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$\theta G^{*}S$-REGULAR AND $\theta G^{*}S$-NORMAL SPACES IN TOPOLOGICAL SPACES

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ABSTRACT. The aim of this paper is to introduce and study the basic properties of two new classes of spaces, called $\theta g^{*}s$-regular and $\theta g^{*}s$-normal spaces.

1. INTRODUCTION

Munshi [7] introduced g-regular and g-normal spaces using g-closed sets in topological spaces. Noiri and Popa [8] have further investigated the concepts introduced by Munshi. In 1975 Maheshwari and Prasad [5, 6] introduced and studied s-regular, s-normal spaces. In 2002 semi-g-regular and semi-g-normal spaces were introduced and studied by Ganster et al. in [3]. The notion of θ-regular introduced by Kohli and Das [4] and the set $\theta g^{*}s$-closed set was introduced by Sathishmohan [9]. The aim of this paper is to introduce and investigate the notions of $\theta g^{*}s$-regularity and $\theta g^{*}s$-normality utilizing $\theta g^{*}s$-closed set.

2. PRELIMINARIES

Throughout this paper the space (X, τ) represents the topological spaces on which no separation axioms are assumed unless otherwise mentioned. Let A be

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a subset of a space (X, τ) then $cl(A)$, $int(A)$ and A^c denote the closure of A, interior of A and complement of A respectively.

Definition 2.1. *A subset A of space (X, τ) is called*

(1) *semi-closed set, if $int(cl(A)) \subseteq A$, [1];*
(2) *regular closed set, if $A = cl(int(A))$, [11].*

Definition 2.2. *[12], A point x of a space (X, τ) is called θ-adherent point of a subset A of X if $cl(U) \cap A \neq \phi$, for every open set U containing x.*
The set of all θ-adherents points of A is called the θ-closure of A and is denoted by $cl_\theta(A)$. A subset A of a space X is called θ-closed if and only if $A = cl_\theta(A)$. The complement of a θ-closed set is called θ-open.

Definition 2.3. *[2], A point x of a space (X, τ) is called semi θ-cluster point of A if $A \cap scl(U) \neq \phi$, for every semi-open set U containing x.*
The set of all semi θ-cluster points of A is called semi θ-closure of A and is denoted by $scl_\theta(A)$. Hence, a subset A is called semi θ-closed if $scl_\theta(A) = A$. The complement of a semi θ-closed set is called semi θ-open set.

Definition 2.4. *[9], A subset A of a topological space (X, τ) is called θ-generalized star semi-closed (briefly $\theta g^* s$-closed) if $scl_\theta(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open. The complement of $\theta g^* s$-closed set is called $\theta g^* s$-open.*
The family of all $\theta g^* s$-open(resp $\theta g^* s$-closed) sets of a space X is denoted by θg^*so(X) (resp. θg^*sc(X)).

The following three definitions are given in [9].

Definition 2.5. *A function $f : (X, \tau) \to (Y, \sigma)$ is called $\theta g^* s$-continuous if $f^{-1}(V)$ is $\theta g^* s$-closed set in (X, τ) for every closed set V in (Y, σ).*

Definition 2.6. *A function $f : (X, \tau) \to (Y, \sigma)$ is called $\theta g^* s$-irresolute if $f^{-1}(V)$ is $\theta g^* s$-closed set in (X, τ) for every $\theta g^* s$-closed set V in (Y, σ).*

Definition 2.7. *A space (X, τ) is called a ${}_{\theta g}T^*_{\frac{1}{2}}$-space if every $\theta g^* s$-closed set of (X, τ) is a closed set.*

Definition 2.8. *[10], A space X is said to be almost normal if for each closed set A and each regular closed set B such that $A \cap B = \phi$, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.*

3. θg^*s-Regular Spaces

In this section, we define and study the concepts of θg^*s-regular and we discuss some of its properties.

Definition 3.1. *A topological space X is said to be θg^*s-regular if for every θg^*s-closed set F and a point $x \notin F$, there exist disjoint open sets U and V such that $F \subseteq U$ and $x \in V$.*

Theorem 3.1. *Every θg^*s-regular space is regular.*

Proof. The proof follows from the definition. □

Theorem 3.2. *If X is regular space and ${}_{\theta g}T^*_{\frac{1}{2}}$-space, then X is θg^*s-regular space.*

Proof. Let X be a regular space. Let $x \in X$ and A be a θg^*s-closed set in X such that $x \notin A$. Since X is ${}_{\theta g}T^*_{\frac{1}{2}}$-space, A is a closed set in X. As X is a regular space, there exist disjoint open sets G and H such that $A \subseteq G$ and $x \in H$. Hence X is a θg^*s-regular space. □

Theorem 3.3. *Let X be a topological space. Then the following statements are equivalent.*

(i) *X is a θg^*s-regular space.*
(ii) *For each $x \in X$ and each θg^*s-open set A of X there exists an open set V containing x such that $cl_\theta(V) \subseteq A$.*

Proof. $(i) \Rightarrow (ii)$. Let A be any θg^*s-open set of X. Then there exists a θg^*s-open set G such that $x \in A \subseteq X$. Since $X - G$ is θg^*s-closed and $x \notin X - G$, by hypothesis, there exist open sets U and V such that $X - G \subseteq U$, $x \in V$ and $U \cap V = \Phi$ and so $V \subseteq X - U$. Now $cl_\theta(V) \subseteq cl_\theta(X - U)$ $= X - U$ and $X - G \subseteq U$ implies $X - U \subseteq G \subseteq A$. Therefore $cl_\theta(V) \subseteq A$.
$(ii) \Rightarrow (i)$. Let F be a θg^*s-closed set and $x \notin F$. Then $x \in X - F$ is θg^*s-open and so $X - F$ is a θg^*s-open set containing x. By hypothesis, there exists an open set V containing x such that $x \in X$ and $cl_\theta \subseteq X - F$, which implies $F \subseteq X - cl_\theta(V)$. Then $X - cl_\theta(V)$ is an open set containing F and $V \cap X - cl_\theta(V) = \phi$. Therefore X is θg^*s-regular space. □

Theorem 3.4. *A space X is a θg^*s-regular if and only if for each θg^*s-closed set F of X and each $x \in X - F$, there exist open sets U and V of X such that $x \in U$ and $F \subseteq V$ and $cl_\theta(U) \cap cl_\theta(V) = \phi$.*

Proof. Necessity: Let F be a θg^*s-closed set of X and $x \notin X$. Then there exist open sets U_0 and V such that $x \in U_0$, $F \subseteq V$ and $U_0 \cap V = \phi$. This implies $U_0 \cap cl_\theta(V)=\phi$. Since X is θg^*s-regular, there exists open sets G and H of X such that $x \in G$, $cl_\theta(V) \subseteq H$ and $G \cap H=\phi$. This implies $cl_\theta(G) \cap H=\phi$. Now, put $U = U_0 \cap G$, then U and V are open sets of X such that $x \in U$, $F \subseteq V$ and $cl_\theta(U) \cap cl_\theta(V)=\phi$.
Sufficiency: If for each θg^*s-closed set F of X and each point $x \in X - F$, there exist open sets G and H such that $x \in G$, $F \subseteq H$ and $cl_\theta(G) \cap cl_\theta(H)=\phi$, then it implies that $x \in G$, $F \subseteq H$ and $G \cap H = \phi$. Hence X is θg^*s-regular. □

Theorem 3.5. *Every subspace of θg^*s-regular space is θg^*s-regular.*

Proof. Let X be a θg^*s-regular space and Y be a subspace of X. Let $x \in Y$ and F be a θg^*s-closed set in Y such that $x \notin F$. Then there is a closed set and so θg^*s-closed set A of X with $F = Y \cap A$ and $x \notin A$. Since X is θg^*s-regular, there exist open sets G and H such that $x \in G$, $A \subseteq H$ and $G \cap H = \phi$. Note that $Y \cap G$ and $Y \cap H$ are open sets in Y. Also $x \in G$ and $x \in Y$ which implies $x \in Y \cap G$ and $A \subseteq H$. This implies $Y \cap A \subseteq Y \cap G$, $F \subseteq Y \cap H$. Also $(Y \cap G) \cap (Y \cap H) = \phi$. Hence Y is a θg^*s-regular space. □

Theorem 3.6. *Let X be a topological space. Then the following statements are equivalent.*

(i) *X is θg^*s-regular space.*
(ii) *For each point $x \in X$ and for each θg^*s-open set U of X, there exists an open set V of X such that $x \in V \subseteq cl_\theta(V) \subseteq U$.*
(iii) *For each point $x \in X$ and for each θg^*s-open set A not containing x, there exists an open set V of X such that $cl_\theta(V) \cap A = \phi$.*

Proof. $(i) \leftrightarrow (ii)$. Follows from the Theorem 3.5.
$(ii) \rightarrow (iii)$. Let $x \in X$ and A be a θg^*s-closed set such that $x \notin A$. Then A^C is a θg^*s-open set such that $x \in A^C$. By hypothesis, there exists an open set V such that $x \in cl_\theta(V) \subseteq A^C$. That is $x \in V$, $V \subseteq cl_\theta(V)$ and $cl_\theta(V) \subseteq A^C$. So $x \in V$ and $cl_\theta(V) \cap A = \phi$.
$(iii) \rightarrow (i)$. Let $x \in X$ and U be a θg^*s-open set in X such that $x \in X$. Then U^C is a θg^*s-closed set such that $x \notin U^C$. Then by hypothesis, there exists an open set V containing such that $cl_\theta(V) \cap U^C = \phi$. Therefore $x \in V$, $cl_\theta(V) \subseteq U$. So $x \in V \subseteq cl_\theta(V) \subseteq U$. □

Theorem 3.7. *If $f : X \to Y$ is bijective, θg^*s-irresolute and open from θg^*s-regular space X into a topological space Y, then Y is θg^*s-regular.*

Proof. Let $y \in Y$ and F be a θg^*s-closed set of Y with $y \notin F$. Since f is θg^*s-irresolute, $f^{-1}(F)$ is θg^*s-closed set in X. As f is bijective, let $f(x) = y$ and $x = f^{-1}(y)$ and $x \notin f^{-1}(F)$. Again since X is θg^*s-regular space, there exist open sets U and V such that $x \in U$ and $f^{-1}(F) \subseteq G$ and $U \cap V = \phi$. Since f is open and bijective, we have $y \in f(U)$, $F \subseteq f(V)$ and $f(U) \cap f(V) = f(U \cap V) = f(\phi) = \phi$. Hence Y is θg^*s-regular space. □

Theorem 3.8. *If $f : X \to Y$ is one-to-one, θg^*s-closed function from a topological space X into θg^*s-regular space Y. If X is ${}_{\theta g}T^*_{\frac{1}{2}}$-space, then X is θg^*s-regular.*

Proof. Let $x \in X$ and F be a θg^*s-closed set in X such that $x \notin F$. Since X is ${}_{\theta g}T^*_{\frac{1}{2}}$-space, F is closed in X. Then $f(F)$ is θg^*s-closed set with $f(x) \notin f(F)$ in Y as f is θg^*s-closed. Again Y is θg^*s-regular, there exist disjoint open sets U and V such that $f(x) \in U$ and $f(F) \subseteq V$. Therefore $x \in f^{-1}(U)$ and $F \subset f^{-1}(V)$. Hence X is θg^*s-regular space. □

4. θg^*s-NORMAL SPACES

In this section, we define and study the concepts of θg^*s-normal and we have discuss some of its properties.

Definition 4.1. *A topological space X is said to be θg^*s-normal if for any pair of disjoint closed sets A and B, there exists disjoint θg^*s-open sets U and V such that $A \subset U$, $B \subset V$.*

Theorem 4.1. *For a space X the follwing are equivalent:*

(i) *X is θg^*s-normal,*
(ii) *for every pair of open sets U and V whose union is X, there exist θg^*s-closed sets A and B such that $A \subset U$, $B \subset V$ and $A \cup B = X$,*
(iii) *for every closed set F and every open set D containing F, there exists a θg^*s-open set U such that $C \subset U \subset \theta g^*scl(U) \subset D$.*

Proof. (i) ⇒ (ii): Let U and V be a pair of open sets in a θg^*s-normal space X such that $X = U \cup V$. Then $X \backslash U$, $X \backslash V$ are disjoint closed sets. Since X is θg^*s-normal, there exist disjoint θg^*s-open sets U_1 and V_1 such that $X \backslash U \subset U_1$

and $X\backslash V \subset V_1$. Let $A = X\backslash U_1$, $B = X\backslash V_1$. Then A and B are θg^*s-closed sets such that $A \subset U$, $B \subset V$ and $A \cup B = X$.

(ii) $\Rightarrow$ (iii): Let C be a closed set and D be an open set containing C. Then $X\backslash C$ and D are open sets whose union is X. Then by (ii), there exist θg^*s-closed sets M_1 and M_2 such that $M_1 \subset X\backslash C$ and $M_2 \subset D$ and $M_1 \cup M_2 = X$. Then $C \subset X\backslash M_1$, $X\backslash D \subset X\backslash M_2$ and $(X\backslash M_1) \cap (X\backslash M_2) = \phi$. Let $U = X\backslash M_1$ and $V = X\backslash M_2$. Then U and V are disjoint θg^*s-open sets such that $C \subset U \subset X\backslash V \subset D$. As $X\backslash V$ is θg^*s-closed set, we have $\theta g^*scl(U) \subset X\backslash V$ and $C \subset U \subset \theta g^*scl(U) \subset D$.

(iii) $\Rightarrow$ (i): Let C_1 and C_2 be any two disjoint closed sets of X. Put $D = X\backslash C_2$, then $C_2 \cap D = \phi$. $C_1 \subset D$ where D is an open set. Then by (iii), there exists a θg^*s-open set U of X such that $C_1 \subset U \subset \theta g^*scl(U) \subset D$. It follows that $C_2 \subset X\backslash \theta g^*scl(U) = V$, then V is θg^*s-open and $U \cap V = \phi$. Hence, C_1 and C_2 are separated by θg^*s-open sets U and V. Therefore X is θg^*s-normal. □

Definition 4.2. *A function $f : X \to Y$ is called strongly θg^*s-closed if $f(U) \in \theta g^*sc(Y)$ for each $U \in \theta g^*sc(X)$.*

Definition 4.3. *A function $f : X \to Y$ is called strongly θg^*s-open if $f(U) \in \theta g^*so(Y)$ for each $U \in \theta g^*so(X)$.*

Theorem 4.2. *A function $f : X \to Y$ is strongly θg^*s-closed if and only if for each subset B in Y and for each θg^*s-open set U in X containing $f^{-1}(B)$, there exists a θg^*s-open set V containing B such that $f^{-1}(V) \subset U$.*

Proof. $(\Rightarrow)$: Suppose that f is strongly θg^*s-closed. Let B be a subset of Y and $U \in \theta g^*so(X)$ containing $f^{-1}(B)$. Put $V = Y\backslash f(X\backslash U)$, then V is a θg^*s-open set of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.

$(\Leftarrow)$: Let K be any θg^*s-closed set of X. Then $f^{-1}(Y\backslash f(K)) \subset X\backslash K$ and $X\backslash K \in \theta g^*so(X)$. There exists a θg^*s-open set V of Y such that $Y\backslash f(K) \subset V$ and $f^{-1}(V) \subset X\backslash K$. Therefore, we have $f(K) \supset Y\backslash V$ and $K \subset f^{-1}(Y\backslash V)$. Hence, we obtain $f(K) = Y\backslash V$ and $f(K)$ is θg^*s-closed in Y. This shows that f is strongly θg^*s-closed. □

Theorem 4.3. *If $f : X \to Y$ is a strongly θg^*s-closed continuous function from a θg^*s-normal space X onto a space Y , then Y is θg^*s-normal.*

Proof. Let K_1 and K_2 be disjoint closed sets in Y. Then $f^{-1}(K_1)$ and $f^{-1}(K_2)$ are closed sets. Since X is θg^*s-normal, then there exist disjoint θg^*s-open sets U and

V such that $f^{-1}(K_1) \subset U$ and $f^{-1}(K_2) \subset V$. By Theorem 4.2 there exist θg^*s-open sets A and B such that $K_1 \subset A$, $K_2 \subset B$, $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Also, A and B are disjoint. Thus, Y is θg^*s-normal. □

Definition 4.4. *A function $f : X \to Y$ is said to be almost θg^*s-irresolute if for each x in X and each θg^*s-neighborhood V of f(x), θg^*s-cl$(f^{-1}(V))$ is a θg^*s-neighborhood of x.*

Lemma 4.1. *Let $f : X \to Y$ be a function. Then f is almost θg^*s-irresolute if and only if f^{-1}(V) $\subset \theta g^*$s–int($\theta g^*scl(f^{-1}(V))$) for every $V \in \theta g^*so(Y)$.*

Theorem 4.4. *A function $f : X \to Y$ is almost θg^*s-irresolute if and only if $f(\theta g^*scl(U)) \subset \theta g^*scl(f(U))$ for every $U \in \theta g^*so(X)$.*

Proof. $(\Rightarrow)$: Let $U \in \theta g^*so(X)$. Suppose $y \notin \theta g^*scl(f(U))$. Then there exists $V \in \theta g^*so(Y, y)$ such that $V \cap f(U) = \phi$. Hence $f^{-1}(V) \cap U = \phi$. Since $U \in \theta g^*so(X)$, we have θg^*s-int($\theta g^*scl(f^{-1}(V))) \cap \theta g^*scl(U) = \phi$. Then by lemma 4.1, $f^{-1}(V) \cap \theta g^*scl(U) = \phi$ and hence $V \cap f(\theta g^*scl(U)) = \phi$. This implies that $y \notin f(\theta g^*scl(U))$.

$(\Leftarrow)$: If $V \in \theta g^*so(Y)$, then $M = X\backslash\theta g^*scl(f^{-1}(V)) \in \theta g^*so(X)$. By hypothesis, $f(\theta g^*scl(M)) \subset \theta g^*scl(f(M))$ and hence

$$\begin{aligned} X\backslash\theta g^*sint(\theta g^*scl(f^{-1}(V))) &= \theta g^*scl(M) \\ &\subset f^{-1}(\theta g^*scl(f(M))) \subset f^{-1}(\theta g^*scl(f(X\backslash f^{-1}(V)))) \\ &\subset f^{-1}(\theta g^*scl(Y\backslash V)) = f^{-1}(Y\backslash V) = X\backslash f^{-1}(V). \end{aligned}$$

Therefore, $f^{-1}(V) \subset \theta g^*sint(\theta g^*scl(f^{-1}(V)))$. By Lemma 4.1, f is almost θg^*s-irresolute. □

Theorem 4.5. *If $f : X \to Y$ is a strongly θg^*s-open continuous almost θg^*s-irresolute function from a θg^*s-normal space X onto a space Y , then Y is θg^*s-normal.*

Proof. Let A be a closed subset of Y and B be an open set containing A. Then by continuity of f, $f^{-1}(A)$ is closed and $f^{-1}(B)$ is an open set of X such that $f^{-1}(A) \subset f^{-1}(B)$. As X is θg^*s-normal, there exists a θg^*s-open set U in X such that $f^{-1}(A \subset U \subset \theta g^*scl(U) \subset f^{-1}(B)$ by Theorem 4.1 Then, $f(f^{-1}(A)) \subset f(U) \subset f(\theta g^*scl(U)) \subset f(f^{-1}(B))$. Since f is strongly θg^*s-open almost θg^*s-irresolute surjection, we obtain $A \subset f(U) \subset \theta g^*scl(f(U)) \subset B$. Then again by Theorem 4.1 the space Y is θg^*s-normal. □

Theorem 4.6. *If X is Normal space and ${}_{\theta g}T^*_{\frac{1}{2}}$-space, then X is θg^*s-Normal space.*

Proof. Let A and B be two disjoint θg^*s-closed sets in X. Since X is ${}_{\theta g}T^*_{\frac{1}{2}}$-space, A and B are disjoint closed sets in X. Again X is Normal space, there exists a pair of disjoint open sets G and H such that $A \subseteq G$ and $B \subseteq H$. Hence X is θg^*s-Normal space. □

Theorem 4.7. *If $f : X \to Y$ is bijective, open, θg^*s-irresolute function from θg^*s-Normal space X into a topological space Y, then Y is θg^*s-normal.*

Proof. Let A and B disjoint θg^*s-closed sets in Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint θg^*s-closed sets in X as f is θg^*s-irresolute. Since X is θg^*s-Normal, there exist disjoint open sets G and H in X such that $f^{-1}(A) \subseteq G$ and $f^{-1}(B) \subseteq H$. Again since f is bijective and open, $f(G)$ and $f(H)$ are disjoint open sets in Y such that $A \subseteq f(G)$ and $B \subseteq f(H)$. Hence Y is θg^*s-Normal. □

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