ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **9** (2020), no.5, 2405–2413
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.5.4

THE TOTAL TRIANGLE FREE DETOUR NUMBER OF A GRAPH

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ABSTRACT. For a connected graph $G = (V, E)$ and u, v any two vertices in G, a $u - v$ path P is said to be a $u - v$ triangle free path if no three vertices of P induce a cycle C_3 in G. The triangle free detour distance $D_{\triangle f}(u, v)$ is the length of a longest $u - v$ triangle free path in G. A $u - v$ triangle free path of length $D_{\triangle f}(u, v)$ is called the $u - v$ triangle free detour. In this article, the concept of total triangle free detour number of a graph G is introduced. It is found that the total triangle free detour number differs from triangle free detour number and connected triangle free detour number. The total triangle free detour number is found for some standard graphs. Their bounds are determined. Certain general properties satisfied by them are studied.

1. INTRODUCTION

For a graph $G = (V, E)$, we mean a finite undirected connected simple graph. The order of G is represented by n. We consider graphs with at least two vertices. For basic definitions we refer [3]. For vertices u and v in a connected graph G, the detour distance $D(u, v)$ is the length of the longest $u - v$ path in G. A $u - v$ path of length $D(u, v)$ is called a $u - v$ detour. This concept was studied by Chartrand et.al, [1].

A vertex x is said to lie on a $u - v$ detour P if x is a vertex of $u - v$ detour path P including the vertices u and v. A set $S \subseteq V$ is called a detour set if

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2010 *Mathematics Subject Classification.* 05C12.

Key words and phrases. triangle free detour set, triangle free detour number, total triangle free detour set, total triangle free detour number.

every vertex v in G lies on a detour joining a pair of vertices of S. The detour number $d_n(G)$ of G is the minimum order of a detour set and any detour set of order $d_n(G)$ is called a minimum detour set of G. These concepts were studied by G.Chartrand et al. in [2].

A chord of a path P is an edge joining two non-adjacent vertices of P. A path P is called a monophonic path if it is a chordless path. A longest $x - y$ monophonic path is called an $x - y$ detour monophonic path. A set S of vertices of G is a detour monophonic set of G if each vertex v of G lies on an $x - y$ detour monophonic path for some x and y in S. The minimum cardinality of a detour monophonic set of G is the detour monophonic number of G and is denoted by $dm(G)$. The detour monophonic number of a graph was introduced in [7] and further studied in [6].

A total detour monophonic set of a graph G is a detour monophonic set S such that the subgraph $G[S]$ induced by S has no isolated vertices. The minimum cardinality of a total detour monophonic set of G is the total detour monophonic number of G and is denoted by $dm_t(G)$. A total detour monophonic set of cardinality $dm_t(G)$ is called a dm_t-set of G. These concepts were studied by A. P. Santhakumaran et.al in [5].

The concept of triangle free detour distance was introduced by Keerthi Asir and Athisayanathan in [4]. A path P is called a triangle free path if no three vertices of P induce a triangle. For vertices u and v in a connected graph G, the triangle free detour distance $D_{\Delta f}(u, v)$ is the length of the longest $u - v$ triangle free path in G. A $u - v$ path of length $D_{\Delta f}(u, v)$ is called a $u - v$ triangle free detour. For any two vertices u and v in a connected graph G, $0 \leq d(u, v) \leq dm(u, v) \leq D_{\Delta f}(u, v) \leq D(u, v) \leq n - 1$.

For a connected graph G, a set $S \subseteq V$ is called a triangle free detour set of G if every vertex of G lies on a triangle free detour joining a pair of vertices of S. The triangle free detour number $dn_{\Delta f}(G)$ of G is the minimum order of its triangle free detour sets and any triangle free detour set of order $dn_{\Delta f}(G)$ is called a triangle free detour basis of G. A vertex v in a graph G is a triangle free detour vertex if v belongs to every triangle free detour basis of G. If G has a unique triangle free detour basis of S, then every vertex in S is a triangle free detour vertex of G.

A set $S \subseteq V$ is a connected triangle free detour set of G if S is a triangle free detour set of G and the subgraph $\langle S \rangle$ induced by S is connected. The connected triangle free detour number $cdn_{\Delta f}(G)$ of G is the minimum cardinality of its connected triangle free detour sets and any connected triangle free detour set of order $cdn_{\Delta f}(G)$ is called a connected triangle free detour basis of G.

A vertex v of a connected graph G is called a support vertex of G if it is adjacent to an end vertex of G. Two adjacent vertices are referred to as neighbors of each other. The set $N(v)$ of neighbors of a vertex v is called the neighborhood of v. A vertex v of a graph G is called extreme vertex if the subgraph induced by its neighborhood is complete. The following theorems will be used in the sequel.

Theorem 1.1. *Every extreme vertex of a connected graph G belongs to every triangle free detour set of G. Also, if the set S of all extreme vertices of G is a triangle free detour set, then S is the unique triangle free detour basis for G.*

Theorem 1.2. *Let G be a connected graph with cut vertices and S a triangle free detour set of G. Then for any cut vertex v of G, every component of $G - v$ contains an element of S.*

Theorem 1.3. *[5] For any connected graph G, $dm_t(G) = 2$ iff $G = K_2$.*

2. The total triangle free detour number

Definition 2.1. *A total triangle free detour set of a graph G is a triangle free detour set S such that the subgraph $G[S]$ induced by S has no isolated vertices. The minimum cardinality of a total triangle free detour set of G is the total triangle free detour number of G. It is denoted by $tdn_{\Delta f}(G)$. A total triangle free detour set of cardinality $tdn_{\Delta f}(G)$ is called $tdn_{\Delta f}$- set of G.*

Example 1. *For the graph G given in Figure 1, $S = \{u_5, u_4, u_1\}$ is a minimum triangle free detour set of G. So that $dn_{\Delta f}(G) = 3$. Clearly S is minimum total triangle free detour set of G, $tdn_{\Delta f}(G) = 3$.*

Remark 2.1. *The total triangle free detour number differs from triangle free detour number and connected triangle free detour number of a graph. For the graph G*

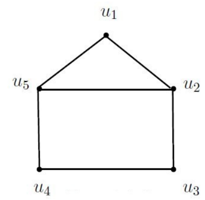

FIGURE 1. G

given in Figure 2, $S = \{u_1, u_4, u_5\}$ *is a minimum triangle free detour set of* G, $dn_{\Delta f}(G) = 3$. $S' = \{u_1, u_2, u_4, u_5\}$ *is a minimum total triangle free detour set of* G, $tdn_{\Delta f}(G) = 4$. $S'' = \{u_1, u_2, u_3, u_4, u_5\}$ *is a minimum connected triangle free detour set of* G, $cdn_{\Delta f}(G) = 5$. *Thus the triangle free detour number, the total triangle free detour number and the connected triangle free detour number are different.*

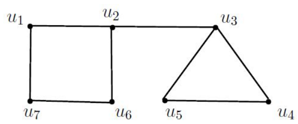

FIGURE 2. G

Note 1. *Every connected triangle free detour set is a total triangle free detour set.*

Remark 2.2. *The opposite of note 1 does not always have to be true. For the graph* G *given in Figure 2,* $S' = \{u_1, u_2, u_4, u_5\}$ *is a total triangle free detour set of* G. *Since the graph induced by* S' *is disconnected,* S' *is not a connected triangle free detour set.*

Note 2. *Every total triangle free detour set is a triangle free detour set.*

Remark 2.3. *The opposite of note 2 does not always have to be true. For the graph* G *given in Figure 2,* $S = \{u_1, u_4, u_5\}$ *is a triangle free detour set of* G, *Since the graph induced by* S *has an isolated vertex,* S *is not a total triangle free detour set.*

Theorem 2.1. *Every total triangle free detour set of a connected graph* G *contains each extreme vertex and each support vertex of* G.

Proof. Since every total triangle free detour set of G is a triangle free detour set of G, by Theorem 1.1 each extreme vertex belongs to every total triangle free detour set of G. Since the graph induced by the total triangle free detour set of G contains no isolated vertices, it follows that each support vertex of G also belongs to every total triangle free detour set of G. □

Theorem 2.2. *If the set of all extreme vertices and support vertices form a total triangle free detour set, then it is the unique minimum total triangle free detour set of G.*

Proof. Let S be the set of all extreme vertices and support vertices of G. By the assumption, S is a total triangle free detour set, then $tdn_{\Delta f}(G) \leq |S|$. By Theorem 2.1 each extreme vertices and each support vertices belong to every total triangle free detour set, then $tdn_{\Delta f}(G) \geq |S|$.

Thus $tdn_{\Delta f}(G) = |S|$ and S is the unique minimum total triangle free detour set of G. □

Corollary 2.1. *If $G = K_n$ $(n \geq 3)$, then $tdn_{\Delta f}(G) = n$.*

Remark 2.4. *The opposite of Corollary 2.1 does not always have to be true. For the graph G given in Figure 3, $tdn_{\Delta f}(G) = 4$, but it is not a complete graph.*

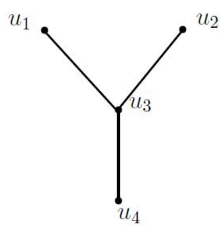

FIGURE 3

Theorem 2.3. *Let G be a connected graph with cut vertices and S be a total triangle free detour set of G. Then for any cut vetex v of G, every component of $G - v$ contains an element of S.*

Proof. Since every total triangle free detour set of G is a triangle free detour set of G. Then by Theorem 2.1, the result follows. □

Corollary 2.2. *Let G be a connected graph with cut vertex v and the number of components of $G-v$ is r, then $tdn_{\Delta f}(G) \geq r$.*

Theorem 2.4. *Let G be a connected graph of order n, then $2 \leq dn_{\Delta f}(G) \leq tdn_{\Delta f}(G) \leq cdn_{\Delta f}(G) \leq n$.*

Proof. Since the given graph is connected and $|V(G)| = n$, thus $cdn_{\Delta f}(G) \leq n$. Any triangle free detour set of G needs at least two vertices. Thus $2 \leq dn_{\Delta f}(G)$. Since every total triangle free detour set is a triangle free detour set of G, $dn_{\Delta f}(G) \leq tdn_{\Delta f}(G)$. Since every connected triangle free detour set is a total triangle free detour set of G, thus $tdn_{\Delta f}(G) \leq cdn_{\Delta f}(G)$. Hence $2 \leq dn_{\Delta f}(G) \leq tdn_{\Delta f}(G) \leq cdn_{\Delta f}(G) \leq n$.

□

Remark 2.5. *The bounds in Theorem 2.4 are sharp. For the graph G given in Figure 2, $dn_{\Delta f}(G) = 3$, $tdn_{\Delta f}(G) = 4$, $cdn_{\Delta f}(G) = 5$ and $n = 7$. so that $2 < dn_{\Delta f}(G) < tdn_{\Delta f}(G) < cdn_{\Delta f}(G) < n$. Hence all the parameters are distinct.*

Definition 2.2. *A vertex v in a graph G is a total triangle free detour vertex if v belongs to every $tdn_{\Delta f}$- set of G. If G has a unique $tdn_{\Delta f}$- set, then every vertex in $tdn_{\Delta f}$- set is a total triangle free detour vertex of G.*

Example 2. *For the graph G given in the figure 4. The sets $S = \{u_2, u_3, u_7, u_4\}$ and $S' = \{u_2, u_3, u_7, u_6\}$ are $tdn_{\Delta f}$- set of G. Thus u_2, u_3, u_7 are total triangle free detour vertices.*

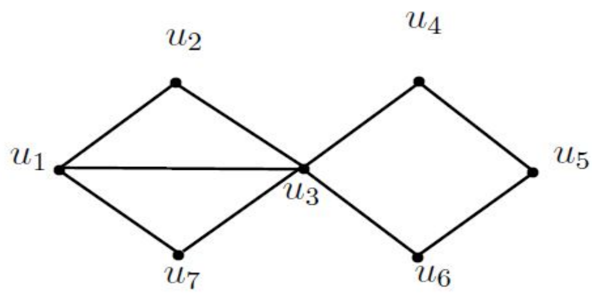

FIGURE 4. G

Corollary 2.3. *For a graph G of order n, if $tdn_{\Delta f}(G) = 2$ then $dn_{\Delta f}(G) = 2$.*

Remark 2.6. *The opposite of the Corollary 2.3 is not always true, i.e. if* $dn_{\Delta f}(G) = 2$ *then* $dn_{\Delta f}(G)$ *need not be 2. For the graph* G *given in the figure 5,* $dn_{\Delta f}(G) = 2$ *but* $tdn_{\Delta f}(G) = 3$.

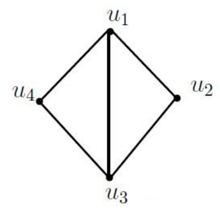

FIGURE 5. G

Corollary 2.4. *For a graph* G *of order* n*, if* $cdn_{\Delta f}(G) = 2$*, then* $tdn_{\Delta f}(G) = 2$.

Theorem 2.5. *For any non-trivial tree* T*, the set of all end vertices and support vertices of* T *is the unique minimum total triangle free detour set of* G.

Proof. Since the set of all extreme vertices and support vertices of T forms a total triangle free detour set and every end vertex is an extreme vertex. The result follows from the Theorem 2.2. □

Theorem 2.6. *If* T *is a tree with* k *end vertices, then* $k + 1 \leq tdn_{\Delta f}(G) \leq 2k$.

Proof. Since $dn_{\Delta f}(T) = k$, we get a minimum triangle free detour set $S = \{u_1, u_2, ..., u_k\}$ where each $u_i,\ i = 1, 2, ...k$ is an end vertex.

Let $S' = \{v_1, v_2, ...v_j\}$, $1 \leq j \leq k$ where each $v_j \in N(u_i)$ for $i = 1, 2, ...k$. Consider the set $S \cup S'$.

Since $S \cup S'$ is the set of all end vertices and support vertices, it is the unique minimum triangle free detour set of T.

Thus $tdn_{\Delta f}(G) = |S \cup S'|$. If support vertex of each end vertex is distinct then $tdn_{\Delta f}(G) = 2k$. If support vertex of any two end vertices is same vertex then $tdn_{\Delta f}(G) < 2k$.

Since S is not a total triangle free detour set, thus $k + 1 \leq tdn_{\Delta f}(G)$.

Hence $k + 1 \leq tdn_{\Delta f}(G) \leq 2k$. □

Theorem 2.7. *For* $G = K_2$*, then* $tdn_{\Delta f}(G) = 2$.

Remark 2.7. *The opposite of the Theorem 2.7 is not true. Consider a cycle $C_n (n > 3)$, for which $tdn_{\Delta f}(G) = 2$.*

Theorem 2.8. *Let G be a connected graph with at least 3 vertices. Then $tdn_{\Delta f}(G) \leq 2dn_{\Delta f}(G)$.*

Proof. Let $S = \{u_1, u_2, ...u_k\}$ triangle free detour basis of G. We have to show that $tdn_{\Delta f}(G) \leq 2dn_{\Delta f}(G)$. This can be proved by induction on number of isolated vertices in $G[S]$.

case:1 If G[S] have no isolated vertices then $tdn_{\Delta f}(G) = dn_{\Delta f}(G) < 2dn_{\Delta f}(G)$.
case:2 If $G[S]$ have one isolated vertex then $tdn_{\Delta f}(G) = dn_{\Delta f}(G)+1 < 2dn_{\Delta f}(G)$.
Assume that the result is true if $G[S]$ have less than k isolated vertices.
case:3 If every vertex in $G[S]$ is isolated, then there exists a set $T = \{v_1, v_2, ...v_j\}$ such that $v_i \in N(u_l),\ \forall l = 1, 2, ...k,\ 1 \leq i \leq k$.

Therefore, $S \cup T$ is the minimum total triangle free detour set of G.

Thus $tdn_{\Delta f}(G) = |S \cup T| \leq 2k = 2dn_{\Delta f}(G)$.

Hence $tdn_{\Delta f}(G) \leq 2dn_{\Delta f}(G)$. □

Theorem 2.9. *Let G be the complete bipartite graph $K_{n,m} (2 \leq n \leq m)$. Then a set S of vertices is $tdn_{\Delta f}(G)$- set of G if and only if $S = \{u, v\}$ where u and v are two adjacent vertices.*

Proof. Let G be the complete bipartite graph $K_{n,m}$. Let X and Y be bipartite sets of G with $|X| = n, |Y| = m$.

Let $S = \{u, v\}$ where u and v are two adjacent vertices.

Without loss of generality assume that $u \in X$. Since u and v are adjacent and given graph is complete bipartite, $v \in Y$.

It is clear that $D_{\Delta f}(u, v) = 2n - 1$ and every vertex $w \in G$ lies on a $u - v$ triangle free detour.

Hence S is $tdn_{\Delta f}(G)$- set of G.

Conversely, let S be $tdn_{\Delta f}(G)$- set of G. Let S' be any set consisting of two adjacent vertices of G.

Then as in the first part of the theorem, S' is a $tdn_{\Delta f}(G)$- set of G.

Hence $|S| = |S'| = 2$ and it follows that the two vertices of S are adjacent. □

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