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$\pi G^*\beta$-CONTINUOUS FUNCTIONS

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ABSTRACT. The purpose of this paper is to introduce a new class of functions, namely, π generalized star β continuous functions and π generalized star β irresolute functions. Also, some of the characterization and basic properties of π generalized star β continuous functions are brought out.

1. INTRODUCTION

Levine [7] introduced the notion of generalized closed sets in 1970. Zaitsev [11] defined the concept of π-closed sets in topological spaces. Abd El-Monsef [1] introduced the notions of β-continuous functions in topological spaces. Dontchev [5] defined the concept of πg-closed sets in topological space and Tahiliani [8] study the concept of $\pi g\beta$-closed sets in topological spaces. Veerakumar [10] introduced g*-closed sets. Devika and Vani has introduced the $\pi g^*\beta$-closed sets and studied some of its relations with the existing sets. In this paper, we define and study $\pi g^*\beta$-continuous functions and prove some theorems which satisfies the definition. The basic definitions are recalled from the papers [1–11].

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2. $\pi g^*\beta$-CONTINUOUS FUNCTIONS

In this section, the notion of π generalized star β continuous functions and π generalized star β irresolute functions are studied and the pasting lemma for $\pi g^*\beta$ - closed maps is also proved.

Definition 2.1. *A function f from a topological space (X,τ) into a topological space (Y,σ) is called π generalized star β continuous ($\pi g^*\beta$- continuous) if $f^{-1}(V)$ is a $\pi g^*\beta$- closed set in (X,τ) for every closed set V in (Y,σ).*

Remark 2.1. *If $\tau_{\pi g^*\beta} = \tau$ in X, then continuity and $\pi g^*\beta$- continuity coincide.*

Theorem 2.1. *If a function $f : (X,\tau) \to (Y,\sigma)$ is r-continuous, then it is $\pi g^*\beta$-continuous.*

Proof. Let $f : X \to Y$ be r-continuous. Let F be any closed set in Y. Then the inverse image $f^{-1}(F)$ is r-closed set in X. Since for every r-closed set is $\pi g^*\beta$ - closed, $f^{-1}(F)$ is $\pi g^*\beta$ - closed set in X. Therefore f is $\pi g^*\beta$- continuous. □

Remark 2.2. *The converse of the above theorem need not be true as seen from the following example.*

Example 1. *Let $X = Y =$ {a,b,c} with $\tau =$ {∅, {c}, {b,c}, {a,c}, X} and $\sigma =$ {∅, {b}, {a,b}, {a,c}, Y}. Let $f : X \to Y$ be the identity function, then f is $\pi g^*\beta$ - continuous but not continuous and r-continuous. Since for the closed set {a,c} in Y, $f^{-1}(\{a,c\})$ = {a,c}, it is $\pi g^*\beta$-closed but not r-closed set in (X,τ).*

Theorem 2.2. *For the function $f : (X,\tau) \to (Y,\sigma)$, the following hold:*

(i) *If f is $\pi g^*\beta$-continuous function, then f is g-continuous.*
(ii) *If f is $\pi g^*\beta$-continuous function, then f is rg-continuous.*
(iii) *If f is $\pi g^*\beta$-continuous function, then f is gs-continuous.*
(iv) *If f is $\pi g^*\beta$-continuous function, then f is gp-continuous.*
(v) *If f is $\pi g^*\beta$-continuous function, then f is gsp-continuous.*
(vi) *If f is $\pi g^*\beta$-continuous function, then f is gpr-continuous.*

Proof. (i) Let U be closed set in Y. Since f is a $\pi g^*\beta$- continuous function, $f^{-1}(U)$ is $\pi g^*\beta$- closed in X. Since every $\pi g^*\beta$-closed set is g-closed, $f^{-1}(U)$ is g-closed in X. Hence f is g-continuous.

(ii) The proof for (ii) to (vi) is similar as (i).

□

Remark 2.3. *The converse of the above theorem need not be true as seen from the following examples.*

Example 2. *Let $X = Y =$ {a,b,c} with $\tau =$ {$\emptyset$, {c}, X} and $\sigma =$ {$\emptyset$, {b}, Y}. Let $f : X \to Y$ be the identity function, then f is g- continuous but not $\pi g^*\beta$-continuous. Since for the closed set {a,c} in Y, $f^{-1}(\{a,c\}) =$ {a,c} is g-closed set but not $\pi g^*\beta$-closed set in (X,τ).*

Example 3. *Let $X = Y =$ {a,b,c} with $\tau =$ {$\emptyset$, {c}, X} and $\sigma =$ {$\emptyset$, {a}, Y}. Let $f : X \to Y$ be the identity function, then f is gs-continuous, rg-continuous, gp-continuous, gsp-continuous, gpr-continuous but not $\pi g^*\beta$ - continuous. Since for the closed set {b,c} in Y, $f^{-1}(\{b,c\}) = \{b,c\}$ is gs-closed, rg- closed, gp-closed, gsp-closed, gpr-closed but not $\pi g^*\beta$-closed in (X,τ).*

Theorem 2.3. *Let $f : (X,\tau) \to (Y,\sigma)$ be the function from a topological space X into a topological space Y. If $f : X \to Y$ is $\pi g^*\beta$-continuous, then $f(\pi g^*\beta cl(A)) \subseteq$ cl(f(A)) for every subset A of X.*

Proof. Since $f(A) \subseteq cl(f(A))$, $A \subset f^{-1}(cl(f(A)))$. Since $cl(f(A))$ is a closed set in Y and f is $\pi g^*\beta$-continuous, $f^{-1}(cl(f(A)))$ is a $\pi g^*\beta$-closed set in X containing A. Hence $\pi g^*\beta cl(A) \subseteq f^{-1}(cl(f(A)))$. Therefore $f(\pi g^*\beta cl(A)) \subseteq cl(f(A))$. □

Lemma 2.1. *For any $x \in X$, $x \in \pi g^*\beta$-cl(A) if and only if $V \cap A \neq \emptyset$, for every $\pi g^*\beta$-open set V containing X.*

Theorem 2.4. *Let A be a subset of a topological space X. Then $x \in \pi g^*\beta cl(A)$ if and only if for any $\pi g^*\beta$-open set U containing x, $A \cap U \neq \emptyset$.*

Proof. Let $x \in \pi g^*\beta cl(A)$ and suppose that, there is a $\pi g^*\beta$-open set U in X such that $x \in U$ and $A \cap U = \emptyset$. Then $A \subset U^c$ which is $\pi g^*\beta$-closed in X. Hence $\pi g^*\beta cl(A) \subseteq \pi g^*\beta cl(U^c) = U^c$. Since $x \in U$, $x \notin U^c$, therefore $x \notin \pi g^*\beta cl(A)$, which is a contradiction.

Conversely, suppose that, for any $\pi g^*\beta$-open set U containing x, $A \cap U \notin \emptyset$. To prove that $x \in \pi g^*\beta cl(A)$. Suppose that $x \notin \pi g^*\beta cl(A)$, then there is a $\pi g^*\beta$-closed set F in X such that $x \notin F$ and $A \subseteq F$. Since $x \notin F$, it implies that $x \in F^c$ which is $\pi g^*\beta$-open in X. Since $A \subseteq F$, it implies that $A \cap F^c = \emptyset$, which is a contradiction. Thus $x \in \pi g^*\beta cl(A)$. □

Theorem 2.5. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function from a topological space X into a topological space Y. Then the following statements are equivalent:*

(i) *For each point x in X and each open set V in Y with $f(x) \in V$, there is a $\pi g^*\beta$-open set U in X such that $x \in U$ and $f(U) \subseteq V$.*
(ii) *For each subset A of X, $f(\pi g^*\beta cl(A)) \subseteq cl(f(A))$.*
(iii) *For each subset B of Y, $\pi g^*\beta cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that (i) holds and let $y \in f(\pi g^*\beta$-cl(A)) and let V be any neighbourhood of Y. Since $y \in f(\pi g^*\beta - cl(A))$, there exists $x \in \pi g^*\beta$-cl(A) such that $f(x) = Y$. Since $f(x) \in V$, by (i) there exists a $\pi g^*\beta$-open set U in X such that $x \in U$ and $f(U) \subseteq V$. Since $x \in f(\pi g^*\beta$-cl(A)), by Theorem 2.4, $U \cap A \neq \emptyset$. Therefore we have $Y = f(x) \in cl(f((A))$. Hence $f(\pi g^*\beta - cl(A)) \subseteq cl(f(A))$.

(ii)$\Rightarrow$(i) Suppose that (ii) holds and let $x \in X$ and V be any open set in Y containing $f(x)$. Let $A = f^{-1}(V^c)$, this implies that $x \notin$ A. Since $f(\pi g^*\beta$-cl(A)) $\subseteq cl(f(A)) \subseteq V^c$, $\pi g^*\beta$-cl(A) $\subseteq f^{-1}(V^c) = A$. Since $x \notin$ A, $x \notin \pi g^*\beta$-cl(A) and by Theorem 2.4 there exists a $\pi g^*\beta$-open set U containing x such that $U \cap A = \emptyset$, that is, $U \subseteq A^c$ and hence $f(U) \subseteq f(A^c) \subseteq V$.

(ii) $\Rightarrow$ (iii) Suppose that (ii) holds and Let B be any subset of Y. Replacing A by $f^{-1}(B)$ in (ii), we get $f(\pi g^*\beta - cl(f^{-1}(B))) \subseteq cl(f(f^{-1}(B))) \subseteq cl(B)$. Hence $\pi g^*\beta - cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

(iii) $\Rightarrow$ (ii) Suppose that (iii) holds, let $B = f(A)$ where A is a subset of X. Then we get from (iii) $\pi g^*\beta cl(A) \subseteq \pi g^*\beta - cl(f^{-1}(f(A)) \subseteq f^{-1}(cl(f(A)))$. Therefore $f(\pi g^*\beta - cl(A)) \subseteq cl(f(A))$. □

Theorem 2.6. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following statements are equivalent:*

(i) *f is $\pi g^*\beta$-continuous.*
(ii) *The inverse image of each open set in Y is $\pi g^*\beta$-open in X.*

Proof. (i)$\Rightarrow$(ii) Assume that $f : X \to Y$ is $\pi g^*\beta$-continuous. Let G be an open set in Y. Then G^c is closed in Y. Since f is $\pi g^*\beta$-continuous, $f^{-1}(G^c)$ is $\pi g^*\beta$-closed in X. But $f^{-1}(G^c) = X - f^{-1}(G)$. Thus $f^{-1}(G)$ is $\pi g^*\beta$-open in X.

(ii)$\Rightarrow$(i) Assume that the inverse image of each open set in Y is $\pi g^*\beta$ - open in X. Let F be any closed set in Y. Then F^c is open in X. But $f^{-1}(F^c) = X - f^{-1}(F)$ is $\pi g^*\beta$ - open in X and so $f^{-1}(F)$ is $\pi g^*\beta$-closed in X. Therefore f is $\pi g^*\beta$ - continuous. □

Theorem 2.7. *If a function $f : (X, \tau) \to (Y, \sigma)$ is $\pi g^*\beta$-continuous, then $f(\pi g^*\beta$-cl(A)) $\subseteq cl(f(A))$ for every subset A of X.*

Proof. Let $f : X \to Y$ be $\pi g^*\beta$-continuous. Let $A \subseteq X$. Then $cl(f(A))$ is closed in Y. Since f is $\pi g^*\beta$-continuous, $f^{-1}(cl(f(A)))$ is $\pi g^*\beta$-closed in X and $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(cl(f(A)))$ implies $\pi g^*\beta - cl(A) \subseteq f^{-1}(cl(f(A)))$. Hence $f(\pi g^*\beta - cl(A)) \subseteq cl(f(A))$. □

Theorem 2.8. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Let (X, τ) and (Y, σ) be any two spaces such that $\tau_{\pi g^*\beta}$ is a topology on X. Then the following statements are equivalent:*

(i) *For every subset A of X, $f(\pi g^*\beta$-cl(A)) $\subseteq cl(f(A))$ holds.*
(ii) *$f : (X, \tau_{\pi g^*\beta}) \to (Y, \sigma)$ is continuous.*

Proof. Suppose (i) holds. Let A be closed set in Y. By hypothesis $f(\pi g^*\beta$-cl($f^{-1}(A))) \subseteq cl(f(f^{-1}(A))) \subseteq cl(A) = A$, i.e., $\pi g^*\beta - cl(f^{-1}(A)) \subseteq f^{-1}(A)$. Also $f^{-1}(A) \subseteq \pi g^*\beta - cl(f^{-1}(A))$. Hence, $\pi g^*\beta - cl(f^{-1}(A)) = f^{-1}(A)$. This implies $(f^{-1}(A))^c \in \tau_{\pi g^*\beta}$. Thus $f^{-1}(A)$ is closed in (X, $\tau_{\pi g^*\beta}$) and so f is continuous. This proves (ii).

Suppose (ii) holds. For every subset A of X, cl(f(A)) is closed in Y. Since $f : (X, \tau_{\pi g^*\beta}) \to (Y, \sigma)$ is continuous, $f^{-1}(cl(A))$ is closed in $(X, \tau_{\pi g^*\beta})$, that implies Theorem 2.3 $\pi g^*\beta$- cl $(f^{-1}(cl(f(A)))) = f^{-1}(cl(f(A)))$. Now we have, $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(cl(f(A)))$ and by Lemma 2.1, $\pi g^*\beta cl(A) \subseteq \pi g^*\beta - cl(f^{-1}(cl(f(A))) = f^{-1}(cl(f(A))$. Therefore $f(\pi g^*\beta - cl(A)) \subseteq cl(f(A))$. □

Theorem 2.9. *Let $f : (X, \tau) \to (Y, \sigma)$ is a $\pi g^*\beta$ - continuous function and $g : (Y, \sigma) \to (Z, \eta)$ is a continuous function. Then $g \circ f : (X, \tau) \to (Z, \eta)$ is $\pi g^*\beta$ - continuous.*

Proof. Let g be a continuous function and V be any open set in Z, then $f^{-1}(V)$ is open in Y. Since f is $\pi g^*\beta$ - continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $\pi g^*\beta$ - open in X. Hence $g \circ f$ is $\pi g^*\beta$ - continuous. □

Now we generalize the pasting lemma for $\pi g^*\beta$ - continuous maps.

Theorem 2.10. *Let $X = A \cup B$ be a topological space with topology τ and Y be a topological space with topology σ. Let $f : (A, \tau/A) \to (Y, \sigma)$ and $g : (B, \tau/B) \to (Y, \sigma)$ be $\pi g^*\beta$ - continuous maps such that f(x) = g(x) for every $x \in A \cap B$.*

Suppose that A and B are g-open and $\pi g^*\beta$ - closed sets in X. Then the combination $\alpha : (X, \tau) \to (Y, \sigma)$ is $\pi g^*\beta$ - continuous.

Proof. Let F be any closed set in Y. Clearly $\alpha^{-1}(F) = f^{-1}(F) \cup g^{-1}(F) = C \cup D$ where $C = f^{-1}(F)$ and $D = g^{-1}(F)$. But C is $\pi g^*\beta$ - closed in A and A is $\pi g^*\beta$ - closed in X and so C is $\pi g^*\beta$ - closed in X. Since we have proved that if $B \subseteq A \subseteq X$, B is $\pi g^*\beta$ - closed in A and A is $\pi g^*\beta$ - closed in X then B is $\pi g^*\beta$ - closed in X. Also $C \cup D$ is $\pi g^*\beta$ - closed in X. Therefore $\alpha^{-1}(F)$ is $\pi g^*\beta$ - closed in X. Hence α is $\pi g^*\beta$-continuous. □

3. $\pi g^*\beta$-IRRESOLUTE FUNCTIONS

Definition 3.1. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $\pi g^*\beta$ - irresolute, if $f^{-1}(V)$ is $\pi g^*\beta$ - open set in (X, τ) for every $\pi g^*\beta$ - open set Vin (Y, σ).*

Theorem 3.1. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be any two functions. Let h =g∘$f : (X, \tau) \to (Z, \eta)$. Then:*

(i) *h is $\pi g^*\beta$ - continuous if f is $\pi g^*\beta$ - irresolute and $g$is $\pi g^*\beta$ - continuous.*
(ii) *h is $\pi g^*\beta$ - continuous if g is continuous and f is $\pi g^*\beta$ - continuous.*

Proof. Let V be a closed set in Z.

(i) Suppose f is $\pi g^*\beta$ - irresolute and g is $\pi g^*\beta$ - continuous, $g^{-1}(V)$ is $\pi g^*\beta$ - closed in Y. Since f is $\pi g^*\beta$ - irresolute, the Definition 3.1 implies that $f^{-1}(g^{-1}(V))$ is $\pi g^*\beta$ - closed in X. This proves (i).
(ii) Let g be continuous and f be $\pi g^*\beta$ - continuous. Then $g^{-1}(V)$ is closed in Y. Since f is $\pi g^*\beta$ - continuous, using the Definition 3.1 $f^{-1}(g^{-1}(V))$ is $\pi g^*\beta$ - closed in X. This proves (ii).

□

Theorem 3.2. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ are $\pi g^*\beta$ - irresolute functions, then $g \circ f : (X, \tau)(Z, \eta)$ is $\pi g^*\beta$-irresolute.*

Proof. Let g be a $\pi g^*\beta$ - irresolute function and V be any $\pi g^*\beta$ - open set in Z, then $g^{-1}(V)$ is $\pi g^*\beta$ - open set in Y. Since f is $\pi g^*\beta$ - irresolute, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $\pi g^*\beta$ - open in X. Hence $g \circ f$ is $\pi g^*\beta$ - irresolute. □

Theorem 3.3. *Let X, Y and Z be any topological spaces. For any πg^*s-irresolute map $f : X \longrightarrow Y$ and any πg^*s-continuous map $g : Y \longrightarrow Z$, the composition $gof : X \longrightarrow Y$ is πg^*s-continuous.*

Proof. It follows from the definitions. □

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