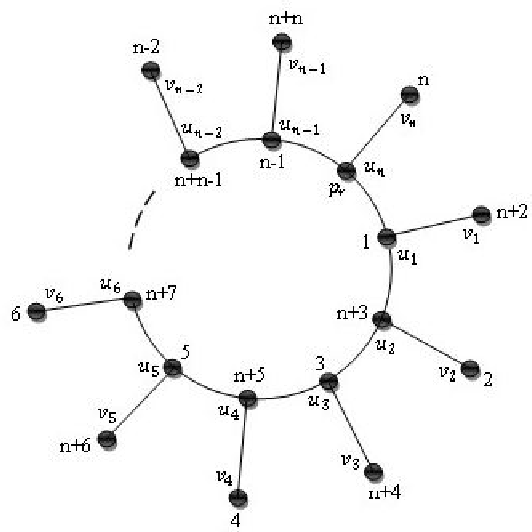

FIGURE 5

Further for $1 \leq i \leq n, N(v_i) = \{u_i\}$ and so

$$|N(v_i)| = 1 \tag{2.11}$$

According to (2.7), (2.8), (2.9), (2.10), (2.11), f is a prime pair labeling.

By Case (i) and Case (ii), f is a prime pair labeling of $C_n \bigodot K_1$. Hence $C_n \bigodot K_1$ is prime pair. □

Theorem 2.7. *The Dragon graph D_{nm} is prime pair.*

Proof. Let $V(D_{nm}) = \{u_i | 1 \leq i \leq n\} \cup \{v_i | 1 \leq i \leq m\}$ be the vertex set where $u_i's$ and $v_i's$ represents the vertices of the cycle and path respectively. Then $E(D_{nm}) = \{u_i u_{i+1} | 1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{u_1 v_1\} \cup \{v_i v_{i+1} | 1 \leq i \leq m-1\}$ is the edge set. The graph has $n+m$ vertices and $n+m$ edges. Let p_r be the first prime number between n and $2n$. Such a prime exists by Theorem 1.3.

Case (i): Both n and m are even.

Define an injective function $f : V \to \{1, 2, 3, \ldots, 2n + 2m\}$ by $f(u_1) = 1$;

For $2 \leq i \leq n-1, f(u_i) = \begin{cases} (\frac{i+4}{2}) \ if \ i \ is \ even \\ (\frac{n+2}{2}) + (\frac{i-1}{2}) \ if \ i \ is \ odd \end{cases}$

$f(u_n) = p_r$; $f(v_1) = 2$;

For $2 \leq i \leq m, f(v_i) = \begin{cases} 2n + (\frac{i-1}{2}) \ if \ i \ is \ odd \\ 2n + (\frac{m-2}{2}) + (\frac{i}{2}) \ if \ i \ is \ even \end{cases}$

Case (ii): Both n and m are odd.

Define an injective function $f : V \to \{1, 2, 3, \ldots, 2n + 2m\}$ by $f(u_1) = 1$;

For $2 \leq i \leq n-1, f(u_i) = \begin{cases} (\frac{i+4}{2}) \ if \ i \ is \ even \\ (\frac{n+3}{2}) + (\frac{i-1}{2}) \ if \ i \ is \ odd \end{cases}$

$f(u_n) = p_r; f(v_1) = 2$;

For $2 \leq i \leq m, f(v_i) = \begin{cases} 2n + (\frac{i-1}{2}) \ if \ i \ is \ odd \\ 2n + (\frac{m-1}{2}) + (\frac{i}{2}) \ if \ i \ is \ even \end{cases}$

Case (iii): n is even and m is odd.

Define an injective function $f : V \to \{1, 2, 3, \ldots, 2n + 2m\}$ by $f(u_1) = 1$;

For $2 \leq i \leq n-1, f(u_i) = \begin{cases} (\frac{i+4}{2}) \ if \ i \ is \ even \\ (\frac{n+2}{2}) + (\frac{i-1}{2}) \ if \ i \ is \ odd \end{cases}$

$f(u_n) = p_r$; $f(v_1) = 2$;

For $2 \leq i \leq m, f(v_i) = \begin{cases} 2n + (\frac{i-1}{2}) \ if \ i \ is \ odd \\ 2n + (\frac{m-1}{2}) + (\frac{i}{2}) \ if \ i \ is \ even \end{cases}$

Case (iv): n is odd and m is even.

Define an injective function $f : V \to \{1, 2, 3, \ldots, 2n + 2m\}$ by $f(u_1) = 1$;

For $2 \leq i \leq n-1, f(u_i) = \begin{cases} (\frac{i+4}{2}) \ if \ i \ is \ even \\ (\frac{n+3}{2}) + (\frac{i-1}{2}) \ if \ i \ is \ odd \end{cases}$

$f(u_n) = p_r$; $f(v_1) = 2$;

For $2 \leq i \leq m, f(u_i) = \begin{cases} 2n + (\frac{i-1}{2}) \ if \ i \ is \ odd \\ 2n + (\frac{m-2}{2}) + (\frac{i}{2}) \ if \ i \ is \ even \end{cases}$

For all the above cases, $N(u_1) = \{u_n, v_1, u_2\}$. Also $gcd[f(u_n), f(v_1)] = gcd[p_r, 2] = 1$; $gcd[f(u_n), f(u_2)] = gcd[p_r, 3] = 1$ and $gcd[f(v_1), f(u_2)] = gcd[2, 3] = 1$. Therefore,

$$gcd[f(x), f(y)] = 1 \forall x, y \in N(u_1) \tag{2.12}$$

$N(u_2) = \{u_1, u_3\}$. Since $f(u_1) = 1$ implies $gcd[f(u_1), f(u_3)] = 1$. Therefore,

$$gcd[f(x)\ , f(y)] = 1 \forall x, y \in N(u_2) \tag{2.13}$$

Let $3 \leq i \leq n-2$. Then $N(u_i) = \{u_{i-1}, u_{i+1}$. Since $f(u_{i-1})$ and $f(u_{i+1})$ are consecutive integers, gcd of every label pair in $N(u_i)$ is equal to 1, where $3 \leq i \leq n-2$. $N(u_{n-1}) = \{u_{n-2}, u_n\}$.

Since $f(u_n) = p_r$ implies $gcd[f(u_{n-2}), f(u_n)] = 1$. Therefore,

$$gcd[f(x), f(y)] = 1 \forall x, y \in N(u_{n-1}) \tag{2.14}$$

Further, $N(u_n) = \{u_{n-1}, u_1\}$. Since $f(u_1) = 1$ implies $gcd[f(u_{n-1}), f(u_1)] = 1$. Therefore,

$$gcd[f(x), f(y)] = 1 \forall x, y \in N(u_n) \tag{2.15}$$

Now, $N(v_1) = \{u_1, v_2\}$. Since $f(u_1) = 1$, it implies $gcd[f(u_1), f(v_2)] = 1$. Therefore,

$$gcd[f(x), f(y)] = 1 \forall x, y \in N(v_1) \tag{2.16}$$

$N(v_2) = \{v_1, v_3\}$. Also $gcd[f(v_1), f(v_3)] = gcd[2, 2n+1] = 1$. Therefore

$$gcd[f(x), f(y)] = 1 \forall x, y \in N(v-2) \tag{2.17}$$

Let $3 \leq i \leq m-1$. Then $N(v_i) = \{v_{i-1}, v_{i+1}\}$. Since $f(v_{i-1})$ and $f(v_{i+1})$ are consecutive integers, gcd of every label pair in $N(v_i)$ is equal to 1, where $3 \leq i \leq m-1$ Further, $N(v_m) = \{v_{m-1}\}$ and so $|N(v_m)| = 1$. According to (2.12), (2.13), (2.14), (2.15), (2.16), (2.17) and the whole discussion, we conclude f is a prime pair labeling of $D_n m$. Hence $D_n m$ is prime pair. $\square$

3. Some new digraphs and results of prime pair labeling on digraphs

In this section we define some new digraphs. In [5] we have introduced Double alternating crown. Now, we find the existence of two kinds of Double alternating crowns due to the alternative directions for the pendent edges. We call them as Sole-Double alternating and Di-Double alternating crowns, as in one type the vertices of the cycle in a crown have either indegree or outdegree as zero and in the other type all the vertices of the cycle have both indegree and outdegree non zero. Hence we have the following definitions.

3.1. **Sole-Double alternating crown** $(SDAC_n \vec{\bigodot} K_1)$**:** Consider a crown graph $C_n \bigodot K_1$. Orient C_n alternatively. Next, orient the pendent edges so that either $d^+(v) = 0$ or $d^-(v) = 0$, $\forall v \in V(C_n)$. The resulting graph is called Sole-Double alternating crown and is denoted as $SDAC_n \vec{\bigodot} K_1$.

3.2. **Di-Double alternating crown** $(DDAC_n \vec{\bigodot} K_1)$**:** In a crown graph $C_n \bigodot K_1$, orient C_n alternatively. Now orient the pendent edges so that neither $d^+(v) = 0$ nor $d^-(v) = 0$, $\forall v \in V(C_n)$. The resulting graph is called Di-Double alternating crown and is denoted as $DDAC_n \vec{\bigodot} K_1$.

Theorem 3.1. *Alternating star* $(A\vec{K_{1,n}})$*, for n even, admits prime pair labeling if* $n \leq 14$.

Proof. : Let $V(A\vec{K_{1,n}}) = \{u, v_1, v_2, v_3, \dots, v_n\}$ be the vertex set where u represent the root vertex and $v_i's$ represent the set of end vertices. Then $A(A\vec{K_{1,n}}) = \{\overrightarrow{v_{2i-1}u} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{uv_{2i}} | 1 \leq i \leq \frac{n}{2}\}$ is the arc set.

When $n = 2$, assigning 1, 2, 3 to the vertices gives the prime pair labeling.

Let $4 \leq n \leq 14$. Define an injective function $f : V \to \{1, 2, 3, \dots, n + q + 1\}$ by $f(u) = 6$; $f(v_1) = 1$; $f(v_2) = 4$; $f(v_3) = 2$; $f(v_4) = 9$; For $5 \leq i \leq n$,

$$f(v_i) = \begin{cases} (\frac{i-1}{2})^{th} prime\ if\ i\ is\ odd \\ ((\frac{n-2}{2}) + (\frac{i-4}{2}))^{th} prime\ if\ i\ is\ even \end{cases}$$

Now, $N^{-(u)} = \{v_{2i-1} | 1 \leq i \leq \frac{n}{2}\}$. Since $f(v_1) = 1$ and the set $f(v_{2i-1})|\ 3 \leq i \leq \frac{n}{2}$ contains only prime numbers, gcd of every label pair in $N^{-(u)}$ is 1.

For $1 \leq i \leq n$ and i is odd, $N^{-(v_i)} = \Phi$ and so

$$|N^-(v_i)| = 0\,. \tag{3.1}$$

Further, for $1 \leq i \leq n$ and i is even, $N^-(v_i) = \{u\}$ and so

$$|N^-(v_i)| = 1\,. \tag{3.2}$$

From (3.1) and (3.2), f is an indegree prime pair labeling.

Now, $N^+(u) = \{v_{2i} | 1 \leq i \leq \frac{n}{2}\}$. Since $f(v_2) = 4$; $f(v_4) = 9$ and the set $f(v_{2i})|\ 6 \leq i \leq \frac{n}{2}$ contains only prime numbers greater than and equal to 5, gcd of every label pair in $N^+(u)$ is 1.

For $1 \leq i \leq n$ and i is odd, $N^+(v_i) = \{u\}$ and so

$$|N^+(v_i)| = 1\,. \tag{3.3}$$

Further, for $1 \leq i \leq n$ and i is even, $N^+(v_i) = \Phi$ and so

$$|N^+(v_i)| = 0\,. \tag{3.4}$$

From (3.3) and (3.4), f is an outdegree prime pair labeling.

So, f is an indegree prime pair labeling and f is an outdegree prime pair labeling, thus f is a prime pair labeling of $A\vec{K_{1,n}}$ where $4 \leq n \leq 14$. Hence the alternating star $(A\vec{K_{1,n}}), n$ even admits prime pair labeling if $n \leq 14$. $\square$

Theorem 3.2. *Let $D(p,q)$ be a digraph with $p > 3$ vertices. Suppose D contains a unique vertex 'u' whose degree sum is 3 and $\max\{d^+(v), d^-(v)\} = 1$, $\forall v \in V(D) - \{u\}$. Then D is prime pair.*

Proof. Let $V(D) = \{v_1, v_2, v_3, \ldots, v_p\}$ be the vertex set. Suppose v_1 is the unique vertex in D whose degree sum is 3, i.e $d^+(v_1) + d^-(v_1) = 3$. Then, we have the following cases:

Case (i): $N^+(v_1) = 2$ and $N^-(v_1) = 1$;
Case (ii): $N^+(v_1) = 1$ and $N^-(v_1) = 2$;
Case (iii): $N^+(v_1) = 3$ and $N^-(v_1) = 0$;
Case (iv): $N^+(v_1) = 0$ and $N^-(v_1) = 3$.

In each case, $N^+(v_1) \cup N^-(v_1)$ contains exactly 3 vertices. Let $N^+(v_1) \cup N^-(v_1) = \{v_i, v_j, v_k\}$ where $i, j, k \in \{2, 3, \ldots, p\}$. Now, define an injective function $f : V \to \{1, 2, 3, \ldots, p+q\}$ such that $f(v_i) = 1, f(v_j) = 2$ and $f(v_k) = 3$. Then, the gcd of every label pair of vertices in $N^+(v_1) \cup N^-(v_1)$ is 1. Therefore, gcd of every label pair of vertices in $N^+(v_1)$ and $N^-(v_1)$ is 1. Also, for every vertex $v \in V(D) - \{v_1\}, \max\{d^+(v)\ ,\ d^-(v)\} = 1$ and so $max\{|N^+(v)|\ ,\ |N^-(v)|\} = 1$. Therefore f is an in and outdegree prime pair labeling of D. Therefore f is a prime pair labeling of D. Hence D is prime pair. $\square$

Theorem 3.3. *Suppose $G(p,q)$ is prime pair, then the digraph $D(p,q)$ obtained from the graph G by orienting the arcs in any direction is prime pair.*

Proof. Let $G(p,q)$ be a prime pair graph. Then there exists an injective function $f : V \to \{1,2,3,\ldots,p+q\}$ such that for every $v \in V(G)$ with $d(v) > 1, gcd\{f(x), f(y)\} = 1 \forall x,y \in N(v)$. Since $N^-(v)$ and $N^+(v)$ are subsets of $N(v)$, for every $v \in V(D)$ with $d^-(v) > 1$, $gcd\{f(x), f(y)\} = 1\ \forall x,y \in N^-(v)$ and for every $v \in V(D)$ with $d^+(v) > 1$, $gcd\{f(x), f(y)\} = 1\ \forall x,y \in N^+(v)$. Therefore f is an in and outdegree prime pair labeling of D. Therefore f is a prime pair labeling of D. Hence $D(p,q)$ is prime pair. □

Remark 3.1. *The converse of the above theorem need not be true. For alternating star $(A\vec{K}_{1,10})$ admits prime pair labeling. But the underlying graph $K_{1,10}$ does not admit prime pair labeling.*

Remark 3.2. *At the end of this paper we have following conclusions.*

(1) *Directed path and alternating path are prime pair.*
(2) *Directed cycle and alternating cycle are prime pair.*
(3) *Upcomb, downcomb, alternating comb, alternating upcomb, alternating downcomb, sole double alternating comb and di-double alternating comb are prime pair.*
(4) *Incrown, outcrown, alternating crown, alternating incrown, alternating outcrown, sole double alternating crown and di-double alternating crown are prime pair.*
(5) *Indragon, outdragon, alternating dragon, alternating indragon, alternating outdragon and double alternating dragon are prime pair.*

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