Advances in Mathematics: Scientific Journal **9** (2020), no.5, 2931–2944
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.5.55

ON NANO SEMI PRE-IRRESOLUTE FUNCTIONS IN NANO TOPOLOGICAL SPACES

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ABSTRACT. The basic objective of this paper is to introduce and investigate the properties of nano β-irresolute in nano topological spaces and studied some of its properties.

1. INTRODUCTION

Lellis Thivagar and Richard in [1] established the notion of nano topology in terms of approximations and boundary region of a subset of an universe using an equivalence relation on it and also make known about nano-closed sets, nano-interior, nano-closure and weak form of nano open sets namely nano semi-open sets, nano pre-open, nano α-open sets and nano β-open sets. Nasef et al. [2] make known about some of nearly open sets in nano topological spaces. Revathy and Gnanambal Illango in [4] gave the idea about the nano β-open sets. Sathishmohan et al. in [7], and in [6] bring up the idea about nano neighourhoods in nano topological spaces. This motivates the author to study the characterizations and properties of different forms of nano β-irresolute functions in nano topological spaces.
The structure of this manuscript is as follows:

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2010 *Mathematics Subject Classification.* 54B05.
Key words and phrases. nano β-continuos, nano β-regular, nano β-θ-open sets and nano β-irresolute functions.

In section 2, we recall some existing definitions and remarks which are more important to prove our main results.

In section 3, we induct and study some theorems which satisfies the conditions of nano β-regular and nano β-θ-open sets.

In section 4, we brings up the concept of nano β-irresolute functions and studied their characterizations and properties of weakly nano β-irresolute functions and also proved some of the theorems which satisfies some existing properties.

In section 5, we make a known about the concept of strongly nano β-irresolute functions and also proved some of the theorems which satisfies some existing properties. Throughout this paper nano semi pre open sets (resp. nano semi pre closed sets) is denoted by $N\beta O(U)$ (resp. $N\beta F(U)$).

2. Preliminaries

Definition 2.1. *[3], Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as indiscernibility relation. Then U is divided into disjoint equivalence classes. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$. Then,*

(i) *The lower approximation of x with respect to R is the set of all objects, which can be for certain classified as X with respect to R and is denoted by $L_R(X)$. $L_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \subseteq X\}$ where $R(x)$ denotes the equivalence class determined by $x \in U$.*

(ii) *The upper approximation of x with respect to R is the set of all objects which can be possibly classified as X with respect to R and is denoted by $U_R(X)$. $U_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \cap X \neq \phi\}$*

(iii) *The boundary region of x with respect to R is the set of all objects which can be classified neither as X nor as not-X with respect to R and it is denoted by $B_R(X)$. $B_R(X) = U_R(X)$ - $L_R(X)$.*

Definition 2.2. *[3], Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then $\tau_R(X)$ satisfies the following axioms:*

(i) *U and $\phi \in \tau_R(X)$.*

(ii) *The union of the elements of any sub-collection of $\tau_R(X)$ is in $\tau_R(X)$.*

(iii) *The intersection of the elements of any finite sub collection of $\tau_R(X)$ is in $\tau_R(X)$.*

Then $\tau_R(X)$ is a topology on U called the nano topology on U with respect to X. We call $(U, \tau_R(X))$ as nano topological space. The elements of $\tau_R(X)$ are called as nano-open sets. The complement of the nano-open sets are called nano-closed sets.

Remark 2.1. *[3], If $\tau_R(X)$ is the nano topology on U with respect to X , then the set $B = \{U, L_R(X), B_R(X)\}$ is the basis for $\tau_R(X)$.*

Definition 2.3. *[1], If $(U, \tau_R(X))$ is a nano topological space with respect to X where $X \subseteq U$ and if $A \subseteq U$, then:*

(i) *The nano interior of A is defined as the union of all nano-open subsets of A is contained in A and is denoted by $Nint(A)$. That is, $Nint(A)$ is the largest nano-open subset of A.*

(ii) *The nano closure of A is defined as the intersection of all nano-closed sets containing A and is denoted by $Ncl(A)$. That is, $Ncl(A)$ is the smallest nano-closed set containing A.*

Definition 2.4. *[2], A function $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is called nano semi pre-continuous if $f^{-1}(B)$ is nano β-open in U for every nano-open set B in V and it is denoted by $N\beta$-continuous.*

Definition 2.5. *[8], A space U is said to be nano semi pre-regular if for each $N\beta$-closed set F and for each $x \in U - F$ there exists nano-open sets G and H such that $F \subset G$ and $x \in H$ and it is denoted by $N\beta$-regular space.*

Definition 2.6. *[5], A point $x \in U$ is called a nano semi pre-θ cluster (briefly, $N\beta$-θ-cluster) point of A if $N\beta cl(G) \cap A \neq \phi$ for every $G \in N\beta O(U, x)$. The set of all nano β-θ-cluster points of A is called nano semi pre-θ-closure of A and it is denoted by $N\beta$-$\theta cl(A)$. If $N\beta$-$\theta cl(A) = A$, then A is said to be nano semi pre-θ-closed (briefly, $N\beta$-θ-closed). The complement of a nano semi pre-θ-closed set is said to be nano semi pre-θ-open (briefly, $N\beta$-θ-open).*

Definition 2.7. *[5], A point $x \in U$ is called a nano β-θ-interior (briefly, $N\beta$-θ-interior) point of A if there exists $N\beta$-open set G containing x such that $G \subset N\beta cl(G) \subset A$. The set of all $N\beta$-θ-interior points of A is called $N\beta$-θ-interior of A and is denoted by $N\beta$-$\theta int(A)$. Thus $N\beta$-$\theta int(A) = \{x \in U / x \in G \subset N\beta cl(G) \subset A, \forall\, G \in N\beta O(U, x)\}$.*

3. $N\beta$-REGULAR SETS AND $N\beta$-θ-OPEN SETS

Lemma 3.1. *The following hold for a subset A of a nano topological space U,*

(1) $N\beta int(A) = A \cap Ncl(Nint(Ncl(A)))$
(2) $N\beta cl(A) = A \cup Nint(Ncl(Nint(A)))$
(3) $x \in N\beta cl(A)$ *if and only if* $A \cap U \neq \phi$ *for every* $U \in N\beta O(U, x)$
(4) $N\beta cl(U - A) = U - N\beta int(A)$
(5) A *is* $N\beta$*-closed if and only if* $A = N\beta cl(A)$.

The following interesting result will play an important role in the sequel.

Theorem 3.1. *Let A be a subset of a nano topological space U. Then*

(1) $A \in N\beta O(U)$ *if and only if* $N\beta cl(A) \in N\beta R(U)$*;*
(2) $A \in N\beta F(U)$ *if and only if* $N\beta int(A) \in N\beta R(U)$.

Proof. (1) Necessity:
Let $A \in N\beta O(U)$. Then we have $A \subset Ncl(Nint(Ncl(A)))$ and hence $N\beta cl(A) \subset N\beta cl[Ncl(Nint(Ncl(A)))] = Ncl(Nint(Ncl(A))) \subset$
$\subset Ncl(Nint(Ncl(N\beta cl(A))))$. Therefore, $N\beta cl(A)$ is $N\beta$-open and also $N\beta$-closed. Hence $N\beta cl(A) \in N\beta R(U)$.
Sufficiency:
Let $N\beta cl(A) \in N\beta R(U)$.
Then we have $A \subset N\beta cl(A) \subset Ncl(Nint(Ncl(N\beta cl(A)))) \subset$
$\subset Ncl(Nint(Ncl(Ncl(A)))) = Ncl(Nint(Ncl(A)))$.
Hence we have $A \in N\beta O(U)$.

(2) This follows from (1) and Lemma 3.1.

□

Theorem 3.2. *For a subset A of a nano topological space U, the following are equivalent:*

(1) $A \in N\beta R(U)$
(2) $A = N\beta int(N\beta cl(A))$
(3) $A = N\beta cl(N\beta int(A))$.

Proof. The proofs of the implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are obvious.
(2) $\Rightarrow$ (1): Since $N\beta cl(A)$ is $N\beta$-closed , by Theorem 3.1 we have

$$N\beta int(N\beta cl(A)) \in N\beta R(U)$$

and $A \in N\beta R(U)$.
(3) $\Rightarrow$ (1): Since $N\beta int(A)$ is $N\beta$-open, by Theorem 3.1 we have

$$N\beta cl(N\beta int(A)) \in N\beta R(U)$$

and $A \in N\beta R(U)$. ☐

Theorem 3.3. *For any subset A of a nano topological space U, the following hold $N\beta$-$\theta cl(A) = \bigcap\{H : A \subset H$ and H is $N\beta$-θ-closed$\} = \bigcap\{H : A \subset H$ and $H \in N\beta R(U)\}$.*

Proof. We prove only the first equation since the other is similarly proved.
First, suppose that $x \notin N\beta$-$\theta cl(A)$. Then there exists $H \in N\beta O(U, x)$ such that $N\beta cl(H) \cap A = \phi$. By Theorem 3.1, $U - N\beta cl(H)$ is $N\beta$-regular. Hence, $U - N\beta cl(H)$ is an $N\beta$-θ-closed set containing A and $x \in U - N\beta cl(H)$. Therefore, we have $x \notin \cap\{H : A \subset H$ and H is $N\beta$-θ-closed$\}$.
Conversely, suppose that $x \notin \cap\{H : A \subset H$ and H is $N\beta$-θ-closed$\}$. There exists an $N\beta$-θ-closed set H such that $A \subset H$ and $x \notin H$. There exists $G \in N\beta O(U)$ such that $x \in G \subset N\beta cl(G) \subset U - H$. Therefore, we have $N\beta cl(U) \cap A \subset N\beta cl(U) \cap H = \phi$. This shows that $x \notin N\beta$-$\theta cl(A)$. ☐

Theorem 3.4. *Let A and B be any subsets of a nano topological space U. Then the following properties hold*

(1) *$x \in N\beta$-$\theta cl(A)$ if and only if $H \cap A \neq \phi$ for each $H \in N\beta R(U, x)$*
(2) *if $A \subset B$, then $N\beta$-$\theta cl(A) \subset N\beta$-$\theta cl(B)$*
(3) *$N\beta$-$\theta cl(N\beta$-$\theta cl(A)) = N\beta$-$\theta cl(A)$*

Proof. The proofs of (1) and (2) are obvious.
(3) Generally we have $N\beta$-$\theta cl(N\beta$-$\theta cl(A)) \supset N\beta$-$\theta cl(A)$. Suppose that $x \notin N\beta$-$\theta cl(A)$. There exist $H \in N\beta R(U, x)$ such that $H \cap A = \phi$. Since $H \in N\beta R(U)$, we have $H \cap N\beta$-$\theta cl(A) = \phi$. This shows that $x \notin N\beta$-$\theta cl(N\beta$-$\theta cl(A))$.
Therefore, we obtain $N\beta$-$\theta cl(N\beta$-$\theta cl(A)) \subset N\beta$-$\theta cl(A)$. ☐

Remark 3.1. *The union of two $N\beta$-θ-closed sets is not necessarily $N\beta$-θ-closed as shown in the following example:*

Example 1. *Let $U = \{a, b, c, d\}$ with $U/R = \{\{a, b\}, \{c\}, \{d\}\}$ and $X = \{a, d\}$ then we have $\tau_R(X) = \{U, \phi, \{d\}, \{a, b, d\}, \{a, b\}\}$. The subsets $\{a, b\}$, and $\{d\}$ are $N\beta$-θ-closed in $(U, \tau_R(X))$ but $\{a, b, d\}$ is not $N\beta$-θ-closed.*

Corollary 3.1. *Let A be any subsets of a nano topological space of U. Then the following hold:*

1. *A is $N\beta$-θ-open in U if and only if for each $x \in A$ there exists $H \in N\beta R(U, x)$ such that $x \in H \subset A$.*
2. *$N\beta$-$\theta cl(A)$ is $N\beta$-θ-closed and $N\beta$-$\theta int(A)$ is $N\beta$-θ-open.*

Theorem 3.5. *For a subset A of a nano topological space U, the following properties hold:*

1. *If $A \in N\beta O(U)$, then $N\beta cl(A) = N\beta$-$\theta cl(A)$.*
2. *$A \in N\beta R(U)$ if and only if A is $N\beta$-θ-open and $N\beta$-θ-closed.*

Proof. (1) Generally we have $N\beta cl(L) \subset N\beta$-$\theta cl(L)$ for every subset L of U. Let $A \in N\beta O(U)$. Suppose that $x \notin N\beta cl(A)$. Then, there exists $H \in N\beta O(U, x)$ such that $H \cap A = \phi$. Since $A \in N\beta O(U)$, we have $N\beta cl(H) \cap A = \phi$. This shows that $x \notin N\beta$-$\theta cl(A)$. Therefore, we obtain $N\beta cl(A) \supset N\beta$-$\theta cl(A)$ and hence $N\beta cl(A) = N\beta$-$\theta cl(A)$.

(2) Let $A \in N\beta R(U)$. Then $N\beta O(U)$ and by (1) $A = N\beta cl(A) = N\beta$-$\theta cl(A)$. Therefore, A is $N\beta$-θ-closed. Since $U - A \in N\beta R(U)$, by the argument above, $U - A$ is $N\beta$-θ-closed and hence A is $N\beta$-θ-open. The converse is obvious.

□

Remark 3.2. *It is obvious that $N\beta$-regular $\Rightarrow$ $N\beta$-θ-open $\Rightarrow$ $N\beta$-open. But the converses are not necessarily true as shown by the following example:*

Example 2. *Let $U = \{a, b, c, d\}$, with $U/R = \{\{a, b\}, \{c\}, \{d\}\}$ and $X = \{a, d\}$ then we have $\tau_R(X) = \{U, \phi, \{d\}, \{a, b, d\}, \{a, b\}\}$. The subset $\{a, b, c\}$ is $N\beta$-regular (resp. $N\beta$-θ-open and $N\beta$-open) but the subset $\{b, c, d\}$ is $N\beta$-θ-open but not $N\beta$-θ-open and $N\beta$-open.*

4. Characterizations and properties of weakly $N\beta$-irresolute functions

Definition 4.1. *A function $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is said to be*

1. *$N\beta$-irresolute if $f^{-1}(H) \in N\beta O(U)$ for each $H \in N\beta O(V)$.*

(2) *weakly $N\beta$-irresolute (resp. strongly $N\beta$-irresolute) if for each point $x \in U$ and each $H \in N\beta O(V, f(x))$ there exists $G \in N\beta O(U, x)$ such that $f(G) \subset N\beta cl(H)$ (resp. $f(N\beta cl(G)) \subset H$).*

Definition 4.2. *A function $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is said to be strongly $N\theta$-β continuous if for each point $x \in U$ and each nano open set H containing $f(x)$, there exists a $G \in N\beta O(U, x)$ such that $f(N\beta cl(G) \subset H)$.*

Remark 4.1. *By the definitions stated above, we have the following diagram. However, none of these implications is reversible as shown by the examples stated below.*

$$\begin{array}{ccccc}
\text{strongly } N\beta\text{-irresolute} & \Rightarrow & N\beta\text{-irresolute} & \Rightarrow & \text{weakly } N\beta\text{-irresolute} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{strongly } N\theta\text{-}\beta\text{-continuous} & \Rightarrow & N\beta\text{-continuous} & \Rightarrow & \text{almost } N\beta\text{-continuous}
\end{array}$$

Example 3. *From previous example,*

(1) *Let $x = b \in U$, $f(b) = \{a\} \subset \{a, b, d\} = H$. We take $G = \{a, b, c\}$, $f(G) = \{a, b, d\}$ is almost $N\beta$-continuous but not strongly $N\theta$-β-continuous.*
(2) *Let $x = b \in U$, $f(b) = \{a\} \subset \{a, b\} = H$. We take $G = \{b, c\}$, $f(G) = \{a, d\}$ is almost $N\beta$-continuous but not weakly $N\beta$-irresolute.*

Theorem 4.1. *For a function $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$, the following properties are equivalent:*

(1) *f is weakly $N\beta$-irresolute.*
(2) *$f^{-1}(H) \subset N\beta int(f^{-1}(N\beta cl(H)))$ for every $H \in N\beta O(V)$.*
(3) *$N\beta cl(f^{-1}(H)) \subset f^{-1}(N\beta cl(H))$ for every $H \in N\beta O(V)$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $H \in N\beta O(V)$ and let $x \in f^{-1}(H)$. It follows from (1) that $f(G) \subset N\beta cl(H)$ for some $G \in N\beta O(U, x)$. Therefore, we have $G \subset f^{-1}(N\beta cl(H))$ and $x \in G \subset N\beta int(f^{-1}(N\beta cl(H)))$. This shows that $f^{-1}(H) \subset N\beta int(f^{-1}(N\beta cl(H)))$.

(2) $\Rightarrow$ (3) Suppose that $H \in N\beta O(V)$ and $x \notin f^{-1}(N\beta cl(H))$. Then $f(x) \notin N\beta cl(H)$. There exists $K \in N\beta O(V, f(x))$ such that $K \cap H = \phi$. Since $H \in N\beta O(V)$, we have $N\beta cl(K) \cap H = \phi$ and hence $N\beta int(f^{-1}(N\beta cl(H))) \cap f^{-1}(H) = \phi$. By (2), we have $x \in f^{-1}(K) \subset N\beta int(f^{-1}(N\beta cl(K))) \in N\beta O(U)$. Therefore, we obtain $x \notin N\beta cl(f^{-1}(H))$. This shows that $N\beta cl(f^{-1}(H)) \subset f^{-1}(N\beta cl(H))$.

(3) $\Rightarrow$ (1) Let $x \in U$ and $H \in N\beta O(V, f(x))$. By Theorem 3.1, $N\beta cl(H) \in N\beta R(V)$ and $x \notin f^{-1}(N\beta cl(V - N\beta cl(H)))$. Since $V - N\beta cl(H) \in N\beta O(V)$, by (3) we have $x \notin N\beta cl(f^{-1}(V - N\beta cl(H)))$. Hence there exists $G \in N\beta O(U, x)$ such that $G \cap f^{-1}(V - N\beta cl(H)) = \phi$. Therefore, we obtain $f(G) \cap (V - N\beta cl(H)) = \phi$ and hence $f(G) \subset N\beta cl(H)$. This shows that f is weakly $N\beta$-irresolute. □

Theorem 4.2. *For a function $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$, the following properties are equivalent:*

(1) *f is weakly $N\beta$-irresolute*
(2) *$N\beta cl(f^{-1}(B)) \subset f^{-1}(N\beta\text{-}\theta cl(B))$ for every subset B of V..*
(3) *$f(N\beta cl(A)) \subset N\beta\text{-}\theta cl(f(A))$ for every subset A of U.*
(4) *$f^{-1}(F) \in N\beta C(U)$ for every $N\beta$-θ-closed set F of V.*
(5) *$f^{-1}(H) \in N\beta O(U)$ for every $N\beta$-θ-open set H of V.*

Proof. (1) $\Rightarrow$ (2) Let B be any subset of V and $x \notin f^{-1}(N\beta\text{-}\theta cl(B))$. Then $f(x) \notin N\beta\text{-}\theta cl(B)$ and there exists $H \in N\beta O(V, f(x))$ such that $N\beta cl(H) \cap B = \phi$. By (1), there exists $G \in N\beta O(U, x)$ such that $f(G) \subset N\beta cl(H)$. Hence $f(G) \cap B = \phi$ and $G \cap f^{-1}(B) = \phi$. Consequently, we obtain $x \notin N\beta cl(f^{-1}(B))$.

(2) $\Rightarrow$ (3) Let A be any subset of U.
By (2), we have $N\beta cl(A) \subset N\beta cl(f^{-1}(f(A))) \subset f^{-1}(N\beta\text{-}\theta cl(f(A)))$ and hence $f(N\beta cl(A)) \subset N\beta\text{-}\theta cl(f(A))$.

(3) $\Rightarrow$ (4) Let F be any $N\beta$-θ-closed set of V.
Then, by (3) we have $f(N\beta cl(f^{-1}(F))) \subset N\beta\text{-}\theta cl(f(f^{-1}(F))) \subset N\beta\text{-}\theta(F) = F$. Therefore, we have $N\beta cl(f^{-1}(F)) \subset f^{-1}(F)$ and hence $N\beta cl(f^{-1}(F)) = f^{-1}(F)$. This shows that $f^{-1}(F) \in N\beta C(U)$.

(4) $\Rightarrow$ (5) This proof is obvious and is omitted.

(5) $\Rightarrow$ (1) Let $x \in U$ and $H \in N\beta O(V, f(x))$. By Theorems 3.1 and 3.3, $N\beta cl(H)$ is $N\beta$-θ-open in V. Set $G = f^{-1}(N\beta cl(H))$. Then by (5), $G \in N\beta O(U, x)$ and $f(G) \subset N\beta cl(H)$. This shows that f is weakly $N\beta$-irresolute. □

Theorem 4.3. *For a function $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$, the following properties are equivalent*

(1) *f is weakly $N\beta$-irresolute;*
(2) *for each $x \in U$ and each $H \in N\beta O(V, f(x))$, there exists $G \in N\beta O(U, x)$ such that $f(N\beta cl(G)) \subset N\beta cl(H)$;*
(3) *$f^{-1}(F) \in N\beta R(U)$ for every $F \in N\beta R(V)$.*

Proof. (1) $\Rightarrow$ (2) Let $x \in U$ and $H \in N\beta O(V, f(x))$. Then $N\beta cl(H)$ is $N\beta$-θ-open and $N\beta$-θ-closed in V by Theorems 3.1 and 3.5. Now, put $G = f^{-1}(N\beta cl(H))$. Then by Theorem 4.2 $G \in N\beta R(U)$. Therefore, we obtain $U \in N\beta O(U, x)$, $G = N\beta cl(G)$ and $f(N\beta cl(U)) \subset N\beta cl(H)$.

(2) $\Rightarrow$ (3) Let $F \in N\beta R(Y)$ and $x \in f^{-1}(F)$. Then $f(x) \in F$. By (2) there exists $G \in N\beta O(U, x)$ such that $f(N\beta cl(G)) \subset F$. Therefore, we have $x \subset G \subset N\beta cl(G) \subset f^{-1}(F)$ and hence $f^{-1}(F) \in N\beta O(U)$. Since $V - F \in N\beta R(V)$, $f^{-1}(V - F) = U - f^{-1}(F) \in N\beta O(U)$. Thus $f^{-1}(F) \in N\beta C(U)$ and hence $f^{-1}(F) \in N\beta R(U)$.

(3) $\Rightarrow$ (1) Let $x \in U$ and $H \in N\beta O(V, f(x))$. By Theorem 3.1, $N\beta cl(H) \in N\beta R(V, f(x))$ and $f^{-1}(N\beta cl(H)) \in N\beta R(U, x)$. Put $G = f^{-1}(N\beta cl(H))$. Then $G \in N\beta O(U, x)$ and $f(G) \subset N\beta cl(H)$. This shows that f is weakly $N\beta$-irresolute. □

Theorem 4.4. *For a function $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$, the following properties are equivalent:*

(1) *f is weakly $N\beta$-θ-irresolute.*
(2) *$f^{-1}(H) \subset N\beta$-$\theta int(f^{-1}(N\beta$-$\theta cl(H))$ for every $H \in N\beta$-$\theta O(V)$.*
(3) *$N\beta$-$\theta cl(f^{-1}(H)) \subset f^{-1}(N\beta$-$\theta cl(H))$ for every $H \in N\beta$-$\theta O(V)$.*

Theorem 4.5. *For a function $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$, the following properties are equivalent:*

(1) *f is weakly $N\beta$-θ-irresolute;*
(2) *$N\beta$-$\theta cl(f^{-1}(B)) \subset f^{-1}(N\beta$-$\theta cl(B))$ for every subset B of V*
(3) *$f(N\beta$-$\theta cl(A)) \subset N\beta$-$\theta(f(A))$ for every subset A of U*
(4) *$f^{-1}(F) \in N\beta$-$\theta C(U)$ for every $N\beta$-θ-closed set F of V*
(5) *$f^{-1}(H) \in N\beta$-$\theta O(U)$ for every $N\beta$-θ-open set H of V.*

Lemma 4.1. *For a nano topological space U, the following properties are equivalent*

1. *U is $N\beta$-regular.*
2. *For each $G \in N\beta O(U)$ and each $x \in G$, there exists $H \in N\beta O(U)$ such that $x \in H \subset N\beta cl(H) \subset G$.*
3. *For each $G \in N\beta O(U)$ and each $x \in G$, there exists $H \in N\beta R(U)$ such that $x \in H \subset G$.*

Proof. This follows easily from Theorem 3.1. □

Theorem 4.6. *Let V be a nano $N\beta$-regular space. Then a function $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is weakly $N\beta$-irresolute if and only if it is $N\beta$-irresolute.*

Proof. Suppose that $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is weakly $N\beta$-irresolute. Let H be any $N\beta$-open set of V and $x \in f^{-1}(H)$. Then $f(x) \in H$. Since V is $N\beta$-regular, by Lemma 4.1 there exists $W \in N\beta O(V)$ such that $f(x) \in W \subset N\beta cl(W) \subset H$. Since f is weakly $N\beta$-irresolute, there exists $G \in N\beta O(U, x)$ such that $f(G) \subset N\beta cl(W)$. Therefore, we have $x \in G \subset f^{-1}(H)$ and $f^{-1}(H) \in N\beta O(U)$. This shows that f is $N\beta$-irresolute. The converse is obvious. □

Theorem 4.7. *For a function $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$, the following properties are equivalent*

1. *f is strongly $N\beta$-irresolute.*
2. *for each $x \in U$ and each $H \in N\beta O(V, f(x))$, there exists $G \in N\beta O(U, x)$ such that $f(N\beta\text{-}\theta cl(U)) \subset H$.*
3. *for each $x \in U$ and each $H \in N\beta O(V, f(x))$, there exists $G \in N\beta R(U, x)$ such that $f((U) \subset H$.*
4. *for each $x \in U$ and each $H \in N\beta O(V, f(x))$, there exists an $N\beta$-θ-open set G in U containing x such that $f(G) \subset H$.*
5. *$f^{-1}(H)$ is $N\beta$-θ-open in U for every $H \in N\beta O(V)$.*
6. *$f^{-1}(F)$ is $N\beta$-θ-closed in U for every $F \in N\beta F(V)$.*
7. *$f(N\beta\text{-}\theta cl(A)) \subset N\beta cl(f(A))$ for every subset A of U.*
8. *$N\beta\text{-}\theta cl(f^{-1}(B)) \subset f^{-1}(N\beta cl(B))$ for every subset B of V.*

Proof. The equivalences of (1) - (4) follow from Theorems 3.1 and 3.5.
(4) $\Rightarrow$ (5) Let $H \in N\beta O(V)$. Suppose that $x \in f^{-1}(H)$. Then $f(x) \in H$ and there exists an $N\beta$-θ-open set G in U containing x such that $f(G) \subset H$.

Therefore, we have $x \in G \subset f^{-1}(H)$. The union of $N\beta$-θ-open sets is $N\beta$-θ-open by Corollary 3.1. Therefore, $f^{-1}(H)$ is $N\beta$-θ-open in U.
(5) $\Rightarrow$ (6) This is obvious.
(6) $\Rightarrow$ (7) Let A be any subset of U. Since $N\beta cl(f(A))$ is $N\beta$-closed in V, by (6) $f^{-1}(N\beta cl(f(A)))$ is $N\beta$-θ-closed in U and we have $N\beta$-$\theta cl(A) \subset N\beta$-$\theta cl((f^{-1}f(A))) \subset N\beta$-$\theta cl(f^{-1}(N\beta cl(f(A)))) = f^{-1}(N\beta cl(f(A)))$. Therefore, we obtain $f(N\beta$-$\theta cl(A)) \subset N\beta cl(f(A))$.
(7) $\Rightarrow$ (8) Let B be any subset of V. By (7), we obtain $f(N\beta$-$\theta cl(f^{-1}(B))) \subset N\beta cl(f(f^{-1}(B))) \subset N\beta cl(B)$ and hence $N\beta$-$\theta cl(f^{-1}(B)) \subset f^{-1}(N\beta cl(B))$.
(8) $\Rightarrow$ (1): Let $x \in U$ and $H \in N\beta O(V, f(x))$. Since $V - H \in N\beta C(V)$, we have $N\beta$-$\theta cl(f^{-1}(V-H)) \subset f^{-1}(N\beta cl(V-H)) = f^{-1}(V-H)$. Therefore, $f^{-1}(V-H)$ is $N\beta$-θ-closed in U and $f^{-1}(H)$ is a $N\beta$-θ-open set containing x. There exists $G \in N\beta O(U, x)$ such that $N\beta cl(G) \subset f^{-1}(H)$, hence $f(N\beta cl(G)) \subset H$. This shows that f is strongly $N\beta$-irresolute. $\square$

Theorem 4.8. *A $N\beta$-irresolute function $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is strongly $N\beta$-irresolute if and only if U is $N\beta$-regular.*

Proof. Necessity. Let $f : (U, \tau_R(X)) \to (U, \tau_R(X))$ be an identity function. Then f is $N\beta$-irresolute and strongly $N\beta$-irresolute by the hypothesis. For any $G \in N\beta O(U)$ and any point x of G, we have $f(x) = x \in G$ and there exists $K \in N\beta O(U, x)$ such that $f(N\beta cl(K)) \subset G$. Therefore, we have $x \in K \subset N\beta cl(K) \subset G$. By Lemma 4.1, U is $N\beta$-regular.
Sufficiency. Suppose that $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is $N\beta$ irresolute and U is $N\beta$-regular. For any $x \in U$ and any $H \in N\beta O(V, f(x))$, $f^{-1}(H)$ is a $N\beta$-open set of U containing x. Since U is $N\beta$-regular, there exists $G \in N\beta O(U)$ such that $x \in G \subset N\beta cl(G) \subset f^{-1}(H)$. Therefore, we have $f(N\beta cl(G)) \subset H$. This shows that f is strongly $N\beta$-irresolute. $\square$

Corollary 4.1. *Let U be a $N\beta$-regular space. Then $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is strongly $N\beta$-irresolute if and only if f is $N\beta$-irresolute.*

Proof. This follows immediately from Theorem 4.4. $\square$

Lemma 4.2. *Let A and B be subsets of a space U.*

(1) *If $A \in N\beta O(U)$ and B is $N\alpha$-open in U, then $A \cap B \in N\beta O(B)$.*
(2) *If $A \in N\beta O(B)$ and $B \in N\beta O(U)$, then $A \in N\beta O(U)$.*

Lemma 4.3. *Let U be a nano topological space and A, B subsets of U such that $A \subset B \subset U$ and B is $N\alpha$-open in U. Then the following properties hold*

(1) *$A \in N\beta O(B)$ if and only if $A \in N\beta O(U)$,*
(2) *$N\beta cl(A) \cap V = N\beta cl_V(A)$, where $N\beta cl_V(A)$ denotes the $N\beta$-closure of A in the subspace V.*

Proof. (1) Let $A \in N\beta O(B)$. Since B is $N\alpha$-open in U, by Lemma 4.2, we have $A \in N\beta O(U)$.
Conversely, Let $A \in N\beta O(U)$. By Lemma 4.2, $A = A \cap B \in N\beta O(B)$.

(2) Let $x \in N\beta cl(A) \cap B$ and $H \in N\beta O(B, x)$. Then, by (1) $H \in N\beta O(U, x)$ and hence $H \cap A \neq \phi$. Therefore, $x \in N\beta cl_B(A)$.
Conversely, let $x \in N\beta cl_B(A)$ and $H \in N\beta O(U, x)$. Then by Lemma 4.2 $x \in H \cap B \in N\beta O(B)$ and hence $\phi \neq A \cap (H \cap B) \subset A \cap H$. Therefore, we obtain $x \in N\beta cl(A) \cap B$.

□

Definition 4.3. *A function $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is said to be nano pre-β-open if $f(G) \in N\beta O(V)$ for each $G \in N\beta O(U)$.*

Lemma 4.4. *If $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is $N\beta$-irresolute and H is $N\beta$-θ-open in V, then $f^{-1}(H)$ is $N\beta$-θ-open in U.*

Proof. Let H be an $N\beta$-θ-open set of V and $x \in f^{-1}(H)$. There exists $L \in N\beta O(V)$ such that $f(x) \in L \subset N\beta cl(L) \subset H$. Since f is $N\beta$-irresolute, we have $f^{-1}(L) \in N\beta O(U)$ and $N\beta cl(f^{-1}(L)) \subset f^{-1}(N\beta cl(L))$. Therefore, we have $x \in f^{-1}(L) \subset N\beta cl(f^{-1}(L)) \subset f^{-1}(H)$ and $f^{-1}(H)$ is $N\beta$-θ-open in U. □

Theorem 4.9. *Let $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ and $g : (V, \tau_{R'}(Y)) \to (W, \tau_{R^*}(Z))$ be functions. Then, the following properties hold*

(1) *If f is strongly $N\beta$-irresolute and g is $N\beta$-irresolute, then the composition $g \circ f : (U, \tau_R(X)) \to (W, \tau_{R^*}(Z))$ is strongly $N\beta$-irresolute.*
(2) *If f is $N\beta$-irresolute and g is strongly $N\beta$-irresolute, then $g \circ f$ is strongly $N\beta$-irresolute.*
(3) *If $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is a nano pre-θ-open bijection and $g \circ f : (U, \tau_R(X)) \to (W, \tau_{R^*}(Z))$ is strongly $N\beta$-irresolute, then g is strongly $N\beta$-irresolute.*

Proof. (1) This is obvious.

(2) This follows immediately from Theorem 4.6 and Lemma 4.4.

(3) Let $L \in N\beta O(W)$. Since $g \circ f$ is strongly $N\beta$-irresolute, $(g \circ f)^{-1}(L)$ is $N\beta$-θ-open in U. Since f is nano pre-β-open and bijective, f^{-1} is $N\beta$-irresolute and by Lemma 4.4 we have $g^{-1}(L) = f((g \circ f)^{-1}(L))$ is $N\beta$-θ-open in V. Hence, by Theorem 4.6 g is strongly $N\beta$-irresolute.

□

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