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HOP DOMINATION NUMBER OF CATERPILLAR GRAPHS

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ABSTRACT. Let $G = (V, E)$ be a graph. A set $S \subset V(G)$ is a hop dominating set of G if for every $v \in VS$, there exists $u \in S$ such that $d(u, v) = 2$. The minimum cardinality of a hop dominating set of G is called a hop dominator number of G and it is denoted by $\gamma_h(G)$. A caterpillar is a graph denoted by $P_k(x_1, x_2, ...x_k)$, where x_i is the number of leaves attached to the ith vertex of the path P_k. In this paper the domination numbers are determined for the hop graphs of $P_n(1, 1, 1)$ and $P_n(2, 2, 2)$ and hop domination number of such caterpillars have been derived.

1. INTRODUCTION

The following two defintions are given in [1,2].

Definition 1.1. *A set $S \subset V$ of a graph G is a hop dominating set of G if for every $v \in V - S$, there exists $u \in S$ such that $d(u, v) = 2$. The minimum cardinality of a hop dominating set of G is called the hop domination number and is denoted by $\gamma_h(G)$.*

Definition 1.2. *The hop graph H(G) of a graph G is the graph obtained from G by taking V(H(G)) = V(G) and joining two vertices u,v in H(G) iff they are at a distance 2 in G.*

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Definition 1.3. *[4], A caterpillar is a graph which can be obtained from the path on k vertices by appending x_i pendant vertices to the ith vertex of the path P_k. The caterpillar with parameters $k, x_1, x_2, ...x_k$ where $x_1, x_k \neq 0$, will be denoted by $P_k(x_1, x_2, ...x_k)$.*

A caterpillar is a tree with the property that the removal of its leaves and incident edges results in a path P_k called the spine of the caterpillar. We say a caterpillar is complete if every vertex on the spine of the caterpillar is adjacent to at least one leaf.

In section 2 we discuss domination number of special types of snake graphs which occur as hop graphs of $P_n(1,1,1)$ and $P_n(2,2,2)$, [3, 5, 6]. In section 3, hop domination number of $P_n(1,1,1)$ and $P_n(2,2,2)$ are determined.

2. Domination number of some special snakes graph

Let $(SN)^n_{K_4}$ denote the snake graph with n copies of K_4, $(SN)^n_{K_4,K_3}$ denote the snake graph with n copies of K_4 followed by one K_3 and $(SN)^n_{K_4,2K_3}$ denote the snake graph with n copies of K_4 starting and ending with K_3's and $(TS_N)^n_{K_3}$ denote the triangular snake graph with n copies of K_3, $(TS_N)^n_{K_3,P_1}$ denote the triangular snake graph with n copies of K_3 followed by 1-pendant vertex and $(TS_N)^n_{K_3,2P_1}$ denote the triangular snake graph with n copies of K_3 starting and ending with 1-pendant vertex.

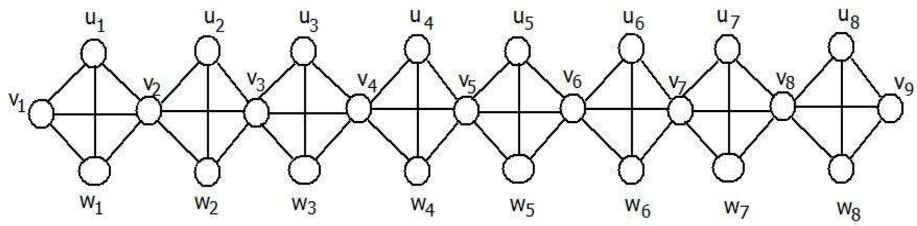
Theorem 2.1. $\gamma((SN)^n_{K_4}) = \left\lceil \frac{n}{2} \right\rceil$.

Proof. Let V($(SN)^n_{K_4}$)= $\{u_i, w_i/i = 1, 2...n\} \cup \{v_i/i = 1, 2...n + 1\}$ and
$E((SN)^n_{K_4})$= $\{v_iv_{i+1}/i = 1, 2...n\} \cup \{v_iu_i, v_{i+1}u_i/i = 1, 2...n\} \cup$
$\{v_iw_i, v_{i+1}w_i/i = 1, 2...n\} \cup \{u_iw_i/i = 1, 2...n\}$

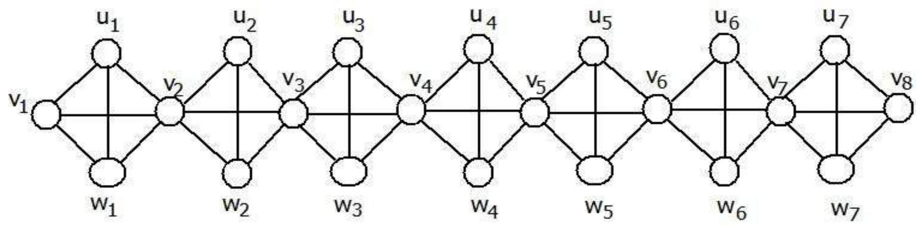
Take $D = \begin{cases} \{v_2, v_4...v_n\} & \text{if n is even} \\ \{v_2, v_4 v_{n-1}, v_{n+1}\} & \text{if n is odd.} \end{cases}$

Clearly D is a minimal dominating set of $(SN)^n_{K_4}$ and hence $|D| = \left\lceil \frac{n}{2} \right\rceil$.
Therefore $\gamma((SN)^n_{K_4}) \leq \left\lceil \frac{n}{2} \right\rceil$ and $\max_{v\in(SN)^n_{K_4}} d(v) = 6$.
So any vertex can dominate at most six vertices apart from it. For any two K_4, atleast one vertex is needed. Hence $\gamma((SN)^n_{K_4}) \geq \left\lceil \frac{n}{2} \right\rceil$. Hence $\gamma((SN)^n_{K_4}) = \left\lceil \frac{n}{2} \right\rceil$.

Illustration1 : Let us consider $(SN)^8_{K_4}$.



$V = \{v_1, v_2, v_3...v_9\} \cup \{u_1, u_2, u_3...u_8\} \cup \{w_1, w_2, v_3...w_8\}$
$D = \{v_2, v_4, v_6, v_8\}$ is a minimal dominating set of $(SN)^8_{K_4}$.
Illustration 2 : Let us consider $(SN)^7_{K_4}$



$V = \{v_1, v_2, v_3...v_8\} \cup \{u_1, u_2, u_3...u_7\} \cup \{w_1, w_2, v_3...w_7\}$
$D = \{v_2, v_4, v_6, v_8\}$ is a minimal dominating set of $(SN)^7_{K_4}$. □

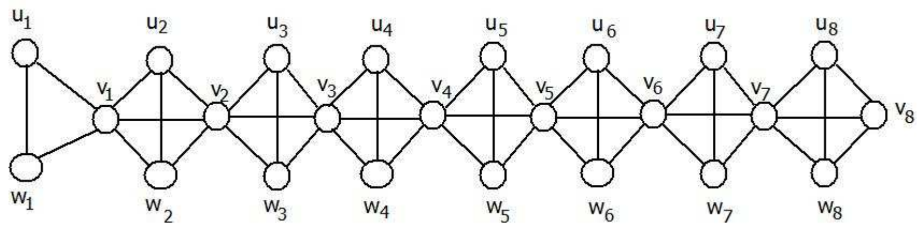
Theorem 2.2. $\gamma((SN)^n_{K_4,K_3}) = \lceil \frac{n}{2} \rceil$.

Proof. Let $V((SN)^n_{K_4,K_3})= \{u_i, v_i, w_i / i = 1, 2...n + 1\}$ and
$E((SN)^n_{K_4,K_3})=\{v_iv_{i+1}/i = 1, 2...n\} \cup \{v_iu_i, v_iw_i, u_iw_i/i = 1, 2...n + 1\} \cup$
$\{v_iu_{i+1}, v_iw_{i+1}/i = 1, 2...n\}$

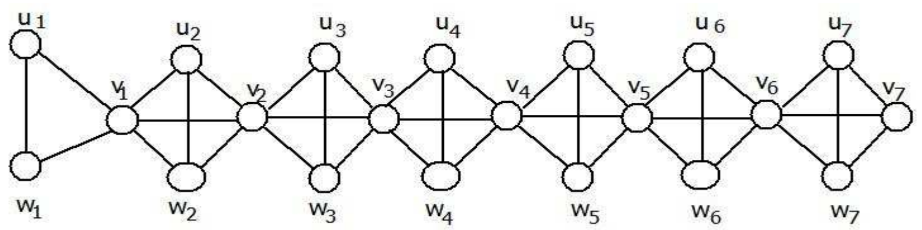
Take $D = \begin{cases} \{v_1, v_3..., v_{n+1}\} & \text{if n is even} \\ \{v_1, v_3..., v_n\} & \text{if n is odd.} \end{cases}$

Clearly D is a minimal dominating set of $(SN)^n_{K_4,K_3}$ and |D| $=\lceil \frac{n}{2} \rceil$.
Therefore $\gamma((SN)^n_{K_4,K_3}) \leq \lceil \frac{n}{2} \rceil$
As in theorem 2.1, $\gamma((SN)^n_{K_4,K_3}) \geq \lceil \frac{n}{2} \rceil$.
Hence $\gamma((SN)^n_{K_4,K_3}) = \lceil \frac{n}{2} \rceil$.

Illustration 1 : Let us consider $(SN)^7_{K_4,K_3}$.



$V = \{v_1, v_2, v_3...v_8\} \cup\{u_1, u_2, u_3...u_8\}\cup\{w_1, w_2, v_3...w_8\}$
$D = \{v_1, v_3, v_5, v_7\}$ is a minimal dominating set of $(SN)^7_{K_4,K_3}$.
Illustration 2 : Let us consider $(SN)^6_{K_4,K_3}$.



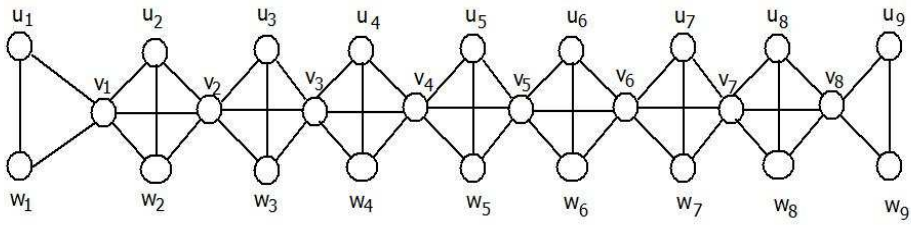
$V = \{v_1, v_2, v_3...v_7\} \cup\{u_1, u_2, u_3...u_7\} \cup \{w_1, w_2, v_3...w_7\}$
D = $\{v_1, v_3, v_5, v_7\}$ is a minimal dominating set of $(SN)^6_{K_4,K_3}$.

□

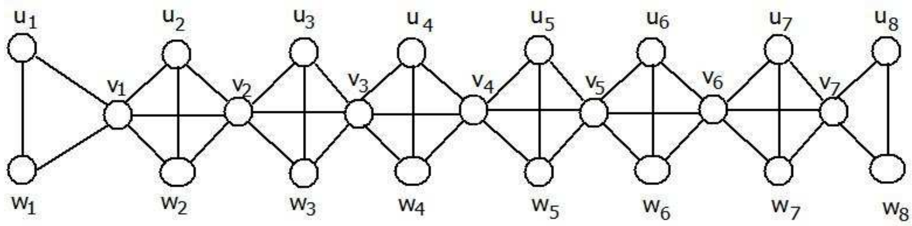
Theorem 2.3. $\gamma((SN)^n_{K_4,2K_3}) = \lceil \frac{n}{2} \rceil + 1.$

Proof. Let $V((SN)^n_{K_4,2K_3})$= $\{v_i/i = 1, 2...n + 1\}$ and $\{u_i, w_i/i = 1, 2...n + 2\}$
$E((SN)^n_{K_4,2K_3})$=$\{v_iv_{i+1}/i = 1, 2...n\} \cup\{v_iu_i, v_iw_i, v_iu_{i+1}v_iw_{i+1}, u_{i+1}w_{i+1}/$
$i = 1, 2, ..., n + 1\}$. Take $D = \begin{cases} \{v_1, v_3...v_{n+1}\} & \text{if n is even} \\ \{v_1, v_3...v_n, w_{n+2}\} & \text{if n is odd.} \end{cases}$
Clearly D is a minimal dominating set of $(SN)^n_{K_4,2K_3}$ and
$\gamma((SN)^n_{K_4,2K_3})\leq\lceil \frac{n}{2} \rceil$+1; $|D|$ =$\lceil \frac{n}{2} \rceil$+1. There are $n + 2$ compartments and hence
$\gamma((SN)^n_{K_4,2K_3})\geq\lceil \frac{n+2}{2} \rceil$+1=$\lceil \frac{n}{2} \rceil + 1$.
Hence $\gamma((SN)^n_{K_4,2K_3}) = \lceil \frac{n}{2} \rceil + 1$.

Illustration 1 : Let us consider $(SN)^7_{K_4,2K_3}$.



$V = \{v_1, v_2, v_3...v_8\} \cup\{u_1, u_2, u_3.....u_8, u_9\}\cup\{w_1, w_2, v_3...w_8, w_9\}$
$D = \{v_1, v_3, v_5, v_7, w_9\}$ is a dominating set of $(SN)^7_{K_4,2K_3}$.
Illustration 2 : Let us consider $(SN)^6_{K_4,2K_3}$.



$V = \{v_1, v_2, v_3...v_7\} \cup\{u_1, u_2, u_3...u_7, u_8\} \cup \{w_1, w_2, w_3...w_7, w_8\}$
$D = \{v_1, v_3, v_5, v_7\}$ is a dominating set of $(SN)^6_{K_4,2K_3}$. □

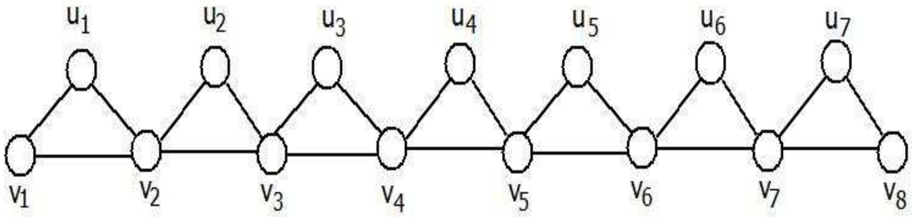
Theorem 2.4. $\gamma((TS_N)^n_{K_3}) = \lceil \frac{n}{2} \rceil$

Proof. Let $V((TS_N)^n_{K_3})= \{v_i/i = 1, 2...n + 1\}\cup\{u_i/i = 1, 2...n\}$ and
$E((TS_N)^n_{K_3})= \{v_iv_{i+1}/i = 1, 2...n\}\cup\{v_iu_i, v_{i+1}u_i/i = 1, 2...n\}$

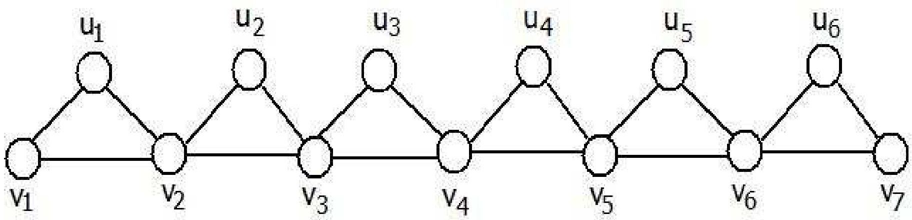
Take $D = \begin{cases} \{v_2, v_4...v_{n+1}\} & \text{if n is odd} \\ \{v_2, v_4...v_n\} & \text{if n is even.} \end{cases}$

Clearly D is a minimal dominating set of $(TS_N)^n_{K_3}$ and
$\gamma((TS_N)^n_{K_3})\leq\lceil \frac{n}{2} \rceil$; $|D| =\lceil \frac{n}{2} \rceil$.
As in theorem 2.1, $\gamma((TS_N)^n_{K_3})\geq\lceil \frac{n}{2} \rceil$
Hence $\gamma((TS_N)^n_{K_3}) = \lceil \frac{n}{2} \rceil$.

Illustration 1 : Let us consider $(TS_N)^7_{K_3}$.



$V = \{v_1, v_2, v_3...v_8\} \cup \{u_1, u_2, u_3...u_7\}$
$D = \{v_2, v_4, v_6, v_8\}$ is a minimal dominating set of $(TS_N)^7_{K_3}$.
Illustration 2 : Let us consider $(TS_N)^6_{K_3}$, when n is even.

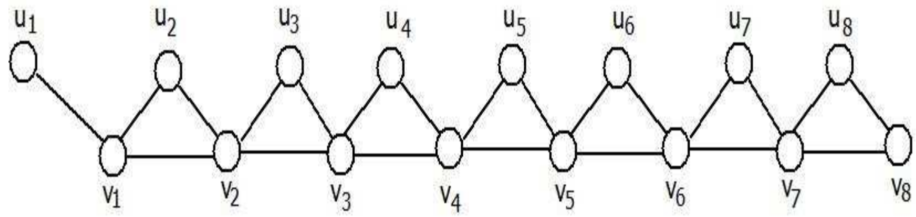


$V = \{v_1, v_2, v_3...v_7\} \cup \{u_1, u_2, u_3.....u_6\}$
$D = \{v_2, v_4, v_6\}$ is a dominating set of $(TS_N)^6_{K_3}$. □

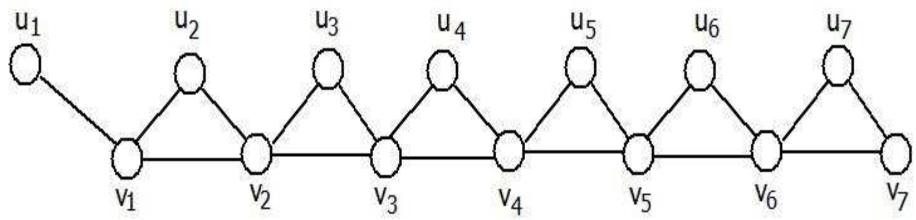
Theorem 2.5. $\gamma((TS_N)^n_{K_3,P_1}) = \lceil \frac{n}{2} \rceil$.

Proof. Let $V((TS_N)^n_{K_3,P_1})$= $\{v_i, u_i/i = 1, 2...n+1\}$nd
$E((TS_N)^n_{K_3,P_1})$=$\{v_iv_{i+1}, v_iu_{i+1}/i = 1, 2...n\} \cup \{v_iu_i/i = 1, 2...n+1\}$ Take $D = \begin{cases} \{v_1, v_3...v_n\} & \text{if n is even} \\ \{v_1, v_3...v_n\} & \text{if n is odd.} \end{cases}$
Clearly D is a minimal dominating set of $(SN)^n_{K_4,K_3}$ and
$\gamma((TS_N)^n_{K_3,P_1}) \leq \lceil \frac{n}{2} \rceil$; |D| $= \lceil \frac{n}{2} \rceil$.
As in previous theorems, $\gamma((TS_N)^n_{K_3,P_1}) \geq \lceil \frac{n}{2} \rceil$
Hence $\gamma((TS_N)^n_{K_3,P_1}) = \lceil \frac{n}{2} \rceil$.

Illustration 1 : Let us consider $(TS_N)^7_{K_3,P_1}$.



V=$\{v_1, v_2, v_3...v_8\}$ $\cup\{u_1, u_2, u_3...u_8\}$
D=$\{v_1, v_3, v_5, v_7\}$ is a minimal dominating set of $(TS_N)^7_{K_3,P_1}$.
Illustration 2 : Let us consider $(TS_N)^6_{K_3,P_1}$.



$V = \{v_1, v_2, v_3...v_7\}$ $\cup\{u_1, u_2, u_3...u_7\}$ $D = \{v_1, v_3, v_5, v_7\}$ is a minimal dominating set of $(TS_N)^6_{K_3,P_1}$. □

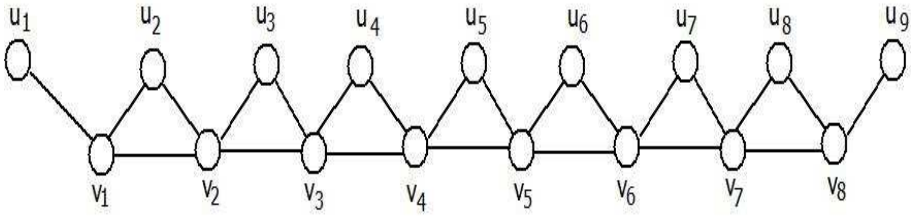
Theorem 2.6. $\gamma((TS_N)^n_{K_3,2P_1}) = \lceil \frac{n}{2} \rceil + 1$.

Proof. Let V($(TS_N)^n_{K_3,2P_1}$)= $\{v_i/i = 1, 2...n + 1\}\cup\{u_i/i = 1, 2...n + 2\}$ and $E((TS_N)^n_{K_3,2P_1})$=$\{v_iv_{i+1}/i = 1, 2...n\}\cup\{v_iu_iv_iu_{i+1}/i = 1, 2...n + 1\}$

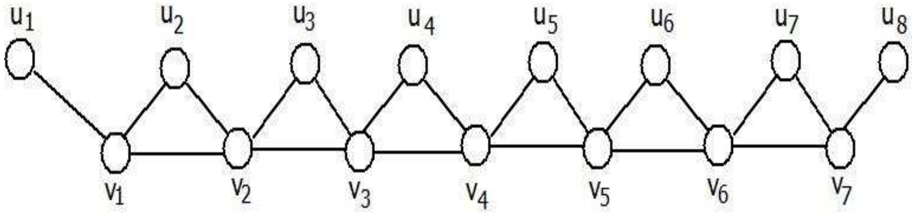
Take $D = \begin{cases} \{v_2, v_4...v_n\} & \text{if n is even} \\ \{v_2, v_4v_{n+1}\} & \text{if n is odd.} \end{cases}$

Clearly D is a minimal dominating set of $(TS_N)^n_{K_3,2P_1}$) and
$\gamma((TS_N)^n_{K_3,2P_1})$≤$\lceil\frac{n}{2}\rceil$+1 ; |D| =$\lceil\frac{n}{2}\rceil$+1.
As in previous theorems, $\gamma((TS_N)^n_{K_3,P_1})$≥$\lceil \frac{n}{2} \rceil + 1$.
Hence $\gamma((TS_N)^n_{K_3,2P_1})$ = $\lceil \frac{n}{2} \rceil$ +1.

Illustration 1 : Let us consider $(TS_N)^7_{K_3,2P_1}$.



$V = \{v_1, v_2, v_3...v_8\} \cup \{u_1, u_2, u_3...u_8, u_9\}$
$D = \{v_1, v_3, v_5, v_7, u_9\}$ is a minimal dominating set of $(TS_N)^7_{K_3,2P_1}$.
Illustration 2 : Let us consider $(TS_N)^6_{K_3,2P_1}$.



$V = \{v_1, v_2, v_3...v_7\} \cup \{u_1, u_2, u_3...u_7, u_8\}$.
$D = \{v_1, v_3, v_5, v_7\}$ is a minimal dominating set of $(TS_N)^6_{K_3,2P_1}$. □

Theorem 2.7. $\gamma_h(P_n(1,1,...,1)) = \begin{cases} 2r+3 & \textit{if n=2r+1} \\ 2r & \textit{if n=2r}. \end{cases}$

Proof. Let $P_n(1,1,...,1)$ be the complete caterpillar with
$V(P_n(1,1,...,1))$=$\{u_i \cup v_i/1 \leq i \leq n\}$ and $E(P_n(1,1,...,1))$=$\{v_iu_i/1 \leq i \leq n\}$.
Hop graph $H(P_n(x_1, x_2...x_n))$ will be the disjoint union of $(TS_N)^{[\frac{n}{2}]}_{K_3}$ and $(TS_N)^{[\frac{n}{2}]-1}_{K_3,2P_1}$ if n is odd and the disjoint union of two $(TS_N)^{\lceil\frac{n}{2}\rceil}_{K_3,P_1}$ if n is even.
When n is odd
H($P_n(1,1,...,1)$)= $(TS_N)^{[\frac{n}{2}]}_{K_3} \cup (TS_N)^{[\frac{n}{2}]-1}_{K_3,2P_1}$
$\gamma(H(P_n(1,1,...,1))) = \gamma((TS_N)^{[}_{K_3}\frac{n}{2}])+ \gamma((TS_N)^{[\frac{n}{2}]-1}_{K_3,2P_1}) = \lceil\frac{n}{2}\rceil + \lceil\frac{n}{2}\rceil + 1 = 2\lceil\frac{n}{2}\rceil + 1$.
If $n = 2r+1$, then $\gamma(H(P_n(1,1,...,1))) = 2\lceil\frac{2r+1}{2}\rceil + 1 = 2(r+1)+1 = 2r+3$.

When n is even

$H(P_n(1,1,...,1))= (TS_N)^{\lceil \frac{n}{2} \rceil}_{K_3,P_1} \cup (TS_N)^{\lceil \frac{n}{2} \rceil}_{K_3,P_1}$

$\gamma(H(P_n(1,1,...,1)))=\gamma((TS_N)^{\lceil \frac{n}{2} \rceil}_{K_3,P_1}) + \gamma((TS_N)^{\lceil \frac{n}{2} \rceil}_{K_3,P_1}) \lceil \frac{n}{2} \rceil + \lceil \frac{n}{2} \rceil = 2 \lceil \frac{n}{2} \rceil$.

If $n = 2r$, then $\gamma(H(P_n(1,1,...,1))) = 2 \lceil \frac{2r}{2} \rceil = 2r$

$\gamma_h(P_n(1,1,...,1))=\gamma(HP_n(1,1,...,1))$.

Hence

$$\gamma_h(P_n(1,1,...,1)) = \begin{cases} 2r+3 & \text{if } n = 2r+1 \\ 2r & \text{if } n = 2r. \end{cases}$$ □

Theorem 2.8. $\gamma_h(P_n(2,2,...,2)) = \begin{cases} 2r+3 & \textit{if } n = 2r+1 \\ 2r & \textit{if } n = 2r. \end{cases}$

Proof. Let $P_n(2,2,...2)$ be a complete caterpillar with
$V(P_n(2,2,...2))=\{u_i,v_i,w_i/1 \leq i \leq n\}$ and
$E(P_n(2,2,...2))=\{v_iv_{i+1}/1 \leq i \leq n-1\}\cup\{u_iv_i/i=1,2...n\} \cup\{w_iv_i/i=1,2...n\}$.
Hop graph H($P_n(2,2,...2)$) will be the disjoint union of $(SN)^{[\frac{n}{2}]}_{K_4}$ and $(SN)^{[\frac{n}{2}]-1}_{K_4,2K_3}$ if n is odd and the disjoint union of two $(SN)^{\lceil \frac{n}{2} \rceil}_{K_4,K_3}$ if n is even .

When n is odd:

H($P_n(2,2,...2)$)= $(SN)^{[\frac{n}{2}]}_{K_4} \cup (SN)^{[\frac{n}{2}]-1}_{K_4,2K_3}$

$\gamma(H(P_n(2,2,...2))) = \gamma((SN)^{[\frac{n}{2}]}_{K_4}) + \gamma((SN)^{[\frac{n}{2}]-1}_{K_4,2K_3}) = \lceil \frac{n}{2} \rceil + \lceil \frac{n}{2} \rceil + 1 = 2 \lceil \frac{n}{2} \rceil + 1$.

If $n = 2r+1$, then $\gamma(H(P_n(2,2,...2))) = 2 \lceil \frac{2r+1}{2} \rceil + 1 = 2(r+1)+1 = 2r+3$.

When n is even

$H(P_n(2,2,...2))= (SN)^{\lceil \frac{n}{2} \rceil}_{K_4,K_3} \cup (SN)^{\lceil \frac{n}{2} \rceil}_{K_4,K_3}$.

$\gamma(H(P_n(2,2,...2)))=\gamma((SN)^{\lceil \frac{n}{2} \rceil}_{K_4,K_3})+ \gamma((SN)^{\lceil \frac{n}{2} \rceil}_{K_4,K_3}) = \lceil \frac{n}{2} \rceil + \lceil \frac{n}{2} \rceil = 2 \lceil \frac{n}{2} \rceil$.

If $n = 2r$, then $\gamma(HP_n(2,2,...2)) = 2 \lceil \frac{2r}{2} \rceil = 2r$

$\gamma_h(P_n(2,2,...2))=\gamma(H(P_n(2,2,...2)))$

Hence $\gamma_h(P_n(2,2,...,2)) = \begin{cases} 2r+3 & \text{if } n = 2r+1 \\ 2r & \text{if } n = 2r. \end{cases}$ □

3. Conclusion

While working on combs $P_n(1,1,...,1)$ and twigs $P_n(2,2,...2)$ it is strongly sensed that the results can be generalized to $P_n(r,r,....r)$ and even to any caterpillar. Our next paper will attempt the generalization process.

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