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SOME NORMALITIES IN FORMS OF NANO TOPOLOGICAL SPACES

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ABSTRACT. The purpose of this paper is to introduce and investigate the properties of nanopre-normal, almost nanopre-normal, nano mildly pre-normal, completely nanopre-normal spaces and further the relationship between them are derived.

1. INTRODUCTION

Sathishmohan et al. in [7] and the authors in [3–5] defined nanopre - neighbourhoods, nanopre- interior, nanopre-limit point, nanopre-derived set, nanopre-frontier and nano-pre-regular in nano topological spaces and obtained some of its properties. Also, in [9] and [2] are introduced and investigated the properties of nano semipre-neighbourhoods, nano semipre-interior, nano semipre-frontier, na-no semipre-exterior, nano-dense and nano-submaximal. Further, the authors in [6] introduced and examined the properties of nano-T_0 space, nano-semi-T_0 space, nano-pre-T_0 space, nano-T_1 space, nanosemi-T_1 space, nanopre-T_1 space, nano-T_2 space, nanosemi-T_2 space, nanopre-T_2 space and obtain some of its basic results. The same authors in [8] have defined and examined the properties of nanopre-irresolute, almost nanopre-irresolute, quasi nanopre-irresolute,

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nano semi-regular, nanopre-regular, strongly nanopre-regular, almost nanopre-regular and obtain some relationship between the existing sets. In [1] are defined quasi nano p-normal spaces.

The structure of this manuscript as follows:

In section 2, we study some additional characterizations of nanopre-normality. Also, we introduce and study the nanopre-normal spaces. In section 3, we introduced and study the characterizations of almost nanopre-normal spaces. In section 4, we introduced and examined the properties of nano mildly pre-normal spaces. In section 5, we study about some strong forms of nanopre-normality.

2. NANO PRE-NORMAL SPACES

Definition 2.1. *A space U is called nano pre-normal if for every pair of disjoint nano closed sets F_1 and F_2, there exist disjoint nano pre-open sets X and Y such that $F_1 \subset X$ and $F_2 \subset Y$.*

Clearly, every nano normal space is nano pre-normal since every nano open set is nano pre-open, but not conversely.

Example 1. *Let $U = \{a,b,c,d\}, U/R = \{\{a,d\},\{b\},\{c\}\}$. $X = \{a,b\}$. $\tau_R(X) = \{\phi, U, \{b\}, \{a,b,d\}, \{a,d\}\}$, $\tau_R^c(X) = \{\phi, U, \{a,c,d\}, \{c\}, \{b,c\}\}$.*
NP-open = $\{\pi, U, \{a\}, \{d\}, \{a,b\}, \{a,d\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}\}$.
NP-closed = $\{\pi, U, \{b,c,d\}, \{a,b,c\}, \{c,d\}, \{b,c\}, \{a,c\}, \{d\}, \{c\}, \{a\}\}$.
$F_1 = \{a\}$, $F_2 = \{c,d\}$, $X = \{a\}, Y = \{b,c,d\}$. Then $F_1 \subset X$ and $F_2 \subset Y$ is nano pre-normal space but not nano normal space.

Theorem 2.1. *For a space U the following are equivalent.*

(a) *U is nano pre-normal.*
(b) *For every pair of nano open sets X and Y whose union is U, there exist nano pre-closed sets A and B such that $A \subset X$, $B \subset Y$ and $A \cup B = U$.*
(c) *For every nano closed set F and every nano open set G containing F, there exists a nano pre-open set X such that $F \subset X \subset X^* \subset G$.*

Proof. (a) $\Rightarrow$ (b): Let X and Y be a pair of nano open sets in a nano pre-normal space U such that $X \cup Y = U$. Then $U - X, U - Y$ are disjoint nano closed sets. Since U is nano pre-normal there exist disjoint nano pre-open sets X_1 and Y_1 such that $U - X \subset X_1$ and $U - Y \subset Y_1$. Let $A = U - X_1$, $B = U - Y_1$. Then

A, B are disjoint nano pre-closed sets such that $A \subset X$, $B \subset Y$ and $A \cup B = U$.
(b) $\Rightarrow$ (c): Let F be a nano closed set and G be an nano open set containing F. Then $U - F$ and G are open sets whose union is U. Then by (b), there exist nano pre-closed sets W_1 and W_2 such that $W_1 \subset U - F$, $W_2 \subset G$ and $W_1 \cup W_2^* = U$. Then, $F \subset U - W_1$ $U - G \subset U - W_2$ and $(U - W_1) \cap (U - W_2) = \phi$. Let $X = U - W_1$ and $Y = U - W_2$. Then X, Y are disjoint nano pre-open sets such that $F \subset X \subset U - Y \subset G$. As $U - Y$ is nano pre-closed, we have $X^* \subset U - Y$ and $F \subset X \subset X^* \subset G$.
(c) $\Rightarrow$ (a): Let F_1 and F_2 be any two disjoint nano closed sets of U. Put $G = U - F_2$, then $F_1 \cap F_2 = \phi$. $F_1 \subset G$ where G is an nano open set. Then by (c), there exists a nano pre-open set X of U such that $F_1 \subset X \subset X^* \subset G$. It follows that $F_2 \subset U - X^* = Y$ say, then Y is nano pre-open and $X \cap Y = \phi$. Hence F_1 and F_2 are separated by disjoint nano pre-open sets X and Y. Therefore U is nano pre-normal. □

Theorem 2.2. *A nano regular closed subspace of a nano pre-normal space is nano pre-normal.*

Proof. Let V be a nano regular closed subspace of a nano pre-normal space U. Let A and B be disjoint nano closed sets of V. As V is nano regular closed and hence nano closed. A, B are closed sets of U. By nano pre-normality of U, there exist disjoint nano pre-open sets X and Y in U such that $A \subset X$ and $B \subset Y$. As every nano regular closed set is nano semi-open, $X \cap V$ and $Y \cap V$ are nano pre-open sets in V such that $A \subset X \cap V$ and $B \subset Y \cap V$ with $(X \cap V) \cap (Y \cap V) = \phi$. Hence V is nano pre-normal. □

A nano pre-normal space need not to be nano pre-regular, as the following example shows.

Example 2. *From example 1, Let $x = \{b\}, F = \{c, d\}, X = \{a\}, Y = \{b, c, d\}$. But $F \not\subset X$, $x \in Y$ is nano pre-normal space but not nano pre-regular space.*

We now prove the following.

Theorem 2.3. *If A and B be two nano gp-open sets then $A \cup B$ is nano gp-open.*

Theorem 2.4. *For a space U, the following are equivalent.*

(a) *U is nano pre-normal.*

(b) *For every pair of disjoint nano closed sets A and B, there exist disjoint nano gp-open sets X and Y such that $A \subset X$ and $B \subset Y$.*

Proof. $(a) \Rightarrow (b)$. Since every nano pre-open set is nano gp-open, the (b) follows from (a) immediately.
$(b) \Rightarrow (a)$. Let A and B be disjoint nano closed sets in U. Then by (b), there exist disjoint nano gp-open sets X and Y such that $A \subset X$ and $B \subset Y$. Since X is nano gp-open and A is nano pre-closed and $A \subset X$, then $A \subset X_*$. Similarly $B \subset Y_*$. Further, $X_* \cap Y_* \subset X \cap Y = \phi$. Hence X_* and Y_* are disjoint nano pre-open sets such that $A \subset X_*$ and $B \subset Y_*$. This shows that U is nano pre-normal. □

3. Almost nano pre-normal spaces.

In this section, we introduce and study the characterizations of almost nanopre-normal spaces.

Definition 3.1. *A space U is said to be almost nano pre-normal if for each nano closed set A and each nano regular closed set B such that $A \cap B = \phi$, there exist disjoint nano pre-open sets X and Y such that $A \subset X$ and $B \subset Y$.*

We have the following characterization of almost nano pre-normality.

Lemma 3.1. *Let A be a subset of U. Then, $Nscl(Nint(A)) = Nint(A^*) = Nint(Ncl(Nint(A)))$.*

Theorem 3.1. *For a space U the following are equivalent.*

(a) *U is almost nano pre-normal.*
(b) *For every pair of sets X and Y, one of which is nano open and the other is regular open whose union is U, there exist nano pre-closed sets G and H such that $G \subset X$, $H \subset Y$ and $G \cup H = U$.*
(c) *For every nano closed set A and every nano regular open set B containing A, there is a nano pre-open set Y such that $A \subset Y \subset Y^* \subset B$.*
(d) *For every nano regular open set B containing a nano closed set A, there exists a nano regular open set X such that $A \subset X \subset Ncl(X) \subset B$.*
(e) *For every pair of disjoint sets A and B where A is nano closed and B is nano regular closed, there exist nano pre-open sets X and Y such that $A \subset X$, $B \subset Y$ and $X^* \cap Y^* = \phi$.*

Proof. The proofs of $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$, are similar to those of Theorem 2.1.
$(c) \Rightarrow (d)$. If B be a nano regular open set containing a nano closed set A, then there exists a nano pre-open set Y such that $A \subset Y \subset Y^* \subset B$ by (c). If $X = Nint(Y^*)$ then $A \subset X \subset Ncl(X) \subset B$, where X is nano regular open.
$(d) \Rightarrow (e)$. Let A be a nano closed set and B be nano regular closed set disjoint from B. Then $(U - B)$ is nano regular open set containing the nano closed set A. Therefore by (d), there exists a nano regular open set W such that $A \subset W \subset Ncl(W) \subset U - B$. Again, since W is a nano regular open set containing a nano closed set A, there exists a nano regular open set X such that $A \subset X \subset Ncl(X) \subset W$. Let $U - Ncl(W) = Y$ say. Then X and Y are nano regular open sets such that $A \subset X$ and $B \subset Y$, with $Ncl(X) \cap Ncl(Y) = \phi$. Since every nano regular open set is nano pre-open and hence X and Y are nano pre-open sets such that $A \subset X$ and $B \subset Y$. Further more, $X^* \subset Ncl(X)$ and $Y^* \subset Ncl(Y)$ and hence $X^* \cap Y^* = \phi$.
$(e) \Rightarrow (a)$. This is obvious. □

Next, we need the following.

Lemma 3.2. *Every nano pre-closed subset of a nano strongly compact space is nano strongly compact.*

Lemma 3.3. *The following are equivalent for any space U.*

(a) *U is almost nano regular.*
(b) *For every nano regular closed set A and $U \notin A$, there exist disjoint nano α-open sets X and Y such that $x \in X$ and $A \subset Y$.*
(c) *For every regular closed set A and $U \notin A$, there exist disjoint sets X_1 and X_2 such that $U \in X_1$. $A \subset X_2$ and for each $i = 1, 2, X_i \cap Y$ is nano semi-open for each nano semi-open set Y.*

Lemma 3.4. *Let U be a space and $\{X_i | i = 1, 2...n\}$ be a finite family of nano semi-open sets. Then $Nint(Ncl(\cap_{i=1}^{n} U_i) = \cap_{i=1}^{n} Nint(Ncl(U_i))$.*

We now prove the following.

Theorem 3.2. *Every almost regular strongly compact space U is almost nano pre-normal.*

Proof. The proof follows from Lemmas 3.2, 3.3 and 3.4. □

4. Nano mildly pre-normal spaces

Definition 4.1. *A space U is said to be mildly nano pre-normal if for every pair of disjoint nano regular closed subsets F_1 and F_2 of U there exist disjoint nano pre-open sets X and Y such that $F_1 \subset X$ and $F_2 \subset Y$.*

Clearly every nano mildly normal space as well as almost nano pre-normal space is nano mildly pre-normal.

We have the following characterization of nano mildly pre-normal spaces.

Theorem 4.1. *For a space U the following are equivalent.*

(a) *U is nano mildly pre-normal.*
(b) *For every pair of nano regular open sets X and Y whose union is U, there exist nano pre-closed sets A and B such that $A \subset X$, $B \subset Y$ and $A \cup B = U$.*
(c) *For every nano regular closed set A and every nano regular open set B containing A, there exist a nano pre-open set X such that $A \subset X \subset X^* \subset B$.*
(d) *For every nano regular open set B containing a nano regular closed set A, there exists a nano regular open set X such that $A \subset X \subset Ncl(X) \subset B$.*
(e) *For every pair of disjoint nano regular closed sets, there exist nano pre-open sets X and Ysuch that $A \subset X$, $B \subset Y$ and $X^* \cap Y^* = \phi$.*

Proof. This theorem may be proved by using arguments similar to those of Theorem 3.1. □

Lemma 4.1. *Let U be a space, the following hold,*
(a) $NSO(U,\tau) = NSO(U,\tau^\alpha)$; $NSF(U,\tau) = NSF(U,\tau^\alpha)$.
(b) $NRO(U,\tau) = NRO(U,\tau^\alpha)$; $NRF(U,\tau) = NRF(U,\tau^\alpha)$.

Theorem 4.2. *If (U,τ) is mildly nano pre-normal iff (U,τ^α) is mildly nano pre-normal.*

Proof. Let (U, τ) be mildly nano pre-normal space. Suppose A and B be two disjoint nano regular closed sets in (U, τ^α). Then by Lemma 4.1 (b), A, B will be disjoint nano regular closed subsets of (U, τ). Since (U, τ) is mildly nano pre-normal, there exist disjoint nano pre-open sets U and V such that $A \subset X$ and $B \subset Y$. X and Y will be disjoint nano pre open sets of (U, τ^α). Hence it follows that (U, τ^α) is mildly nano pre-normal space.

Converse follows on similar lines using Lemma 4.1. □

We need the following.

Definition 4.2. *We say that a subset A of space U is said to be:*

(i) *Nπ-open if A is a finite union of nano regular open sets.*
(ii) *Nπ-closed if A is a finite intersection of nano regular closed sets.*
(iii) *Ncl-open if A is both nano closed and nano open.*

Clearly, if A is Ncl-open, $U - A$ also is a Ncl-open. And, A is Nπ-open $\rightarrow$ A is Nδ-open.

Example 3. *Let $U = \{a, b, c, d\}$, $U/R = \{\{a, b\}, \{c\}, \{d\}\}$, $X = \{a, d\}$. $\tau_R(X) = \{\phi, U, \{d\}, \{a, b, d\}, \{a, b\}\}$, $\tau_R(X)^c = \{\phi, U, \{b, c, a\}, \{c\}, \{c, d\}\}$. Let $A = \{d\}$. Then A is Nδ-open but not Nπ-open.*

Definition 4.3. *A space U is said to be quasi nano normal if any two disjoint Nπ-closed sets are separated by nano open sets.*

Definition 4.4. *A space U is said to be quasi nano pre-normal if any two disjoint Nπ-closed sets F_1 and F_2, there exist disjoint nano pre-open sets X and Y such that $F_1 \subset X$ and $F_2 \subset Y$.*

Theorem 4.3. *For any space U, the following are equivalent.*

(a) *U is quasi nano pre-normal.*
(b) *For every Nπ-closed set G, and every Nπ-open set H containing G, there exists a nano pre-open set X such that $G \subset X \subset X^* \subset H$.*
(c) *For every Nπ-closed set F, and every Nπ-open set G containing F, there exists a nano regular open set Y such that $F \subset Y \subset Ncl(Y) \subset G$.*

Proof. The proof is similar to the proof of Theorem 3.1. □

5. Strong forms of nano pre-normality

In this section we introduce and study, some more strong forms of normality.

Definition 5.1. *A space U is said to be completely nano normal if for any two separated subsets A and B of U, there exist disjoint nano open sets X and Y such that* $A \subset X$ *and* $B \subset Y$.

Definition 5.2. *A space U is said to be completely nano pre-normal if for any two separated subsets A and B of U, there exist disjoint nano pre-open sets X and Y such that* $A \subset X$ *and* $B \subset Y$.

Theorem 5.1. *Complete nano pre-normality is a nano topological property.*

Proof. Let U be a completely nano pre-normal space and $f : U \to V$ be a nano homeomorphism. Let A and B any two separated nano sets in V. Then $A \cap Ncl(B) = \phi$ and $Ncl(A) \cap B = \phi$. Since f is nano continuous, $Ncl(f^{-1}(A)) \subset f^{-1}(Ncl(A))$ and $Ncl(f^{-1}(B)) \subset f^{-1}(Ncl(A))$. Therefore $f^{-1}(A)$ and $f^{-1}(B)$ are two separated nano sets in U. Since U is a completely nano pre-normal space, there exist disjoint nano pre-open sets X and Y such that $f^{-1}(A) \subset X$ and $f^{-1}(B) \subset Y$. As f is bijective, $A \subset f(X)$ and $B \subset f(Y)$ and $f(X) \cap f(Y) = \phi$. Since f is nano open and nano continuous and hence f is nano pre-open and nano semi-open continuous function. Hence $f(X), f(Y) \in NPO(V)$. Therefore V is completely nano pre-normal. □

Definition 5.3. *A space U is said to be strongly nano pre-normal if for every pair of disjoint nano pre-closed sets A and B in U, there exist disjoint nano pre-open sets X and Y such that* $A \subset X$ *and* $B \subset Y$.

Definition 5.4. *A space U is said to be nano generalized pre-normal if for every pair of disjoint nano generalized pre-closed sets A and B, there exist disjoint nano pre-open sets X and Y such that* $A \subset X$ *and* $B \subset Y$.

Definition 5.5. *A space U is said to be nano generalized pre-regular if for every Ngp-closed set F and a point* $x \notin F$*, there exist disjoint nano pre-open sets X and Y such that* $F \subset X$ *and* $U \subset Y$.

Theorem 5.2. *For a space* U*, the following are equivalent.*

(a) U *is strongly nano normal.*

(b) *For every nano pre-closed set F and every nano generalized pre-open set G containing F, there is a nano pre-open set X such that $F \subset X \subset X^* \subset G$.*

(c) *For every nano generalized pre-closed set F and every nano pre-open set G containing F, there is a nano pre-open set X such that $F \subset X \subset X^* \subset G$.*

(d) *For every pair of disjoint sets F_1 and F_2, one of which is nano pre-closed and the other is nano generalized pre-closed, there exist nano pre-open sets X and Y such that $F_1 \subset X$ and $F_2 \subset Y$ and $X^* \cap Y^* = \phi$.*

Proof. The proof is similar to the proof of Theorem 2.1. □

Theorem 5.3. *For a space U, the following are equivalent.*

(a) *U is nano generalized pre-normal.*

(b) *For every generalized pre-closed set$F$ and every generalized pre-open set G containing F, there is a nano pre-open set X such that $F \subset X \subset X^* \subset G$.*

Proof. $(a) \Rightarrow (b)$: Let F be nano generalized pre-closed and G be nano generalized pre-open set containing F, then $F \cap (U - G) = \phi$, $U - G$ is also nano generalized pre-closed set. Then by (a), there exist disjoint nano pre-open sets X and Y such that $F \subset X$ and $U - G \subset Y$ that is, $U - Y \subset G$. Now, $X \cap Y = \phi$ implies $X \subset U - Y$. As $U - Y$ is nano pre-closed, $X^* \subset U - Y$. Therefore $F \subset X \subset X^* \subset G$.

$(b) \Rightarrow (a)$ Obvious. □

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