

COMMON FIXED POINT THEOREMS USING CLR AND E.A. PROPERTIES IN COMPLEX PARTIAL B-METRIC SPACE

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ABSTRACT. In this paper, we prove a existence and uniqueness of common fixed point theorems using (CLR) and (E.A.) properties in complex partial b-metric space. The proved results generalize and extend some of the well known results in the literature. An example to support our result is presented.

1. INTRODUCTION

In many branches of science, economics, computer science, engineering and the development of nonlinear analysis, the fixed point theory is one of the most important tool. In 1989, Backhtin in [1] introduced the concept of b-metric space. In 1993, Czerwik in [2] extended the results of b-metric spaces. Azam et al. in [3] introduced new spaces called complex valued metric spaces and established the existence of fixed point theorems under the contraction condition. Rao et al. in [5] introduced the concept of complex valued b-metric space which was more general than the well known complex valued metric space. P. Dhivya and M. Marudai in [6] introduced new spaces called complex partial metric space and established the existence of common fixed point theorems under the contraction condition of rational expression. M.Gunaseelan in [4] introduced new spaces called complex partial b-metric space and established the existence

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of fixed point theorem under the contractive condition. In this paper, we prove a existence and uniqueness of common fixed point theorems using (CLR) and (E.A.) properties in complex partial b-metric space.

2. PRELIMINARIES

Let \mathbb{C} be the set of complex numbers and $d_1, d_2 \in \mathbb{C}$. Define a partial order \preceq on \mathbb{C} as follows:

$d_1 \preceq d_2$ if and only if $Re(d_1) \leq Re(d_2), Im(d_1) \leq Im(d_2)$.

Consequently, one can infer that $d_1 \preceq d_2$ if one of the following conditions is satisfied:

- (i) $Re(d_1) = Re(d_2), Im(d_1) < Im(d_2)$,
- (ii) $Re(d_1) < Re(d_2), Im(d_1) = Im(d_2)$,
- (iii) $Re(d_1) < Re(d_2), Im(d_1) < Im(d_2)$,
- (iv) $Re(d_1) = Re(d_2), Im(d_1) = Im(d_2)$.

In particular, we write $d_1 \prec d_2$ if $d_1 \neq d_2$ and one of (i), (ii) and (iii) is satisfied and we write $d_1 \prec d_2$ if only (iii) is satisfied. Notice that:

- (a) If $0 \preceq d_1 \prec d_2$, then $|d_1| < |d_2|$,
- (b) If $d_1 \preceq d_2$ and $d_2 \prec d_3$ then $d_1 \prec d_3$,
- (c) If $e, f \in \mathbb{R}$ and $e \leq f$ then $ec_1 \preceq fc_1$ for all $c_1 \in \mathbb{C}$.

Definition 2.1. [6] A complex partial metric on a non-empty set Q is a function $\vartheta_c : Q \times Q \rightarrow \mathbb{C}^+$ such that for all $p, r, s \in Q$:

- (i) $0 \preceq \vartheta_c(p, p) \preceq \vartheta_c(p, r)$ (small self-distances)
- (ii) $\vartheta_c(p, r) = \vartheta_c(r, p)$ (symmetry)
- (iii) $\vartheta_c(p, p) = \vartheta_c(p, r) = \vartheta_c(r, r)$ if and only if $p = r$ (equality)
- (iv) $\vartheta_c(p, r) \preceq \vartheta_c(p, s) + \vartheta_c(s, r) - \vartheta_c(s, s)$ (triangularity).

A complex partial metric space is a pair (Q, ϑ_c) such that Q is a non empty set and ϑ_c is complex partial metric on Q .

Definition 2.2. [4] A complex partial b-metric on a non-empty set Q is a function $\kappa_{cb} : Q \times Q \rightarrow \mathbb{C}^+$ such that for all $x, y, z \in Q$:

- (i) $0 \preceq \kappa_{cb}(x, x) \preceq \kappa_{cb}(x, y)$ (small self-distances)
- (ii) $\kappa_{cb}(x, y) = \kappa_{cb}(y, x)$ (symmetry)
- (iii) $\kappa_{cb}(x, x) = \kappa_{cb}(x, y) = \kappa_{cb}(y, y) \Leftrightarrow x = y$ (equality)

(iv) \exists a real number $s \geq 1$ such that $\kappa_{cb}(x, y) \preceq s[\kappa_{cb}(x, z) + \kappa_{cb}(z, y)] - \kappa_{cb}(z, z)$ (triangularity).

A complex partial b-metric space is a pair (Q, κ_{cb}) such that Q is a non empty set and κ_{cb} is complex partial b-metric on Q . The number s is called the coefficient of (Q, κ_{cb}) .

Remark 2.1. [4] In a complex partial b-metric space (Q, κ_{cb}) if $x, y \in Q$ and $\kappa_{cb}(x, y) = 0$, then $x = y$, but the converse may not be true.

Remark 2.2. [4] It is clear that every complex partial metric space is a complex partial b-metric space with coefficient $s = 1$ and every complex valued b-metric is a complex partial b-metric space with the same coefficient and zero self-distance. However, the converse of this fact need not hold.

Example 1. [4] Let $Q = \mathbb{R}^+$, $q > 1$ a constant and $\kappa_{cb} : Q \times Q \rightarrow \mathbb{C}^+$ be defined by $\kappa_{cb}(x, y) = [\max\{x, y\}]^q + |x - y|^q + i\{[\max\{x, y\}]^q + |x - y|^q\} \forall x, y \in Q$. Then (Q, κ_{cb}) is a complex partial b-metric space with coefficient $s = 2^q > 1$, but it is neither a complex valued b-metric nor a complex partial metric. Indeed, for any $x > 0$ we have $\kappa_{cb}(x, x) = x^q(1 + i) \neq 0$. Therefore, κ_{cb} is not a complex valued b-metric on Q . Also, for $x = 6, y = 2, z = 5$,

$$\begin{aligned}\kappa_{cb}(x, y) &= (6^q + 5^q)(1 + i), \\ \kappa_{cb}(x, z) + \kappa_{cb}(z, y) - \kappa_{cb}(z, z) &= (6^q + 1^q)(1 + i) + (5^q + 3^q)(1 + i) - 5^q(1 + i) \\ &= (6^q + 1 + 3^q)(1 + i).\end{aligned}$$

So, $\kappa_{cb}(x, y) > \kappa_{cb}(x, z) + \kappa_{cb}(z, y) - \kappa_{cb}(z, z) \forall q > 1$. Therefore κ_{cb} is not a complex partial metric on Q .

Proposition 2.1. [4] Let Q be a non-empty set such that ϑ_c is a complex partial and d is a complex valued b-metric with coefficient $s > 1$ on Q . Then the function $\kappa_{cb} : Q \times Q \rightarrow \mathbb{C}^+$ defined by $\kappa_{cb}(x, y) = \vartheta_c(x, y) + d(x, y) \forall x, y \in Q$ is a complex partial b-metric on Q , that is, (Q, κ_{cb}) is a complex partial b-metric space.

Proposition 2.2. [4] Let (Q, ϑ_c) be a complex partial metric space, $r \geq 1$, then (Q, κ_{cb}) is a complex partial b-metric space with coefficient $s = 2^{r-1}$, where κ_{cb} is defined by $\kappa_{cb}(x, y) = [\vartheta_c(x, y)]^r$.

Every complex partial b-metric κ_{cb} on a non-empty set Q generates a topology τ_{cb} on Q whose base is the family of open κ_{cb} -balls $B_{\kappa_{cb}}(x, \epsilon)$ where

$\tau_{cb} = \{B_{\kappa_{cb}}(x, \epsilon) : x \in Q, \epsilon > 0\}$ and $B_{\kappa_{cb}}(x, \epsilon) = \{y \in Q : \kappa_{cb}(x, y) < \epsilon + \kappa_{cb}(x, x)\}$.

Obviously, the topological space (Q, τ_{cb}) is T_0 , but need not be T_1 .

Now, we define Cauchy sequence and convergent sequence in complex partial b-metric spaces.

Definition 2.3. [4] Let (Q, κ_{cb}) be a complex partial b-metric space with coefficient s . Let $\{x_n\}$ be any sequence in Q and $x \in Q$. Then

- (i) The sequence $\{x_n\}$ is said to be convergent with respect to τ_{cb} and converges to x , if $\lim_{n \rightarrow \infty} \kappa_{cb}(x_n, x) = \kappa_{cb}(x, x)$.
- (ii) The sequence $\{x_n\}$ is said to be Cauchy sequence in (Q, κ_{cb}) if $\lim_{n, m \rightarrow \infty} \kappa_{cb}(x_n, x_m)$ exists and is finite.
- (iii) (Q, κ_{cb}) is said to be a complete complex partial b-metric space if for every Cauchy sequence $\{x_n\}$ in Q there exists $x \in Q$ such that $\lim_{n, m \rightarrow \infty} \kappa_{cb}(x_n, x_m) = \lim_{n \rightarrow \infty} \kappa_{cb}(x_n, x) = \kappa_{cb}(x, x)$.
- (iv) A mappings $R : Q \rightarrow Q$ is said to be continuous at $x_0 \in Q$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $R(B_{\kappa_{cb}}(x_0, \delta)) \subset B_{\kappa_{cb}}(R(x_0, \epsilon))$.

Let Q be a complex partial b-metric space and $A \subseteq Q$. A point $x \in Q$ is called an interior of set A , if there exists $0 < r \in \mathbb{C}$ such that $B_{\kappa_{cb}}(x, r) = \{y \in Q : \kappa_{cb}(x, y) < \kappa_{cb}(x, x) + r\} \subseteq A$. A subset A is called open, if each point of A is an interior point of A . A point $x \in Q$ is said to be a limit point of A , for every $0 < r \in \mathbb{C}$, $B_{\kappa_{cb}}(x, r) \cap (A - \{x\}) \neq \emptyset$. A subset $B \subseteq Q$ is called closed, B contains all its limit points.

Example 2. [4] Let $Q = \mathbb{R}^+$, $a > 0$ be any constant and define $\kappa_{cb} : Q \times Q \rightarrow \mathbb{C}^+$ by $\kappa_{cb}(x, y) = (\max\{x, y\} + a)(1 + i) \forall x, y \in Q$.

Then (Q, κ_{cb}) is a complex partial b-metric space with arbitrary coefficient $s \geq 1$. Now, define a sequence $\{x_n\}$ in Q by $x_n = 1$ for all $n \in \mathbb{N}$. Note that, if $y \geq 1$, we have $\kappa_{cb}(x_n, y) = (y + a)(1 + i) = \kappa_{cb}(y, y)$. Therefore $\lim_{n \rightarrow \infty} \kappa_{cb}(x_n, y) = \kappa_{cb}(y, y)$ for all $y \geq 1$. Thus, the limit of convergent sequence in complex partial b-metric space need not be unique.

Example 3. [4] Let $Q = [0, \infty)$ endowed with complex partial b-metric $\kappa_{cb} : Q \times Q \rightarrow \mathbb{C}^+$ with $\kappa_{cb} = (\max\{x, y\})^2 + i(\max\{x, y\})^2 \forall x, y \in Q$.

It is easy to verify that (Q, κ_{cb}) is a complex partial b-metric space and note that self distance need not be zero, for example $\kappa_{cb}(1, 1) = 1 + i \neq 0$. Now the

complex valued b-metric is not induced by κ_{cb} is follows, $d_{\kappa_{cb}}(x, y) = 2\kappa_{cb}(x, y) - \kappa_{cb}(x, x) - \kappa_{cb}(y, y)$ without loss of generality suppose $x \geq y$ then $d_{\kappa}(x, y) = 2[(\max\{x, y\})^2 + i(\max\{x, y\})^2] - (x^2 + ix^2) - (y^2 + iy^2)$. Therefore, $d_{\kappa}(x, y) = x^2 - y^2 + i(x^2 - y^2)$.

Therefore, we have the following proposition.

Proposition 2.3. [4] Every complex partial b-metric κ_{cb} is not defines complex b-metric $d_{\kappa_{cb}}$, where $d_{\kappa_{cb}}(x, y) = 2\kappa_{cb}(x, y) - \kappa_{cb}(x, x) - \kappa_{cb}(y, y) \forall x, y \in Q$.

So, we introduce the new notion generalized complex partial b-metric space.

Definition 2.4. [4] A generalized complex partial b-metric on a non-empty set Q is a function $\kappa_{cb} : Q \times Q \rightarrow \mathbb{C}^+$ such that for all $x, y, z \in Q$:

- (i) $0 \preceq \kappa_{cb}(x, x) \preceq \kappa_{cb}(x, y)$ (small self - distances)
- (ii) $\kappa_{cb}(x, y) = \kappa_{cb}(y, x)$ (symmetry)
- (iii) $\kappa_{cb}(x, x) = \kappa_{cb}(x, y) = \kappa_{cb}(y, y) \Leftrightarrow x = y$ (equality)
- (iv) \exists a real number $s \geq 1$ such that $\kappa_{cb}(x, y) \preceq s[\kappa_{cb}(x, z) + \kappa_{cb}(z, y) - \kappa_{cb}(z, z)] + (\frac{1-s}{2})(\kappa_{cb}(x, x) - \kappa_{cb}(y, y))$ (triangularity).

A generalized complex partial b-metric space is a pair (Q, κ_{cb}) such that Q is a non empty set and κ_{cb} is generalized complex partial b-metric on Q . The number s is called the coefficient of (Q, κ_{cb}) .

Since $s \geq 1$, from (iv) from the previous definition we have:

$$\kappa_{cb}(x, y) \leq s[\kappa_{cb}(x, z) + \kappa_{cb}(z, y) - \kappa_{cb}(z, z)] \leq s[\kappa_{cb}(x, z) + \kappa_{cb}(z, y)] - \kappa_{cb}(z, z).$$

Hence, a complex partial b-metric space is also a generalized complex partial b-metric space.

Proposition 2.4. [4] Every generalized complex partial b-metric κ_{cb} is defines complex b-metric $d_{\kappa_{cb}}$, where $d_{\kappa_{cb}}(x, y) = 2\kappa_{cb}(x, y) - \kappa_{cb}(x, x) - \kappa_{cb}(y, y) \forall x, y \in Q$.

Definition 2.5. Let G and F be self maps on a set Q , if $t = Fq = Gq$ for some q in Q , then q is called coincidence point of F and G and t is called a point of coincidence of F and G .

Definition 2.6. Let F and G be two self maps defined on a set Q , then F and G are said to be weakly compatible if they commute at coincidence points.

Definition 2.7. Let $L, M : D \rightarrow D$ be two self mappings of a complex partial b-metric space (D, κ_{cb}) . The pair (L, M) is said to satisfy (E.A) property if there exists a sequence $\{p_n\}$ in D such that $\lim_{n \rightarrow \infty} Mp_n = \lim_{n \rightarrow \infty} Lp_n = t$, for some $t \in D$.

Definition 2.8. The self mappings L and M from D to D are said to satisfy the common limit in the range of M property (CLR_s property) if $\lim_{n \rightarrow \infty} Lp_n = \lim_{n \rightarrow \infty} Mp_n = Mp$, for some $p \in D$.

3. MAIN RESULTS

3.1. Common Fixed Point Theorem. In this section, we prove common fixed point theorem in the complex partial b-metric space.

Theorem 3.1. Let (Q, κ_{cb}) be a complete complex partial b-metric space with coefficient $s \geq 1$ and let F, G, L and M are four self maps of Q such that $M(Q) \subseteq F(Q)$ and $L(Q) \subseteq G(Q)$ and satisfying

$$\begin{aligned} \kappa_{cb}(Lp, Mq) \preceq & a\kappa_{cb}(Fp, Gq) + b[\kappa_{cb}(Fp, Lp) + \kappa_{cb}(Gq, Mq)] \\ & + c[\kappa_{cb}(Fp, Mq) + \kappa_{cb}(Gq, Lp)] \end{aligned}$$

for all $p, q \in Q$ where $a, b, c \geq 0$ and $a + 2sb + 2sc < 1$. Suppose that the pairs $\{F, L\}$ and $\{G, M\}$ are weakly compatible. Then F, G, L and M have a unique common fixed point.

Proof. Suppose p_0 is an arbitrary point of Q . Define the sequence $\{q_n\}$ in Q such that

$$\begin{aligned} q_{2n} &= Lp_{2n} = Gp_{2n+1} \\ q_{2n+1} &= Mp_{2n+1} = Fp_{2n+2}. \end{aligned}$$

Now,

$$\begin{aligned} \kappa_{cb}(q_{2n}, q_{2n+1}) &= \kappa_{cb}(Lp_{2n}, Mp_{2n+1}) \\ &\preceq a\kappa_{cb}(Fp_{2n}, Gp_{2n+1}) + b[\kappa_{cb}(Fp_{2n}, Lp_{2n}) + \kappa_{cb}(Gp_{2n+1}, Mp_{2n+1})] \\ &\quad + c[\kappa_{cb}(Fp_{2n}, Mp_{2n+1}) + \kappa_{cb}(Gp_{2n+1}, Lp_{2n})] \\ &\preceq a\kappa_{cb}(q_{2n-1}, q_{2n}) + b[\kappa_{cb}(q_{2n-1}, q_{2n}) + \kappa_{cb}(q_{2n}, q_{2n+1})] \\ &\quad + c[\kappa_{cb}(q_{2n-1}, q_{2n+1}) + \kappa_{cb}(q_{2n}, q_{2n})] \\ &\preceq a\kappa_{cb}(q_{2n-1}, q_{2n}) + b[\kappa_{cb}(q_{2n-1}, q_{2n}) + \kappa_{cb}(q_{2n}, q_{2n+1})] \\ &\quad + c[s\kappa_{cb}(q_{2n-1}, q_{2n}) + s\kappa_{cb}(q_{2n}, q_{2n+1}) - \kappa_{cb}(q_{2n}, q_{2n}) + \kappa_{cb}(q_{2n}, q_{2n})] \\ &\preceq \frac{a + b + sc}{1 - b - sc} \kappa_{cb}(q_{2n-1}, q_{2n}), \end{aligned}$$

$$|\kappa_{cb}(q_{2n}, q_{2n+1})| \leq k |\kappa_{cb}(q_{2n-1}, q_{2n})|,$$
$$|\kappa_{cb}(q_{2n+1}, q_{2n+2})| \leq k |\kappa_{cb}(q_{2n}, q_{2n+1})|.$$
$$|\kappa_{cb}(q_{n+1}, q_{n+2})| \leq k|\kappa_{cb}(q_n, q_{n+1})| + \cdots + k^{n+1}|\kappa_{cb}(q_0, q_1)|$$
$$\begin{aligned}
& \kappa_{cb}(q_n, q_{n+m}) \preceq s[\kappa_{cb}(q_n, q_{n+1}) + \kappa_{cb}(q_{n+1}, q_{n+m})] - \kappa_{cb}(q_{n+1}, q_{n+1}) \\
& \dots\dots\dots \\
& \preceq sk^n \kappa_{cb}(q_0, q_1) + s^2 k^{n+1} \kappa_{cb}(q_0, q_1) \\
& + \dots + s^m k^{n+m-1} \kappa_{cb}(q_0, q_1) - \kappa_{cb}(q_{n+1}, q_{n+1}) - s\kappa_{cb}(q_{n+2}, q_{n+2}) \\
& - s^2 \kappa_{cb}(q_{n+3}, q_{n+3}) - \dots - s^{m-2} \kappa_{cb}(q_{n+m-1}, q_{n+m-1}) \\
& = [sk^n + s^2 k^{n+1} + \dots + s^m k^{n+m-1}] \kappa_{cb}(q_0, q_1) \\
& - [\kappa_{cb}(q_{n+1}, q_{n+1}) + s\kappa_{cb}(q_{n+2}, q_{n+2}) \\
& + s^2 \kappa_{cb}(q_{n+3}, q_{n+3}) + \dots + s^{m-2} \kappa_{cb}(q_{n+m-1}, q_{n+m-1})] \\
& = sk^n[1 + (sk) + (sk)^2 + \dots + (sk)^{m-1}] \kappa_{cb}(q_0, q_1) \\
& - \sum_{i=1}^{m-1} s^{m-1-i} \kappa_{cb}(q_{n+m-i}, q_{n+m-i})
\end{aligned}$$

$$\begin{aligned}
|\kappa_{cb}(q_n, q_{n+m})| &\leq sk^n |\kappa_{cb}(q_0, q_1)| (1 + (sk) + (sk)^2 + \dots + (sk)^{m-1}) \\
&\quad - \sum_{i=1}^{m-1} s^{m-1-i} |\kappa_{cb}(q_{n+m-i}, q_{n+m-i})| \\
&\leq sk^n |\kappa_{cb}(q_0, q_1)| (1 + (sk) + (sk)^2 + \dots + (sk)^{m-1}) \\
&\leq sk^n |\kappa_{cb}(q_0, q_1)| (1 + (sk) + (sk)^2 + \dots) \\
&\leq \frac{sk^n}{1 - sk} |\kappa_{cb}(q_0, q_1)|
\end{aligned}$$

Since $k < 1$, and $s \geq 1$. Therefore, $sk < 1$. Taking limit $n \rightarrow \infty$, we have $k^n \rightarrow 0$. This implies $|\kappa_{cb}(q_n, q_{n+m})| \rightarrow 0$ as $n, m \rightarrow \infty$. Therefore, the sequence $\{q_n\}$ is

Cauchy sequence in Q . Since Q is a complete complex partial b-metric. Therefore, there exists $z \in Q$ such that $q_n \rightarrow z$ and $\kappa_{cb}(z, z) = \lim_{n \rightarrow \infty} \kappa_{cb}(z, q_n) = \lim_{n \rightarrow \infty} \kappa_{cb}(q_n, q_n) = 0$. Since $M(Q) \subseteq F(Q)$, there exists a point $u \in Q$ such that $z = Fu$. Suppose that $\kappa_{cb}(Lu, z) > 0$. Then

$$\begin{aligned} \kappa_{cb}(Lu, z) &\preceq s\kappa_{cb}(Lu, Mp_{2n-1}) + s\kappa_{cb}(Mp_{2n-1}, z) - \kappa_{cb}(Mp_{2n-1}, Mp_{2n-1}) \\ &\preceq s[a\kappa_{cb}(Fu, Gp_{2n-1}) + b(\kappa_{cb}(Fu, Lu) + \kappa_{cb}(Gp_{2n-1}, Mp_{2n-1})) \\ &\quad + c(\kappa_{cb}(Fu, Mp_{2n-1}) + \kappa_{cb}(Gp_{2n-1}, Lu))] + s\kappa_{cb}(Mp_{2n-1}, z). \end{aligned}$$

As $n \rightarrow \infty$,

$$\begin{aligned} |\kappa_{cb}(Lu, z)| &\leq sa|\kappa_{cb}(z, z)| + sb|\kappa_{cb}(z, Lu)| + sb|\kappa_{cb}(z, z)| \\ &\quad + cs|\kappa_{cb}(z, z)| + sc|\kappa_{cb}(z, Lu)| + s|\kappa_{cb}(z, z)| \\ &\leq (sb + sc)|\kappa_{cb}(z, Lu)|. \end{aligned}$$

Since $a + 2sb + 2sc < 1$. Therefore which is a contradiction. Hence $Lu = Fu = z$. Since $L(Q) \subseteq G(Q)$, there exists a point $v \in q$ such that $z = Gv$. Suppose that $\kappa_{cb}(z, Mv) > 0$. Then

$$\begin{aligned} \kappa_{cb}(z, Mv) &\preceq \kappa_{cb}(Lu, Mv) \\ &\preceq a\kappa_{cb}(Fu, Gv) + b[\kappa_{cb}(Fu, Lu) + \kappa_{cb}(Gv, Mv)] \\ &\quad + c[\kappa_{cb}(Fu, Mv) + \kappa_{cb}(Gv, Lu)], \end{aligned}$$

which implies that,

$$\begin{aligned} |\kappa_{cb}(z, Mv)| &\leq a|\kappa_{cb}(z, z)| + b[|\kappa_{cb}(z, z)| + |\kappa_{cb}(z, Mv)|] \\ &\quad + c[|\kappa_{cb}(z, Mv)| + |\kappa_{cb}(z, z)|] \\ &\leq (b + c)|\kappa_{cb}(z, Mv)|. \end{aligned}$$

Since $b + c < 1$, which is a contradiction. Therefore $Mv = Gv = z$. Hence $Lu = Fu = Mv = Gv = z$. Since F and L are weakly compatible maps, then $LFu = FLu$. Therefore $Lz = Fz$. Now we claim that z is a fixed point of L if

$Lz \neq z$. We have

$$\begin{aligned}
 \kappa_{cb}(Lz, z) &\preceq \kappa_{cb}(Lz, Mv) \\
 &\preceq a\kappa_{cb}(Fz, Gv) + b[\kappa_{cb}(Fz, Lz) + \kappa_{cb}(Gv, Mv)] \\
 &\quad + c[\kappa_{cb}(Fz, Mv) + \kappa_{cb}(Gv, Lz)] \\
 &= a\kappa_{cb}(Lz, z) + b[\kappa_{cb}(Lz, Lz) + \kappa_{cb}(z, z)] + c[\kappa_{cb}(Lz, z) + \kappa_{cb}(z, Lz)] \\
 &\preceq (a + b + 2c)\kappa_{cb}(Lz, z),
 \end{aligned}$$

which implies that

$$|\kappa_{cb}(Lz, z)| \leq (a + b + 2c)|\kappa_{cb}(Lz, z)|.$$

Since $a + b + 2c < 1$, which is a contradiction. Therefore $Lz = z$. Hence $Lz = Fz = z$. Similarly, G and M are weakly compatible maps, we have $Mz = Gz$. Now we claim that z is a fixed point of M . Suppose that $Mz \neq z$. Then we have

$$\begin{aligned}
 \kappa_{cb}(z, Mz) &\preceq \kappa_{cb}(Lz, Mz) \\
 &\preceq a\kappa_{cb}(Fz, Gz) + b\kappa_{cb}(Fz, Lz) + \kappa_{cb}(Gz, Mz) \\
 &\quad + c[\kappa_{cb}(Fz, Mz) + \kappa_{cb}(Gz, Lz)] \\
 &\preceq a\kappa_{cb}(z, Mz) + b\kappa_{cb}(z, z) + \kappa_{cb}(Mz, Mz) \\
 &\quad + c[\kappa_{cb}(z, Mz) + \kappa_{cb}(z, Mz)] \\
 &\preceq a\kappa_{cb}(z, Mz) + b\kappa_{cb}(z, Mz) \\
 &\quad + c[\kappa_{cb}(z, Mz) + \kappa_{cb}(z, Mz)] \\
 &= (a + b + 2c)\kappa_{cb}(z, Mz),
 \end{aligned}$$

which implies that

$$\kappa_{cb}(z, Mz) \leq (a + b + 2c)|\kappa_{cb}(z, Mz)|.$$

Since $a + b + 2c < 1$, which is a contradiction. Therefore $Mz = z$. Thus $Mz = Gz = z$. Hence $Mz = Gz = Fz = Gz = z$ and it follows that z is a common fixed point of F, G, L and M . Next we claim that the uniqueness of z . Let z and

w are distinct common fixed points of F, G, L and M . Suppose not, we have

$$\begin{aligned}\kappa_{cb}(z, w) &= \kappa_{cb}(Lz, Mw) \\ &\preceq a\kappa_{cb}(Fz, Gw) + b[\kappa_{cb}(Fz, Lz) + \kappa_{cb}(Gw, Mw)] \\ &\quad + c[\kappa_{cb}(Fz, Mw) + \kappa_{cb}(Gw, Lz)] \\ &\preceq a\kappa_{cb}(z, w) + b[\kappa_{cb}(z, z) + \kappa_{cb}(w, w)] + c[\kappa_{cb}(z, w) + \kappa_{cb}(w, z)] \\ &= (a + 2c)\kappa_{cb}(z, w),\end{aligned}$$

which implies that

$$|\kappa_{cb}(z, w)| \leq (a + 2c)|\kappa_{cb}(z, w)|.$$

Since $a + 2c < 1$, which is a contradiction. Therefore $z = w$. Hence z is the unique common fixed point of F, G, L and M . \square

Corollary 3.1. *Let (Q, κ_{cb}) be a complex partial b -metric space with coefficient $s \geq 1$ and let F, L and M are three self maps of Q such that $M(Q) \subseteq F(Q)$ and $L(Q) \subseteq F(Q)$ and satisfying*

$$\begin{aligned}\kappa_{cb}(Lp, Mq) &\preceq a\kappa_{cb}(Fp, Fq) + b[\kappa_{cb}(Fp, Lp) + \kappa_{cb}(Fq, Mq)] \\ &\quad + c[\kappa_{cb}(Fp, Mq) + \kappa_{cb}(Fq, Lp)]\end{aligned}$$

for all $p, q \in Q$ where $a, b, c \geq 0$ and $a + 2sb + 2sc < 1$. Suppose that the pairs $\{F, L\}$ and $\{F, M\}$ are weakly compatible. Then F, G, L and M have a unique common fixed point.

Proof. The result follows on putting $F = G$ in Theorem 3.1. \square

3.2. Common Fixed Point Theorem Using (E.A) Property. In this section, we prove common fixed point theorems using (E.A.) property in the complex partial b -metric space.

Theorem 3.2. *Let F, G, L and M be four self mappings of a complex partial b -metric space (Q, κ_{cb}) satisfying:*

- (i) $F(Q) \subseteq M(Q)$ and $G(Q) \subseteq L(Q)$;
- (ii) for all $p, q \in Q$ and $a, b \geq 0$, $4s(a + b) < 1$,

$$\kappa_{cb}(Fp, Gq) \preceq a[\kappa_{cb}(Fp, Lp) + \kappa_{cb}(Gq, Mq)] + b[\kappa_{cb}(Fp, Mq) + \kappa_{cb}(Gq, Lp)]$$

- (iii) the pairs (F, L) and (G, M) are weakly compatible;
- (iv) one of the pairs (F, L) or (G, M) satisfies (E.A)-property.

If the range of one of the mappings $L(Q)$ or $M(Q)$ is a closed subspace of Q , then the mappings F, G, L and M have common fixed point in Q .

Proof. Suppose that the pair (G, M) satisfies $(E.A.)$ property. Then, by definition, there exists a sequence $\{p_n\}$ in Q such that $\lim_{n \rightarrow \infty} Gp_n = \lim_{n \rightarrow \infty} Mp_n = z$ for some $z \in Q$. Since $G(Q) \subseteq L(Q)$, there exists a sequence $\{q_n\}$ in Q such that $Gp_n = Lq_n$. Hence, $\lim_{n \rightarrow \infty} Lq_n = z$. We claim that $\lim_{n \rightarrow \infty} Fq_n = z$. Let $\lim_{n \rightarrow \infty} Fq_n = z_1 \neq z$, then putting $p = q_n$, $q = p_n$ in condition (ii), we have

$$\begin{aligned} \kappa_{cb}(Fq_n, Gp_n) &\preceq a[\kappa_{cb}(Fq_n, Lq_n) + \kappa_{cb}(Gp_n, Mp_n)] \\ &\quad + b[\kappa_{cb}(Fq_n, Mp_n) + \kappa_{cb}(Gp_n, Lq_n)]. \end{aligned}$$

As $n \rightarrow \infty$, we have

$$\begin{aligned} \kappa_{cb}(z_1, z) &\preceq a[\kappa_{cb}(z_1, z) + \kappa_{cb}(z, z)] \\ &\quad + b[\kappa_{cb}(z_1, z) + \kappa_{cb}(z, z)] \\ &\preceq a[\kappa_{cb}(z_1, z) + 2s\kappa_{cb}(z, z_1) - \kappa_{cb}(z_1, z_1)] \\ &\quad + b[\kappa_{cb}(z_1, z) + 2s\kappa_{cb}(z, z_1) - \kappa_{cb}(z_1, z_1)] \\ &\prec a[\kappa_{cb}(z_1, z) + 2s\kappa_{cb}(z, z_1)] \\ &\quad + b[\kappa_{cb}(z_1, z) + 2s\kappa_{cb}(z, z_1)]. \end{aligned}$$

Then, $|\kappa_{cb}(z_1, z)| < 0$; hence, $z_1 = z$ and that is, $\lim_{n \rightarrow \infty} Fq_n = \lim_{n \rightarrow \infty} Gp_n = z$.

Now suppose that $L(Q)$ is closed subspace of Q , then $z = Lu$ for some $u \in Q$. Consequently, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} Fq_n &= \lim_{n \rightarrow \infty} Gp_n = \lim_{n \rightarrow \infty} Mp_n \\ &= \lim_{n \rightarrow \infty} Lq_n = z = Lu. \end{aligned}$$

We claim that $Fu = Lu$. Put $p = u$ and $q = p_n$ in contractive condition (ii), and we have

$$\begin{aligned} \kappa_{cb}(Fu, Gp_n) &\preceq a[\kappa_{cb}(Fu, Lu) + \kappa_{cb}(Gp_n, Mp_n)] \\ &\quad + b[\kappa_{cb}(Fu, Mp_n) + \kappa_{cb}(Gp_n, Lu)]. \end{aligned}$$

As $n \rightarrow \infty$, we have

$$\begin{aligned}
 \kappa_{cb}(Fu, z) &\preceq a[\kappa_{cb}(Fu, z) + \kappa_{cb}(z, z)] \\
 &\quad + b[\kappa_{cb}(Fu, z) + \kappa_{cb}(z, z)] \\
 &\preceq a[\kappa_{cb}(Fu, z) + 2s\kappa_{cb}(Fu, z) - \kappa_{cb}(Fu, Fu)] \\
 &\quad + b[\kappa_{cb}(Fu, z) + 2s\kappa_{cb}(Fu, z) - \kappa_{cb}(Fu, Fu)] \\
 &\prec a[\kappa_{cb}(Fu, z) + 2s\kappa_{cb}(Fu, z)] \\
 &\quad + b[\kappa_{cb}(Fu, z) + 2s\kappa_{cb}(Fu, z)].
 \end{aligned}$$

Then, $|\kappa_{cb}(Fu, z)| < 0$, which is contradiction. Hence, u is a coincidence point of (F, L) . Now the weak compatibility of pair (F, L) implies that $FLu = LFu$ or $Fz = Lz$.

On the other hand, since $F(Q) \subseteq M(Q)$, there exists v in Q such that $Fu = Mv$. Thus, $Fu = Lu = Mv = z$. Now, we prove that v is a coincidence point of (G, M) ; that is, $Gv = Mv = t$. put $p = u, q = v$ in contractive condition (ii), and we have

$$\begin{aligned}
 \kappa_{cb}(z, Gv) &\preceq a[\kappa_{cb}(z, z) + \kappa_{cb}(Gv, z)] \\
 &\quad + b[\kappa_{cb}(z, z) + \kappa_{cb}(Gv, z)] \\
 &\preceq a[2s\kappa_{cb}(z, Gv) - \kappa_{cb}(Gv, Gv) + \kappa_{cb}(Gv, z)] \\
 &\quad + b[2s\kappa_{cb}(z, Gv) - \kappa_{cb}(Gv, Gv) + \kappa_{cb}(Gv, z)] \\
 &\prec a[2s\kappa_{cb}(z, Gv) + \kappa_{cb}(Gv, z)] \\
 &\quad + b[2s\kappa_{cb}(z, Gv) + \kappa_{cb}(Gv, z)].
 \end{aligned}$$

Then, $|\kappa_{cb}(z, Gv)| < 0$, which is a contradiction. Thus, $Gv = z$. Hence, $Gv = Mv = z$, and v is the coincidence point of G and M .

Further, the weak compatibility of pair (G, M) implies that $GMv = MGv$, or $Gz = Mz$. Therefore, z is a common coincidence point of F, G, L and M .

Now, we prove that z is a common fixed point. Put $p = u$ and $q = z$ in contractive

condition (ii), we have

$$\begin{aligned}
 \kappa_{cb}(z, Gz) &\preceq a[\kappa_{cb}(z, z) + \kappa_{cb}(Gz, z)] \\
 &\quad + b[\kappa_{cb}(z, Gz) + \kappa_{cb}(Gz, z)] \\
 &\preceq a[2s\kappa_{cb}(z, Gz) - \kappa_{cb}(Gz, Gz) + \kappa_{cb}(Gz, z)] \\
 &\quad + b[\kappa_{cb}(z, Gz) + \kappa_{cb}(Gz, z)] \\
 &\prec a[2s\kappa_{cb}(z, Gz) + \kappa_{cb}(Gz, z)] \\
 &\quad + b[\kappa_{cb}(z, Gz) + \kappa_{cb}(Gz, z)],
 \end{aligned}$$

which implies that

$$|\kappa_{cb}(z, Gz)| < 0.$$

This is a contradiction. Thus, $Gz = z$. Hence, $Fz = Gz = Lz = Mz = z$.

Similarly if we assume that $M(Q)$ is closed subspace of Q and $(E.A)$ -property of the pair (F, L) will give a similar result.

We claim that uniqueness of the common fixed point. Let us assume that r is another common fixed point of F, G, L and M . Then, put $p = r, q = z$ in contractive condition (ii), we have

$$\begin{aligned}
 \kappa_{cb}(r, z) = \kappa_{cb}(Fr, Gz) &\preceq a[\kappa_{cb}(Fr, Lr) + \kappa_{cb}(Gz, Mz)] \\
 &\quad + b[\kappa_{cb}(Fr, Mz) + \kappa_{cb}(Gz, Lr)] \\
 &\preceq a[\kappa_{cb}(r, r) + \kappa_{cb}(z, z)] \\
 &\quad + b[\kappa_{cb}(r, z) + \kappa_{cb}(z, r)] \\
 &\preceq a[2s\kappa_{cb}(r, z) - \kappa_{cb}(z, z) + 2s\kappa_{cb}(z, r) - \kappa_{cb}(r, r)] \\
 &\quad + b[\kappa_{cb}(r, z) + \kappa_{cb}(z, r)] \\
 &\prec a[2s\kappa_{cb}(r, z) + 2s\kappa_{cb}(z, r)] \\
 &\quad + b[\kappa_{cb}(r, z) + \kappa_{cb}(z, r)].
 \end{aligned}$$

Then, $|\kappa_{cb}(r, z)| < 0$, which is a contradiction. Thus $r = z$. Hence, $Fz = Gz = Lz = Mz = z$, and z is the unique common fixed point of F, G, L and M . \square

Remark 3.1. *Completeness of Q is relaxed in Theorem 3.1.*

If $F = G$ and $L = M$ in Theorem 3.1, we have the following result.

Corollary 3.2. *Let F, M be the self mappings of a complex partial b -metric space (Q, κ_{cb}) satisfying:*

- (i) $F(Q) \subseteq M(Q)$;
- (ii) for all $p, q \in Q$ and $a, b \geq 0$, $4s(a + b) < 1$,

$$\begin{aligned} \kappa_{cb}(Fp, Fq) \preceq & a[\kappa_{cb}(Fp, Mp) + \kappa_{cb}(Fq, Mq)] \\ & + b[\kappa_{cb}(Fp, Mq) + \kappa_{cb}(Fq, Mp)] \end{aligned}$$

- (iii) the pairs (F, M) is weakly compatible;
- (iv) the pairs (F, M) satisfies (E.A)-property.

If the range of one of the mapping $M(Q)$ is a closed subspace of Q , then the mappings F and M have common fixed point in Q .

Example 4. Let $Q = \{-\frac{1}{3}\} \cup (0, 3]$ and $\kappa_{cb}(u, v) = \{\max\{u, v\}\}^2(1 + i)$ where $u, v \in Q$; then (U, δ_{cb}) is a complex partial b -metric space. Let $F, G, L, M: Q \rightarrow Q$ be defined by

$$\begin{aligned} F(q) &= \begin{cases} 1 & \text{if } q \in \{-\frac{1}{3}\} \cup [1, 3], \\ \frac{1}{3} & \text{if } q \in (0, 1). \end{cases} \\ G(q) &= \begin{cases} 1 & \text{if } q \in \{-\frac{1}{3}\} \cup [1, 3], \\ \frac{1}{2} & \text{if } q \in (0, 1). \end{cases} \\ L(q) &= \begin{cases} 1 & \text{if } q = 1, \\ \frac{3}{2} & \text{if } q \in (0, 1), \\ \frac{q-1}{2} & \text{if } q \in \{-\frac{1}{3}\} \cup (1, 3]. \end{cases} \end{aligned}$$

and

$$M(q) = \begin{cases} 1 & \text{if } q = 1, \\ \frac{4}{3} & \text{if } q \in (0, 1), \\ \frac{q+1}{2} & \text{if } q \in \{-\frac{1}{3}\} \cup (1, 3]. \end{cases}$$

Then $F(Q) = \{1, \frac{1}{3}\}$, $G(Q) = \{1, \frac{1}{2}\}$, $L(Q) = \{-\frac{2}{3}, \frac{3}{2}\} \cup (0, 1]$, $M(Q) = \{\frac{1}{3}\} \cup [1, 2]$, and

- (i) $F(Q) \subseteq M(Q)$ and $G(Q) \subseteq L(Q)$;
- (ii) for all $p, q \in Q$ and $a, b \geq 0$, $4s(a + b) < 1$, one can verify that

$$\begin{aligned} \kappa_{cb}(Fp, Gq) \preceq & a[\kappa_{cb}(Fp, Lp) + \kappa_{cb}(Gq, Mq)] \\ & + b[\kappa_{cb}(Fp, Mq) + \kappa_{cb}(Gq, Lp)] \end{aligned}$$

- (iii) the pairs (F, L) and (G, M) are weakly compatible;

(iv) let $\{q_n\} = \{3 - \frac{1}{n}\}_{n \geq 1}$ be a sequence in Q .

Then

$$\lim_{n \rightarrow \infty} Fq_n = \lim_{n \rightarrow \infty} Lq_n = 1 \in Q,$$

and

$$\lim_{n \rightarrow \infty} Gq_n = \lim_{n \rightarrow \infty} Mq_n = 1 \in Q.$$

Therefore, one of the pairs (F, L) or (G, M) satisfies (E.A)-property.

(v) $L(Q)$ or $M(Q)$ is a closed subspace of Q . Hence by theorem 3.2, 1 is a unique common fixed point of F, G, L and M .

3.3. Common Fixed Point Theorem Using (CLR) Property. In this section, we prove common fixed point theorems using (CLR) property in the complex partial b-metric space.

Theorem 3.3. Let F, G, L and M be four self mappings of a complex partial b-metric space (Q, κ_{cb}) satisfying :

- (i) $F(Q) \subseteq M(Q)$ and $G(Q) \subseteq L(Q)$;
- (ii) for all $p, q \in Q$ and $a, b \geq 0$, $4s(a + b) < 1$,

$$\begin{aligned} \kappa_{cb}(Fp, Gq) &\preceq a[\kappa_{cb}(Fp, Lp) + \kappa_{cb}(Gq, Mq)] \\ &\quad + b[\kappa_{cb}(Fp, Mq) + \kappa_{cb}(Gq, Lp)] \end{aligned}$$

- (iii) the pairs (F, L) and (G, M) are weakly compatible.

If the pair (F, L) satisfies (CLR_F) property or (G, M) satisfies (CLR_G) property, then F, G, L and M have a unique common fixed point in Q .

Proof. Suppose that the pair (G, M) satisfies (CLR_G) property. Then, by definition, there exists a sequence $\{p_n\}$ in Q such that

$$\lim_{n \rightarrow \infty} Gp_n = \lim_{n \rightarrow \infty} Mp_n = Gp,$$

for some $p \in Q$. Since $G(Q) \subseteq L(Q)$, we have $Gp = Lu$, for some $u \in Q$. We show that $Fu = Lu = z$ (say). Put $p = u$ and $q = p_n$ in contractive condition (ii), we have

$$\begin{aligned} \kappa_{cb}(Fu, Gp_n) &\preceq a[\kappa_{cb}(Fu, Lu) + \kappa_{cb}(Gp_n, Mp_n)] \\ &\quad + b[\kappa_{cb}(Fu, Mp_n) + \kappa_{cb}(Gp_n, Lu)]. \end{aligned}$$

As $n \rightarrow \infty$, we have

$$\begin{aligned}
 \kappa_{cb}(Fu, Gp) &\preceq a[\kappa_{cb}(Fu, Gp) + \kappa_{cb}(Gp, Gp)] \\
 &\quad + b[\kappa_{cb}(Fu, Gp) + \kappa_{cb}(Gp, Gp)] \\
 &\preceq a[\kappa_{cb}(Fu, Gp) + 2s\kappa_{cb}(Fu, Gp) - \kappa_{cb}(Fu, Fu)] \\
 &\quad + b[\kappa_{cb}(Fu, Gp) + 2s\kappa_{cb}(Fu, Gp) - \kappa_{cb}(Fu, Fu)] \\
 &\prec a[\kappa_{cb}(Fu, Gp) + 2s\kappa_{cb}(Fu, Gp)] \\
 &\quad + b[\kappa_{cb}(Fu, Gp) + 2s\kappa_{cb}(Fu, Gp)].
 \end{aligned}$$

Then, $|\kappa_{cb}(Fu, Gp)| < 0$, which is a contradiction. Thus, $Fu = Lu$.

Hence, $Fu = Lu = Gp = t$.

Now, the weak compatibility of pair (F, L) implies that, $FLu = LFu$ or $Fz = Lz$.

Since $F(Q) \subseteq M(Q)$, there exists v in Q such that $Fu = Mv$. Thus, $Fu = Lu = Mv = z$.

Next, we claim that v is a coincidence point of (G, M) that is, $Gv = Mv = z$.

Put $p = u, q = v$ in contractive condition (ii), we have

$$\begin{aligned}
 \kappa_{cb}(z, Gv) = \kappa_{cb}(Fu, Gv) &\preceq a[\kappa_{cb}(Fu, Lu) + \kappa_{cb}(Gv, Mv)] \\
 &\quad + b[\kappa_{cb}(Fu, Mv) + \kappa_{cb}(Gv, Lu)] \\
 &\preceq a[\kappa_{cb}(z, z) + \kappa_{cb}(Gv, z)] \\
 &\quad + b[\kappa_{cb}(z, z) + \kappa_{cb}(Gv, z)] \\
 &\preceq a[2s\kappa_{cb}(z, Gz) - \kappa_{cb}(Gz, Gz) + \kappa_{cb}(Gv, z)] \\
 &\quad + b[2s\kappa_{cb}(z, Gz) - \kappa_{cb}(Gz, Gz) + \kappa_{cb}(Gv, z)] \\
 &\prec a[2s\kappa_{cb}(z, Gz) + \kappa_{cb}(Gv, z)] \\
 &\quad + b[2s\kappa_{cb}(z, Gz) + \kappa_{cb}(Gv, z)].
 \end{aligned}$$

Then, $|\kappa_{cb}(z, Gv)| < 0$, which is a contradiction. Thus, $Bv = z$. Hence, $Bv = Mv = z$, and v is coincidence point of B and M .

Further, the weak compatibility of pair (G, M) implies that $GMv = MGv$ or $Gz = Mz$. Therefore, z is a common coincidence point of F, G, L and M . Now, we claim that z is a common fixed point. Put $p = u$ and $q = z$ in contractive

condition (ii), we have

$$\begin{aligned}
 \kappa_{cb}(z, Gz) &\preceq a[\kappa_{cb}(z, z) + \kappa_{cb}(Gz, z)] \\
 &\quad + b[\kappa_{cb}(z, z) + \kappa_{cb}(Gz, z)] \\
 &\preceq a[2s\kappa_{cb}(Gz, z) - \kappa_{cb}(Gz, Gz) + \kappa_{cb}(Gz, z)] \\
 &\quad + b[2s\kappa_{cb}(Gz, z) - \kappa_{cb}(Gz, Gz) + \kappa_{cb}(Gz, z)] \\
 &\prec a[2s\kappa_{cb}(Gz, z) + \kappa_{cb}(Gz, z)] \\
 &\quad + b[2s\kappa_{cb}(Gz, z) + \kappa_{cb}(Gz, z)].
 \end{aligned}$$

Then, $|\kappa_{cb}(z, Gz)| < 0$, which is a contradiction. Thus, $Gz = z$. Hence, $Fz = Gz = Lz = Mz = z$. Easy to verify, uniqueness of the common fixed point.

In a similar way, the argument that the pair (F, L) satisfies property (CLR_F) will also give the unique common fixed point of F, G, L and M .

Example 5. Let $Q = (0, 3]$ and $\kappa_{cb}(u, v) = \{\max\{u, v\}\}^2(1 + i)$ where $u, v \in Q$; then (U, δ_{cb}) is a complex partial b -metric space. Let $F, G, L, M: Q \rightarrow Q$ be defined by

$$F(q) = \begin{cases} 1 & \text{if } q \in [1, 3], \\ \frac{2}{3} & \text{if } q \in (0, 1). \end{cases}$$

$$G(q) = \begin{cases} 1 & \text{if } q \in [1, 3], \\ \frac{1}{2} & \text{if } q \in (0, 1). \end{cases}$$

$$L(q) = \begin{cases} 1 & \text{if } q = 1, \\ \frac{3}{2} & \text{if } q \in (0, 1), \\ \frac{q}{3} & \text{if } q \in (1, 3]. \end{cases}$$

and

$$M(q) = \begin{cases} 1 & \text{if } q = 1, \\ \frac{4}{3} & \text{if } q \in (0, 1), \\ \frac{q}{2} & \text{if } q \in (1, 3]. \end{cases}$$

Then $F(Q) = \{1, \frac{2}{3}\}$, $G(Q) = \{1, \frac{1}{2}\}$, $L(Q) = (\frac{1}{3}, 1] \cup \{\frac{3}{2}\}$, $M(Q) = (\frac{1}{2}, \frac{3}{2}]$, and
(i) $F(Q) \subseteq M(Q)$ and $G(Q) \subseteq L(Q)$;

(ii) for all $p, q \in Q$ and $a, b \geq 0$, $4s(a + b) < 1$, one can verify that

$$\begin{aligned}\kappa_{cb}(Fp, Gq) &\preceq a[\kappa_{cb}(Fp, Lp) + \kappa_{cb}(Gq, Mq)] \\ &\quad + b[\kappa_{cb}(Fp, Mq) + \kappa_{cb}(Gq, Lp)]\end{aligned}$$

(iii) the pairs (F, L) and (G, M) are weakly compatible.

(iv) let $\{q_n\} = \{3 - \frac{1}{n}\}_{n \geq 1}$ be a sequences in Q . Then

$$\lim_{n \rightarrow \infty} Fq_n = \lim_{n \rightarrow \infty} Lq_n = 1 = F(1)$$

Therefore, the pair (F, L) satisfies (CLR_F) property or (G, M) satisfies (CLR_G) property. Hence, by theorem 3.3, 1 is a unique common fixed point of F, G, L and M . Note that $L(Q)$ or $M(Q)$ are not closed subspace of Q .

□

REFERENCES

- [1] I. A. BAKHTIN: *The contraction mappings principle in quasi-metric spaces*, Functional Analysis, **30** (1989), 26-37, (Russian).
- [2] S. CZERWICK: *Contraction mappings in b-metric spaces*, Acta Mathematica et Informatica Universitatis Ostraviensis, **1**(1993), 5–11.
- [3] A. AZAM B. FISHER, M. KHAN: *Common fixed point theorems in complex valued metric spaces*, Numer.Funct.Anal.Optim., **32**(3) (2011), 243–253.
- [4] M. GUNASEELAN: *Generalized fixed point Theorems on Complex Partial b-Metric Space*, International Journal of Research and Analytical Reviews, **6**(2) (2019).
- [5] M. A. GERAGHTY: *On contractive mappings*, Proc. Amer. Math. Soc., **40**(1973), 604–608.
- [6] E. KARAPINAR, B. SAMET: *Generalized $\alpha - \psi$ contractive type mappings and related fixed point theorems with applications*, Abstr. Appl. Anal., (2012), 17 pages.

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