

EFFICIENTLY STRONG AND WEAK DOMINATING DETOUR GRAPHS

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ABSTRACT. Let $G = (V, E)$ be a simple graph. A detour dominating set S of $V(G)$ is called an efficiently strong (or weak) dominating detour set of G if for every $v \in V(G)$, $|N_s[v] \cap S| = 1$ (or $|N_w[v] \cap S| = 1$), where $N_s(v) = \{u \in V(G) : uv \in E(G), \deg(u) \geq \deg(v)\}$ (or $N_w(v) = \{u \in V(G) : uv \in E(G), \deg(v) \geq \deg(u)\}$). The minimum cardinality of an efficiently strong (or weak) dominating detour set of G is called the efficiently strong (or weak) dominating detour number of G and denoted $es\gamma_d(G)$ ($ew\gamma_d(G)$). In this paper, we introduce the concepts efficiently strong and weak dominating detour number of graphs. Also, this number is found for some standard graphs and subdivision graphs.

1. INTRODUCTION

We consider finite graphs without loops and multiple edges. For any graph G , the set of vertices is denoted by $V(G)$ and the edge set by $E(G)$.

The order and size of G are denoted by p and q respectively. We consider connected graphs with at least two vertices. For basic definitions and terminologies, we refer to [1, 5]. For vertices u and v in a connected graph G , the detour distance $D(u, v)$ is the length of the longest $u - v$ path in G . A $u - v$ path of length $D(u, v)$ is called a $u - v$ detour.

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It is known that the detour distance is a metric on the vertex set $V(G)$. These concepts were studied by Chartrand et al. [2]. A vertex x is said to lie on a $u - v$ detour P if x is a vertex of a $u - v$ detour path P including the vertices u and v . A set $S \subseteq V$ is called a detour set if every vertex v in G lies on a detour joining a pair of vertices of S . The detour number $dn(G)$ is called a minimum order of a detour set and any detour set of order $dn(G)$ is called a minimum detour set of G . These concepts were studied by Chartrand [3,4]. The concept of domination in graphs was introduced by Ore in [8].

Let $G = (V, E)$ be a connected graph with at least two vertices. A set $S \subseteq V(G)$ is called a dominating set of G if every vertex in $V(G) - S$ is adjacent to some vertex in S . The domination number $\gamma(G)$ of G is the minimum order of its dominating sets and any dominating set of order $\gamma(G)$ is called γ -set of G .

A detour dominating graph were introduced and studied by J. John and N. Arianayagam in [6]. A detour dominating set is a subset S of $V(G)$ which is both a dominating and a detour set of G . A detour dominating set is said to be minimal detour dominating set of G if no proper subset of S is a detour dominating set of G . A detour dominating set S is said to be minimum detour dominating set of G if there exists no detour dominating set S' such that $|S'| < |S|$. The smallest cardinality of a detour dominating set of G is called the detour domination number of G . It is denoted by $\gamma_d(G)$. Any detour dominating set S of G of cardinality $\gamma_d(G)$ is called a (γ, d) -set of G .

For a connected graph G , let S be a γ_d -set of G . Then S is efficiently dominating detour dominating set of G if for every $v \in V(G)$, $|N[v] \cap S| = 1$. The cardinality of S is the efficiently dominating detour number of G and is denoted by $e(\gamma, d)$. An efficiently dominating detour set of cardinality $e\gamma_d(G)$ is called a $e\gamma_d$ -set of G .

A subset S of $V(G)$ is called a efficiently strong(or weak) dominating detour graph for every $v \in V(G)$, $|N_s[v] \cap S| = 1$ (or $|N_w[v] \cap S| = 1$) where $N_s(v) = \{u \in V(G)/uv \in E(G), \deg u \geq \deg v\}$ (or $N_w(v) = \{u \in V(G)/uv \in E(G), \deg v \geq \deg u\}$ and $N_s[v] = \{v\} \cup N_s(v)$ (or $N_w[v] = \{v\} \cup N_w(v)$). The minimum cardinality of S is a efficiently strong(or weak) dominating detour number of G and is denoted by $es\gamma_d(G)$ ($ew\gamma_d(G)$).

Strong(weak) efficient dominating graphs were introduced and studied by Meena, Subramanian and Swaminathan in [7]. A subdivision of an edge $e = uv$ of a graph G is the replacement of the edge e by a path $\{u, v, w\}$. If every edge of

G is subdivided exactly once, then the resulting graph is called the subdivision graph $S(G)$. The following results are given in [6].

Theorem 1.1. *For a non-trivial tree, $dn(G) = k$, where k is the number of end vertices of G .*

Theorem 1.2.

$$\gamma_d(P_n) = \begin{cases} \lceil \frac{n-4}{3} \rceil + 2 & \text{if } n \geq 5 \\ 2 & \text{if } n = 2, 3 \text{ or } 4. \end{cases}$$

Theorem 1.3. *For if $n \geq 5$, $\gamma_d(C_n) = \lceil \frac{n}{3} \rceil + 2$.*

Theorem 1.4. *For the complete graph K_n ($n \geq 2$), $\gamma_d(K_n) = 2$.*

Theorem 1.5. *For the complete bipartite graph $G = K_{m,n}$*

$$\gamma_d(G) = \begin{cases} 2 & \text{if } m = n \text{ and } m = 2, n \geq 3 \\ n - 1 & \text{if } m = 1, n \geq 2 \\ 3 & \text{if } 3 \leq m \leq n \end{cases}.$$

Theorem 1.6. *Every end vertex of G belongs to every efficiently dominating (γ, d) -graph.*

Theorem 1.7. *C_{3n} is an efficiently dominating detour graph and $e\gamma_d(C_{3n}) = n$ if $n > 1$, $n \in \mathbb{N}$.*

Theorem 1.8. *For cycle C_n , $n \equiv 1(\text{mod } 3)$ or $n \equiv 2(\text{mod } 3)$, there is no efficiently dominating detour set.*

2. EFFICIENTLY STRONG AND WEAK DOMINATING DETOUR GRAPHS

Definition 2.1. *Let $G = (V, E)$ be a simple graph. A detour dominating set S of $V(G)$ is called an efficiently strong(or weak) dominating detour set of G if for every $v \in V(G)$, $|N_s[v] \cap S| = 1$ (or $|N_w[v] \cap S| = 1$), where $N_s(v) = \{u \in V(G)/uv \in E(G), \deg u \geq \deg v\}$ (or $N_w(v) = \{u \in V(G)/uv \in E(G), \deg v \geq \deg u\}$.*

The minimum cardinality of an efficiently strong(or weak) dominating detour set of G is called the efficiently strong(or weak) dominating detour number of

G and denoted by $es\gamma_d(G)$ ($ew\gamma_d(G)$). A graph G is called an efficiently strong (or weak) dominating detour graph if G has an efficiently strong (or weak) dominating detour set.

Example 1. Here, $S = \{v_3, v_6\}$ is an efficiently strong (or weak) dominating detour set of G . Also, $\gamma_d(G) = e\gamma_d(G) = es\gamma_d(G) = ew\gamma_d(G) = 2$.

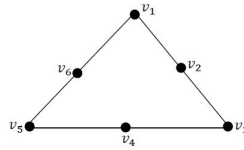


FIGURE 1

Theorem 2.1. Each end vertex of G belongs to every efficiently strong (or weak) dominating detour set.

Proof. Every efficiently strong (or weak) dominating detour set of G is a detour dominating set of G . Therefore, by Theorem 1.6, each end vertex of G belongs to every efficiently strong (or weak) dominating detour set. \square

Remark 2.1. A detour dominating graph need not be an efficiently strong (or weak) dominating detour graph. $S = \{v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ is a detour dominating set. Therefore, $\gamma_d(G) = 7$. But, S is not an efficiently strong (or weak) dominating detour set, since $|N_s[v] \cap S| \neq 1$ ($|N_w[v] \cap S| \neq 1$).

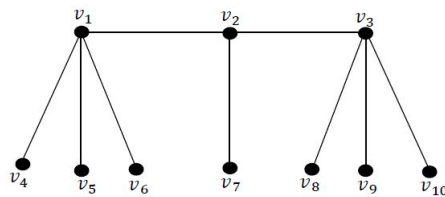


FIGURE 2

Remark 2.2. (1) Not all graphs possess efficiently strong (or weak) dominating detour set.

- (2) An efficiently strong (or weak) dominating detour graph need not be an efficiently dominating detour graph.

For example, P_6 is an efficiently weak dominating detour graph but not an efficiently dominating detour graph and P_5 is an efficiently strong dominating detour graph but not an efficiently dominating detour graph.

- (3) A regular graph which is an efficiently dominating detour graph is obviously an efficiently strong (or weak) dominating detour graph.

Further, $es\gamma_d(G) = ew\gamma_d(G)(G) = ew\gamma_d(G) = e\gamma_d(G)$.

For, $N_s[v] = N_w[v] = N[v]$ for every $v \in G$, where G is regular.

- (4) Any efficiently strong dominating detour set contains atleast one vertex of deg Δ .
- (5) Any efficiently weak dominating detour set contains atleast one vertex of degree δ .

Theorem 2.2. Every efficiently strong (or weak) dominating detour set is independent.

Proof. Let G be a graph and S be an efficiently strong (or weak) dominating detour set of G .

Let $u, v \in S$. Suppose u and v are adjacent.

Without loss of generality, assume $d(u) \geq d(v)$.

Then $|N_s[v] \cap S| \geq 2$ (or $|N_w[v] \cap S| \geq 2$), which is a contradiction.

Therefore, u and v are not adjacent. Hence, S is independent.

Hence, every efficiently strong (or weak) dominating detour set is independent. \square

Theorem 2.3. The star graph $K_{1,n}$ is not an efficiently strong (or weak) dominating detour graph for all $n \geq 2$.

Proof. Suppose $K_{1,n}$ is an efficiently strong dominating detour graph.

- (1) Let S be an efficiently strong dominating detour set of $K_{1,n}$.

Clearly, v is the unique vertex of degree Δ in $K_{1,n}$.

Then, by Theorem 2.1, $v \in S$. Further, $\{v_1, v_2, v_3, \dots, v_n\} \subseteq S$. But, $d(v, v_i) = 1$ and hence, $|N_s[v_i] \cap S| \neq 1$ for all i , which is a contradiction. Therefore, the star graph is not an efficiently strong dominating detour graph.

(2) Let S be an efficiently weak dominating detour set of $K_{1,n}$.

By Theorem 2.1, $\{v_1, v_2, v_3, \dots, v_n\} \subseteq S$.

Therefore, $|N_w[v] \cap S| \neq 1$, which is a contradiction.

Therefore, the star graph is not an efficiently weak dominating detour graph. \square

Theorem 2.4. P_n , $n \equiv 0(\text{mod } 3)$ is neither an efficiently weak dominating detour graph nor an efficiently strong dominating detour graph.

Proof. In any weak dominating detour set of P_{3n} , either v_2 or v_{3k-1} does not satisfy the efficient condition and also every strong dominating detour set fails to be efficient.

Hence, P_n , $n \equiv 0(\text{mod } 3)$ is neither an efficiently weak dominating detour graph nor an efficiently strong dominating detour graph. \square

Theorem 2.5. The path P_n , $n \equiv 1(\text{mod } 3)$ is an efficiently weak dominating detour graph but not an efficiently strong dominating detour graph.

Proof. Let $P_n = (v_1, v_2, v_3, \dots, v_{3k}, v_{3k+1})$ for some k .

Here, $S = \{v_1, v_4, v_7, \dots, v_{3k+1}\}$ is the unique minimum detour dominating set of P_n . And, is also a minimum efficiently weak dominating detour set of P_n .

Therefore, by Theorem 1.2, $ew\gamma_d(P_n) = \gamma_d(P_n) = \lceil \frac{n-4}{3} \rceil + 2$.

The possible minimum strong detour dominating sets are

$S_1 = \{v_1, v_3, v_6, \dots, v_{3k}, v_{3k+1}\}$, $S_2 = \{v_1, v_2, v_5, \dots, v_{3k-1}, v_{3k+1}\}$. But, both are not efficiently strong dominating detour sets since in each S_i either v_1 or v_{3k+1} does not satisfy the efficiently strong dominating condition.

Therefore, P_n , $n \equiv 1(\text{mod } 3)$ has no efficiently strong dominating detour set.

Hence, the path P_n , $n \equiv 1(\text{mod } 3)$ is an efficiently weak dominating detour graph but not an efficiently strong dominating detour graph. \square

Theorem 2.6. The path P_n , $n \equiv 2(\text{mod } 3)$ is an efficiently strong dominating detour graph but not an efficiently weak dominating detour graph.

Proof. Let $P_n = (v_1, v_2, v_3, \dots, v_{3k+2})$, where $n \equiv 2(\text{mod } 3)$.

(1) Then, $S = \{v_1, v_{3k+2}\} \cup \{v_3, v_6, \dots, v_{3k}\}$ is the unique minimum efficiently strong dominating detour set of P_n .

Therefore, $es\gamma_d(P_n) = 2 + k$.

- (2) P_n , $n \equiv 2(\text{mod } 3)$ has no efficiently weak dominating detour set, since in any minimum efficiently weak dominating detour set S , either v_2 or v_{3k+1} is weak dominated by two vertices.

Hence, the path P_n , $n \equiv 2(\text{mod } 3)$ is an efficiently strong dominating detour graph but not an efficiently weak dominating detour graph. \square

Theorem 2.7. C_n is an efficiently strong(or weak) dominating detour graph if and only if $n \equiv 0(\text{mod } 3)$.

Proof. Since C_n is regular, by Remark 2.2(3), efficiently strong (or weak) dominating detour graph if and only if C_n is efficiently dominating detour graph.

Therefore, the result follows from theorems 1.7 and 1.8. \square

Theorem 2.8. The Complete graph K_p , $p > 2$, is neither an efficiently strong dominating nor an efficiently weak dominating detour graph.

Proof. Let $G \simeq K_p$. Let S be an detour dominating set of K_p . Then, $|S| = 2$. Let $u, v \in S$.

$S \subseteq V(K_p)$ implies that u and v are adjacent. Further, u and v are of same degree. Therefore, $|N_s[u] \cap S| \geq 2$ and $|N_w[u] \cap S| \geq 2$. Hence, S is not an efficiently strong or weak dominating detour set of K_p . Therefore, it is not an efficiently strong or weak dominating detour graph. \square

Theorem 2.9. Complete bipartite graphs $K_{m,n}$ are not efficiently strong or weak dominating detour graphs.

Proof.

Case (i): $m = n = 1$.

Then, $K_{m,n} \simeq K_2$. Therefore, by previous theorem, $K_{m,n}$ is not an efficiently strong or weak dominating detour graph.

Case (ii): $n \geq 2$, $m = 1$.

We get a star graph. By Theorem 2.3, $K_{1,n}$ is not an efficiently strong or weak dominating detour graph.

Case (iii): $m, n \geq 2$.

Let $V = (V_1, V_2)$ be the bipartition of $K_{m,n}$.

In this case, any detour dominating set S of $K_{m,n}$ contains at least one vertex from each of V_1 and V_2 .

Hence, S is not independent.

Therefore, S is neither an efficiently strong dominating detour set nor an efficiently weak dominating detour set.

Since, S is arbitrary, $K_{m,n}$ is neither an efficiently strong dominating detour graph nor an efficiently weak dominating detour graph. \square

Theorem 2.10. *The wheel graph W_n , $n \geq 4$ is neither an efficiently strong nor an efficiently weak dominating detour graph.*

Proof. Let $V(W_n) = \{v, v_1, v_2, v_3, \dots, v_n\}$, where v is the central vertex of the wheel. Obviously $d(v) = \Delta$.

Therefore, any strong detour dominating set, S contains v . But, v is adjacent to every other vertex of the wheel.

Also, S contains at least one v_i . Therefore, there is at least one vertex in the outer cycle say v_i such that $|N_s[v_i] \cap S| \neq 1$. Therefore, no strong dominating detour set is an efficiently strong dominating detour set. Hence, W_n has no efficiently strong dominating detour set.

Similarly, one can prove W_n is not an efficiently weak dominating detour graph.

Therefore, wheel graph is neither an efficiently strong nor an efficiently weak dominating detour graph. \square

Remark 2.3. *If S is an efficiently strong dominating detour set of a connected graph G , then $V - S$ is a dominating set of G .*

Proof. Since, every efficiently strong dominating detour set is independent and G is connected, every vertex in S is adjacent to at least one vertex in $V - S$. Therefore, $V - S$ is a dominating set of G . \square

3. EFFICIENTLY STRONG (OR WEAK) DOMINATING DETOUR NUMBER OF SUBDIVISION GRAPHS

Theorem 3.1. *$S(P_n)$ is an efficiently strong dominating detour graph if $n \equiv 0(\text{mod } 3)$; an efficiently weak dominating detour graph if $n \equiv 1(\text{mod } 3)$ and is neither an efficiently strong dominating detour graph nor an efficiently weak dominating detour graph if $n \equiv 2(\text{mod } 3)$.*

Proof.

Case (i): $n \equiv 0(\text{mod } 3)$.

Here, $S(P_n)$ is isomorphic to P_m with $m \equiv 2(\text{mod } 3)$.

Therefore, by Theorem 2.5, $S(P_n)$ where $n \equiv 0(\text{mod } 3)$ is an efficiently strong dominating detour graph.

Case (ii): $n \equiv 1(\text{mod } 3)$.

Here, $S(P_n)$ is isomorphic to P_m with $m \equiv 1(\text{mod } 3)$.

Therefore, by Theorem 2.4, $S(P_n)$ where $n \equiv 1(\text{mod } 3)$ is an efficiently weak dominating detour graph.

Case (iii): $n \equiv 2(\text{mod } 3)$.

Here, $S(P_n)$ is isomorphic to P_m with $m \equiv 0(\text{mod } 3)$.

Therefore, by Theorem 2.3, $S(P_n)$ where $n \equiv 2(\text{mod } 3)$ is neither an efficiently strong dominating detour graph nor an efficiently weak dominating detour graph. \square

Theorem 3.2. $S(C_n)$ is an efficiently strong(or weak) dominating detour graph if and only if $n \equiv 0(\text{mod } 3)$.

Proof. $S(C_n)$ is isomorphic to C_{3k+1} , C_{3k+2} and C_{3k} accordingly $n \equiv 0, 1, 2(\text{mod } 3)$.

Therefore, by Theorem 2.7, $S(C_n)$ is an efficiently strong(or weak) dominating detour graph if and only if $n \equiv 0(\text{mod } 3)$. \square

Theorem 3.3. The subdivision of the star graph $S(K_{1,n})$, where $n \geq 2$ has efficiently strong and weak dominating detour sets and $es\gamma_d(S(K_{1,n})) = ew\gamma_d(S(K_{1,n})) = n + 1$.

Proof. Let $V(K_{1,n}) = \{v, v_1, v_2, v_3, \dots, v_n\}$ with v is the central vertex of the star and let $u_1, u_2, u_3, \dots, u_n$ be the vertices which subdivide the n edges of the star graph.

Then, $V(S(K_{1,n})) = \{v, v_1, v_2, v_3, \dots, v_n, u_1, u_2, u_3, \dots, u_n\}$ and $S = \{v, v_1, v_2, v_3, \dots, v_n\}$ is the minimum detour dominating set of $S(K_{1,n})$.

(1) Here, $N_s[v] = \{v\}$ and so, $N_s([v] \cap S) = 1$. Then, for each v_i , $N_s[v_i] = \{v_i, u_i\}$ and so $N_s([v_i] \cap S) = 1$ for all $i = 1, 2, \dots, n$. Also for each u_i , $N_s[u_i] = \{u_i, v\}$ and so $N_s([u_i] \cap S) = 1$ for all $i = 1, 2, \dots, n$. Therefore, every vertex v of $S(K_{1,n})$ satisfies the condition $N_s([v] \cap S) = 1$ and so S is a minimum efficiently strong dominating detour set.

Hence, $es\gamma_d(S(K_{1,n})) = n + 1$.

- (2) Let S be as defined above. Then, $N_w[v] = \{v, u_1, u_2, \dots, u_n\}$ and so, $N_w([v] \cap S) = 1$. Then, for each v_i , $N_w[v_i] = \{v_i\}$ and so $N_w[v_i] \cap S = 1$ for all $i = 1, 2, \dots, n$. Also for each u_i , $N_w[u_i] = \{u_i, v_i\}$ and so $N_w[u_i] \cap S = 1$ for all $i = 1, 2, \dots, n$. Therefore, every vertex v of $S(K_{1,n})$ satisfies the condition that $N_w([v] \cap S) = 1$. Hence, S is the minimum efficiently weak dominating detour set and $ew\gamma_d(S(K_{1,n})) = n + 1$. Therefore, the subdivision of the star graph $S(K_{1,n})$, where $n \geq 2$ is both efficiently strong and weak domination detour graph and $es\gamma_d(S(K_{1,n})) = ew\gamma_d(S(K_{1,n})) = n + 1$.

□

Theorem 3.4. *The subdivision graph $S(K_n)$, where $n \geq 3$ has no efficiently strong and weak dominating detour set.*

Proof. Let $V(K_n) = \{v_1, v_2, v_3, \dots, v_n\}$ and let $\{u_1, u_2, \dots, u_{\binom{n}{2}}\}$ be the vertices which subdivide the edges of K_n . Then, $V(S(K_n)) = \{v_1, v_2, v_3, \dots, v_n, u_1, u_2, \dots, u_{\binom{n}{2}}\}$ and $S = \{v_1, v_2, v_3, \dots, v_n\}$ is the unique minimum detour dominating set of K_n and $\gamma_d(S(K_n)) = n$.

- (1) Here, $N_s([u_i] \cap S) = 2$ for every $i = 1, 2, \dots, \binom{n}{2}$. Also, there exists no other efficiently strong dominating detour set for $S(K_n)$. Therefore, $S(K_n)$ has no efficiently strong dominating detour set.

- (2) Here, $N_w[u_i] = \{u_i\}$ and $N_w([u_i] \cap S) = 0$ for every $i = 1, 2, \dots, \binom{n}{2}$.

Also, there exists no other efficiently weak dominating detour set for $S(K_n)$. Therefore, $S(K_n)$ has no efficiently weak dominating detour set.

□

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