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GENERALIZED REGULAR BLOCK INTUITIONISTIC FUZZY MATRICES

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ABSTRACT. In this paper, equivalent conditions for k-regularity of block triangular intuitionistic fuzzy matrices are obtained. Necessary and sufficient conditions are established for the k-regularity of block intuitionistic fuzzy matrices in terms of the Schur complements of its k-regular diagonal blocks.

1. INTRODUCTION

Matrices partition is easy way to find sum and product of smaller matrices. Blocks or cells of the matrix $A = [a_{ij}]_{m\times n}$ are obtained by using horizontal lines between rows and vertical lines between colums. If a matrix is with large order the primary memory of a computer is unable to store the entire matrix. In this case partitioning the matrices is useful for matrix operation. In intuitionistic fuzzy matrices, partitioning of matrices is very useful to perform the matrix operations.

In [19], Zadeh introduced the concept of fuzzy set with the membership function and the operations on fuzzy sets are developed. Atanssov [1], defined the concept of intuitionistic fuzzy set and the generalization of fuzzy set, also the operations and relations on intuitionistic fuzzy sets are defined. In [4], Cho introduced regularity properties and ranks of fuzzy matrices, the generalized inverses of a fuzzy matrix and the solutions of fuzzy equations are studied.

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Cen [3], studied the concept of generalized inverses of fuzzy matrices. In [7], Kim and Roush defined the basis of row and column and also the concept of rank of row and column for the fuzzy matrices is introduced. Further the condition for a fuzzy matrix to be regular is discussed. In [10], Meenakshi and Jenita introduced the concept of k-regular fuzzy matrix and regularity index of a matrix. In [11], Meenakshi and Jenita introduced the concept of k-regularity of block fuzzy matrices. Pradhan and Pal [14], studied the decomposition of block intuitionistic fuzzy matrix to upper triangular idempotent IFM and lower triangular IFM. In [15], Pradhan and Pal defined the pseudo-similar IFM and properties of pseudo-similar and semi-similar IFMs are discussed. Paul, Khan and Shyamal [13], studied the concept of intuitionistic fuzzy matrices and its properties. In [12], Meenakshi and Gandhimathi studied the regularity and various g-inverse of intuitionistic fuzzy matrices over Intuitionistic fuzzy algebra. Khan and Paul [8], discussed the concept of g-inverse for IFM and defined the partial orderings for IFMs. In [2] Bhowmik and Pal studied IFM and distinguish the valid and invalid operations between intuitionistic fuzzy matrices and generalized intuitionistic fuzzy matrices.

In [16], a problem of reducing intuitionistic fuzzy matrices is examined and some useful properties are obtained with respect to nilpotent intuitionistic fuzzy matrices. In [17], Szpilrajn's theorem on ordering is generalized to intuitionistic fuzzy orderings. In [18], Riyaz Ahmad Padder and Murugadas introduced the max-max operations on intuitionistic fuzzy matrices to study the conditions for convergence of intuitionistic fuzzy matrices. Recently, Jenita, Karuppusamy and Thangamani [5] introduced the concept of k-regular IFM as a generalization of regular IFM. In this paper, we introduce the concept of k-regular block intuitionistic fuzzy matrices as a generalization of results found in [11] and [14].

2. PRELIMINARIES

Here, we are concerned with fuzzy matrices, that is matrices over a fuzzy algebra FM(FN) with support $[0, 1]$, under maxmin(minmax) operations and the usual ordering of real numbers. Let $(IF)_{m\times n}$ be the set of all intuitionistic fuzzy matrices of order $m \times n$, $F^M_{m\times n}$ be the set of all fuzzy matrices of order $m \times n$, under the maxmin composition and $F^N_{m\times n}$ be the set of all fuzzy matrices of order $m \times n$, under the minmax composition.

Let F be a fuzzy algebra over the support $[0,1]$ with max-min operations $(+,\cdot)$ defined as $a+b=max\{a,b\}$ and $a\cdot b=min\{a,b\}$ for all $a,b\in[0,1]$. Let $F_{m\times n}$ be the set of all $m\times n$ fuzzy matrices over F. In short F_n denotes $F_{n\times n}$. $A\in F_{m\times n}$ is said to be regular if there exists X such that $AXA=A$, X is called g-inverse of A. If $A=(a_{ij})_{m\times n}\in(IF)_{m\times n}$, then $A=(\langle a_{ij\mu},a_{ij\vartheta}\rangle)_{m\times n}$, where $a_{ij\mu}$ and $a_{ij\vartheta}$ are the membership values and non membership values of a_{ij} in A respectively with respect to the fuzzy sets μ and ϑ, maintaining the condition $0\le a_{ij\mu}+a_{ij\vartheta}\le 1$.

We shall follow the matrix operations on intuitionistic fuzzy matrices as defined in [12]. For $A,B\in(IF)_{m\times n}$, the following operations are defined:

$$A+B=(\langle\max\{a_{ij\mu},b_{ij\mu}\},\min\{a_{ijv},b_{ijv}\}\rangle)\ ,$$

$$AB=\left(\left\langle\max_k\min\{a_{ik\mu},b_{kj\mu}\},\min_k\max\{a_{ikv},b_{kjv}\}\right\rangle\right)\ .$$

Let us define the order relation on $(IF)_{m\times n}$ as: $A\le B\Leftrightarrow a_{ij\mu}\le b_{ij\mu}$ and $a_{ij\vartheta}\ge b_{ij\vartheta}$, for all i and j.

In this work, we shall represent $A\in(IF)_{m\times n}$ as Cartesian product of fuzzy matrices. For $A=(a_{ij})_{m\times n}\in(IF)_{m\times n}$, let $A=(a_{ij})_{m\times n}=(\langle a_{ij\mu},a_{ij\vartheta}\rangle)_{m\times n}\in(IF)_{m\times n}$. We define $A_\mu=(a_{ij\mu})_{m\times n}\in F^M_{m\times n}$ as the membership part of A and $A_\vartheta=(a_{ij\vartheta})_{m\times n}\in F^N_{m\times n}$ as the non-membership part of A. Thus A is written as the Cartesian product of A_μ and A_ϑ, $A=<A_\mu,A_\vartheta>$ with $A_\mu\in F^M_{m\times n},A_\vartheta\in F^N_{m\times n}$.

For $A\in(IF)_{m\times n}$, $R(A),C(A)$ and A^T denotes the row space, column space and transpose of A, respectively.

Definition 2.1. *[12] For $A,B\in(IF)_{m\times n}$, if $A=<A_\mu,A_\vartheta>$ and $B=<B_\mu,B_\vartheta>$ then*

$$A+B=<A_\mu+B_\mu,A_\vartheta+B_\vartheta>$$

Definition 2.2. *[12] For $A\in(IF)_{m\times p}$ and $B\in(IF)_{p\times n}$ if $A=<A_\mu,A_\vartheta>$ and $B=<B_\mu,B_\vartheta>$, then:*

(i) *$AB=<A_\mu B_\mu,A_\vartheta B_\vartheta>$, where $A_\mu B_\mu$ is the max min product in $F^M_{m\times n}$ and $A_\vartheta B_\vartheta$ is the min max product in $F^N_{m\times n}$.*
(ii) *$A^T=<A^T_\mu,A^T_\vartheta>$.*

Definition 2.3. *[12] A matrix $A\in(IF)_n$ is said to be invertible iff there exists $X\in(IF)_n$ such that $AX=XA=I_n=<I^M_n,I^N_n>$, where I_n is the identity matrix in $(IF)_n$.*

Definition 2.4. *[12] A square intuitionistic fuzzy matrix is called intuitionistic fuzzy permutation matrix if every row and column contains exactly one $< 1, 0 >$ and all the other entries are $< 0, 1 >$.*

Let P_n be the set of all $n \times n$ permutation matrices in $(IF)_n$.

Definition 2.5. *[8] An $A \in (IF)_{m\times n}$ is said to be regular if there exists $X \in (IF)_{m\times n}$ satisfying $AXA = A$ and X is called a generalized inverses (g-inverse) of A, which is denoted by $\overline{A}$.*

Let $A\{1\}$ be the set of all g-inverses of A.

Theorem 2.1. *[12] Let $A \in (IF)_{m\times n}$ be of the form $A = \langle A_\mu, A_\vartheta \rangle$. Then A is regular $\Leftrightarrow$ A_μ is regular in $F^M_{m\times n}$ under max-min composition and A_ϑ is regular in $F^N_{m\times n}$ under min-max composition. $A_\mu = (a_{ij\mu})_{m\times n} \in F^M_{m\times n}$ as the membership part of A and $A_\vartheta = (a_{ij\vartheta})_{m\times n} \in F^N_{m\times n}$ as the non-membership part of A.*

Definition 2.6. *[5] A matrix $A \in (IF)_n$, is said be right k-regular if there exists a matrix $X \in (IF)_n$ such that $A^kXA = A^k$, for some positive integer k. X is called a right k-g-inverse of A.*

Let $A_r\left\{1^k\right\} = \left\{X/A^kXA = A^k\right\}$.

Definition 2.7. *[5] A matrix $A \in (IF)_n$, is said be left k-regular if there exists a matrix $Y \in (IF)_n$ such that $AYA^k = A^k$, for some positive integer k. Y is called a left k-g-inverse of A.*

Let $A_\ell\left\{1^k\right\} = \left\{Y/AYA^k = A^k\right\}$. $A\left\{1^k\right\} = A_r\left\{1^k\right\} \cup A_\ell\left\{1^k\right\}$.

Lemma 2.1. *[12] For $A, B \in (IF)_{m\times n}$, $R(B) \subseteq R(A) \Leftrightarrow B = XA$ for some $X \in (IF)_m$, $C(B) \subseteq C(A) \Leftrightarrow B = AY$ for some $Y \in (IF)_n$.*

Lemma 2.2. *[12] If $A \in (IF)_{m\times n}$ is of the form $A = \langle A_\mu, A_\vartheta \rangle$, it hold:*

(i) $R(A) = \langle R(A_\mu), R(A_\vartheta)\rangle$
(ii) $C(A) = \langle C(A_\mu), C(A_\vartheta)\rangle$.

Lemma 2.3. *[6] For $A, B \in (IF)_n$, and a positive integer k, it hold:*

(i) *If A is a right k-regular and for some $R(B) \subseteq R(A^k) \Rightarrow B = BXA$ for each right k-g inverse X of A.*
(ii) *If A is a left k-regular and for some $C(B) \subseteq C(A^k) \Rightarrow B = AYB$ for each left k-g inverse Y of A.*

Theorem 2.2. *[5] Let $A = \langle A_\mu, A_\vartheta \rangle \in (IF)_n$. Then A is right(left) k-regular IFM $\Leftrightarrow A_\mu, A_\vartheta \in F_n$ are right(left) k-regular.*

Remark 2.1. *Each element of the set $A\{1^k\} = A\{1_r^k\} \cup A\{1_\ell^k\}$ is called a k-g inverse of A. If A is k-regular then A is q-regular for all integer $q \geq k$.*

Theorem 2.3. *[9] Let M be of the form $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $R(C) \subseteq R(A)$ and $C(B) \subseteq C(A)$ .Then the following are equivalent:*

(i) *$R(B) \subseteq R(D), C(C) \subseteq C(D)$, the Schur complements M/A and M/D are fuzzy matrices.*

(ii) *M is regular; BD^-C is invariant and*

$$m = \begin{bmatrix} A^- + A^-BD^-CA & A^-BD^- \\ D^-CA^- & D^- \end{bmatrix}$$

is a g-inverse of M for some g-inverse A^- of A and D^- of D.

Definition 2.8. *[7] A set S of vectors over a semiring R is independent if and only if for no $v \in S$ is a linear combination of elements of $S/\{v\}$. If v is a linear combination of elements of $S/\{v\}$ it is said to be dependent.*

Definition 2.9. *[7] A basis C over the fuzzy algebra is a standard basis if and only if whenever $c_i = \sum a_{ij}c_j$ for $c_i, c_j \in C$ then $a_{ii}c_i = c_i$.*

3. k-REGULARITY OF BLOCK AND TRIANGULAR BLOCK INTUITIONISTIC FUZZY MATRICES

In this section, we shall derive the equivalent conditions for k-regularity of a block IFM of the form $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with the diagonal blocks A and D are k-regular with respect to this partitioning. A Schur complement of A in M is a matrix of the form $M/A = D - CXB$, where X is some k-g inverse of A. Similarly $M/D = A - BYC$ is a Schur complement of D in M, where Y is some k-g inverse of D. In Theorem 3.1, under certain conditions, it is shown that CXB is invariant for all choices of k-g inverse of A. By M/A is an IFM, we mean that CXB is invariant and $D \geq CXB$. Therefore M/A is an IFM $\Leftrightarrow CXB$ is invariant and

$$D = D + CXB\,. \tag{3.1}$$

Similarly, M/D is an IFM $\Leftrightarrow BYC$ is invariant and $A = A + BYC$.

Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ can be expressed as $M = \langle M_\mu, M_\vartheta \rangle$, where $M_\mu = \begin{bmatrix} A_\mu & B_\mu \\ C_\mu & D_\mu \end{bmatrix}$ and $M_\vartheta = \begin{bmatrix} A_\vartheta & B_\vartheta \\ C_\vartheta & D_\vartheta \end{bmatrix}$ are block IFM. $A = \langle A_\mu, A_\vartheta \rangle, B = \langle B_\mu, B_\vartheta \rangle, C = \langle C_\mu, C_\vartheta \rangle$ and $D = \langle D_\mu, D_\vartheta \rangle$. Since A and D are k- regular, $A_\mu, A_\vartheta, D_\mu$ and D_ϑ are all k-regular IFMs.

Also, we inestigate the k-regularity of block triangular IFM of the form:

$$\begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \text{ or } \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}.$$

In Theorem 3.3, it is shown that, M is k-regular $\Leftrightarrow M^T$ is k-regular and the transpose of a lower block triangular matrix is an upper block triangular matrix, throughout we shall only investigate the case of lower block triangular IFM. We derive the equivalent conditions for k-regularity of block IFM of the form:

$$M = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \; withA, D \in (IF)_n. \tag{3.2}$$

For any positive integer s, it can be easily verified that,

$$M^s = \begin{bmatrix} A^s & 0 \\ \sum_{i=0}^{s-1} D^i C A^{s-1-i} & D^s \end{bmatrix}. \tag{3.3}$$

Theorem 3.1. *Let $A \in (IF)_n$ be a k-regular intuitionistic fuzzy matrix, $C \in (IF)_n$ and $B \in (IF)_n$ if $R(C) \subseteq R(A^k)$ and $C(B) \subseteq C(A^k)$, then CXB is invariant for all choice of k-g inverses of A.*

Proof.

Case (i): A is right k-regular.
By Lemma 2.1, $R(C) \subseteq R(A^k) \Rightarrow C = YA^k$ for some $Y \in (IF)_n$ and $C(B) \subseteq C(A^k) \subseteq C(A) \Rightarrow B = AU$ for some $U \in (IF)_n$. Since $A \in (IF)_n$ is a right k-regular intuitionistic fuzzy matrix, by Lemma 2.3, $R(C) \subseteq R(A^k) \Rightarrow C = CZA$ for each $Z \in A\left\{1_r^k\right\}$.

Hence for any $X \in A\{1_r^k\}$, $CXB = (YA^k)X(AU) = YA^kXAU = Y(A^kXA)U = YA^kU = CU = CZAU = CZ(AU) = CZB$. Thus $CXB = CZB$ for all $X, Z \in A\{1_r^k\}$

Case (ii): A is left k-regular.
By Lemma 2.1, $R(C) \subseteq R(A^k) \subseteq R(A) \Rightarrow C = YA$ for some $Y \in (IF)_n$ and $C(B) \subseteq C(A^k) \Rightarrow B = A^kU$ for some $U \in (IF)_n$. Since $A \in (IF)_n$ is a left k- regular intuitionistic fuzzy matrix, by Lemma 2.3, $C(B) \subseteq C(A^k) \Rightarrow B = AZB$ for each $Z \in A\{1_\ell^k\}$. Hence for any $X \in A\{1_\ell^k\}$, $CXB = (YA)X(A^kU) = Y(AXA^k)U = YA^kU = YB = Y(AZB) = (YA)(ZB) = CZB$. Thus $CXB = CZB$ for all $X, Z \in A\{1_\ell^k\}$.

Case (iii): A is both right and left k-regular.
By Lemma 2.1, $R(C) \subseteq R(A^k) \Rightarrow C = YA^k$ for some $Y \in (IF)_n$. Since $A \in (IF)_n$ is a left k-regular intuitionistic fuzzy matrix, by Lemma 2.3, $C(B) \subseteq C(A^k) \Rightarrow B = AZB$ for each $Z \in A\{1_\ell^k\}$. Since $A \in (IF)_n$ is a right k-regular intuitionistic fuzzy matrix, for any $X \in A\{1_r^k\}$, $CXB = (YA^k)X(AZB) = Y(A^kXA)ZB = YA^kZB = CZB$. Thus $CXB = CZB$ for all $X \in A\{1_r^k\}$ and $Z \in A\{1_\ell^k\}$. Thus CXB is invariant for all choices of $k-g$ inverses of A.

□

Example 1. *Let us consider* $A = \begin{bmatrix} \langle 0.3, 0\rangle & \langle 0, 1\rangle \\ \langle 0.5, 0\rangle & \langle 0.2, 0\rangle \end{bmatrix} \in (IF)_2$, *where* $A_\mu = \begin{bmatrix} 0.3 & 0 \\ 0.5 & 0.2 \end{bmatrix} \in F_2^M$ *and* $A_\vartheta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in F_2^N$. *Since each row of* A_μ *cannot be expressed as linear combination of the other row, by Definition 2.8, the rows are linearly independent. By Definition 2.9, they form a standard basis for the row space of* A_ϑ.

For both permutation matrices $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A_\mu P_1 A_\mu = \begin{bmatrix} 0.3 & 0 \\ 0.3 & 0.2 \end{bmatrix} \neq A_\mu$ *and* $A_\mu P_2 A_\mu = \begin{bmatrix} 0.3 & 0.2 \\ 0.5 & 0.2 \end{bmatrix} \neq A_\mu$. *Hence* A_μ *is regular. Namely,* A_μ *is regular iff* $A_\mu P A_\mu = A_\mu$ *for some permutation matrix* P. *Since* A_ϑ *is idempotent,* A_ϑ *itself is a* g*-inverse of* A_ϑ, *therefore* A_ϑ *is regular under min max composition. Hence by Theorem 2.1,* A *is not regular.*

For this A, $A^2 = \begin{bmatrix} \langle 0.3,0\rangle & \langle 0,1\rangle \\ \langle 0.3,0\rangle & \langle 0.2,0\rangle \end{bmatrix}$. For $X = \begin{bmatrix} \langle 1,0\rangle & \langle 0,1\rangle \\ \langle 0,0\rangle & \langle 0.3,0\rangle \end{bmatrix}$, $A^2XA = A^2 = AXA^2$ holds. A is 2-regular. Hence X is a 2-g inverse of A.

Consider another 2-g inverse Y of A satisfies the equation $A^2YA = A^2$.

$$A^2YA = \begin{bmatrix} \langle 0.3,0\rangle & \langle 0,1\rangle \\ \langle 0.3,0\rangle & \langle 0.2,0\rangle \end{bmatrix} \begin{bmatrix} \langle 1,0\rangle & \langle 0,1\rangle \\ \langle 0.5,0.5\rangle & \langle 0.2,0.2\rangle \end{bmatrix} \begin{bmatrix} \langle 0.3,0\rangle & \langle 0,1\rangle \\ \langle 0.5,0\rangle & \langle 0.2,0\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle 0.3,0\rangle & \langle 0,1\rangle \\ \langle 0.3,0\rangle & \langle 0.2,0\rangle \end{bmatrix} = A^2$$

Since $C(B) \subseteq C(A^2), B = AYB, Y \in A\{1_r^2\}$. So, we take:

$$B = AYB = \begin{bmatrix} \langle 0.2,0.5\rangle & \langle 0.3,0.5\rangle \\ \langle 0.2,0.3\rangle & \langle 0.3,0.3\rangle \end{bmatrix}$$

Since $R(C) \subseteq R(A^2), C = UA^2$ for $U \in (IF)_2$. Take $U = \begin{bmatrix} \langle 0.6,0.2\rangle & \langle 0.5,0.4\rangle \\ \langle 0.7,0.3\rangle & \langle 0.5,0.4\rangle \end{bmatrix}$.

$C = UA^2 \Rightarrow$

$$C = \begin{bmatrix} \langle 0.6,0.2\rangle & \langle 0.5,0.4\rangle \\ \langle 0.7,0.3\rangle & \langle 0.5,0.4\rangle \end{bmatrix} \begin{bmatrix} \langle 0.3,0\rangle & \langle 0,1\rangle \\ \langle 0.3,0\rangle & \langle 0.2,0\rangle \end{bmatrix} = \begin{bmatrix} \langle 0.3,0.2\rangle & \langle 0.2,0.4\rangle \\ \langle 0.3,0.3\rangle & \langle 0.2,0.4\rangle \end{bmatrix}$$

Now,

$$CXB = \begin{bmatrix} \langle 0.3,0.2\rangle & \langle 0.2,0.4\rangle \\ \langle 0.3,0.3\rangle & \langle 0.2,0.4\rangle \end{bmatrix} \begin{bmatrix} \langle 1,0\rangle & \langle 0,1\rangle \\ \langle 0,0\rangle & \langle 0.3,0\rangle \end{bmatrix} \begin{bmatrix} \langle 0.2,0.5\rangle & \langle 0.3,0.5\rangle \\ \langle 0.2,0.3\rangle & \langle 0.3,0.3\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle 0.2,0.4\rangle & \langle 0.3,0.4\rangle \\ \langle 0.2,0.4\rangle & \langle 0.3,0.4\rangle \end{bmatrix}$$

and

$$CYB = \begin{bmatrix} \langle 0.3,0.2\rangle & \langle 0.2,0.4\rangle \\ \langle 0.3,0.3\rangle & \langle 0.2,0.4\rangle \end{bmatrix} \begin{bmatrix} \langle 1,0\rangle & \langle 0,1\rangle \\ \langle 0,0\rangle & \langle 0.3,0\rangle \end{bmatrix} \begin{bmatrix} \langle 0.2,0.5\rangle & \langle 0.3,0.5\rangle \\ \langle 0.2,0.3\rangle & \langle 0.3,0.3\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle 0.2,0.4\rangle & \langle 0.3,0.4\rangle \\ \langle 0.2,0.4\rangle & \langle 0.3,0.4\rangle \end{bmatrix}$$

Hence CXB is invariant for any k-g inverses of A.

Theorem 3.2. *Let $A \in (IF)_n$ and k be a positive integer, then $X \in A\left\{1_r^k\right\} \Leftrightarrow X^T \in A^T\left\{1_\ell^k\right\}$.*

Proof.

$$\begin{aligned} X \in A\left\{1_r^k\right\} &\Leftrightarrow A^kXA = A^k \\ &\Leftrightarrow (A^kXA)^T = (A^k)^T \\ &\Leftrightarrow A^TX^T(A^T)^k = (A^T)^k \\ &\Leftrightarrow X^T \in A^T\left\{1_\ell^k\right\} \end{aligned}$$

□

Theorem 3.3. *For any block triangular IFM,* $M = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$ *with* $A, D \in (IF)_n$*, if* M *is* k*-regular for some positive integer* k *and* M *has a lower block triangular* k*-g inverse, then the block* A *and* D *are* k*-regular IFM.*

Proof. Since M is k-regular, let us assume M is right k-regular and M has a lower block triangular right k-g inverse $U = \begin{bmatrix} X & 0 \\ Z & Y \end{bmatrix} \in (IF)_n$ satisfying $M^kUM = M^k$. Since M is of the form (3.1), M^k is of the form (3.2)), comparing the corresponding diagonal blocks on both sides, we get $A^kXA = A^k$ and $D^kYD = D^k$. Thus both A and D are right k-regular IFM.

Similarly, if M is left k-regular and M has a lower block triangular left k-g inverse, $U = \begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix} \in (IF)_n$ is satisfying $MUM^k = M^k$ on comparing the blocks $AXA^k = A^k$ and $DYD^k = D^k$.

Thus both A and D are left k-regular IFM. □

Theorem 3.4. *Let* M *be of the form* (3.1) *with* A *is right* k_1*-regular and* D *right* k_2*-regular. If* $R(C) \subseteq R(A^k)$ *and* $C(C) \subseteq C(D)$*, then* M *is right* k*-regular IFM, where* $k = max\{k_1, k_2\}$*.*

Proof. Since $k > k_1, k_2$, by Remark 2.1, both A and D are right k-regular. Hence,

(3.4) $$A^kXA = A^k \; and \; D^kYD = D^k, \; for some \; X, Y \in (IF)_n$$

Let us take $Z = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in (IF)_n$. Since $R(C) \subseteq R(A^k)$,

$$R(C) \subseteq R(A^k) \subseteq R(A^{k-1}) \subseteq R(A^{k-2}) \subseteq \dots\dots \subseteq R(A)$$

By Lemma 2.1, there exist matrices $X_k, X_{k-1}, \ldots\ldots.., X_2, X_1 \in (IF)_n$ such that

$$C = X_k A^k = X_{k-1}A^{k-1} \ldots\ldots = X_2 A^2 = XA. \tag{3.5}$$

Since $C(C) \subseteq C(D)$, by Lemma 2.1

$$C = DV \ for some \ V \in (IF)_n. \tag{3.6}$$

From (3.2) and (3.3), we get:

$$M^k Z M = \begin{bmatrix} A^k & 0 \\ (\sum_{i=0}^{k-1} D^i C A^{k-1-i})XA + D^k Y C & D^k \end{bmatrix}.$$

We claim that Z is a right k-g inverse of M. It is enough to prove that:

$$(\sum_{i=0}^{k-1} D^i C A^{k-1-id})XA + D^k Y C = \sum_{i=0}^{k-1} D^i C A^{k-1-i}. \tag{3.7}$$

Using equations (3.3) and (3.4), we have:

$$\begin{aligned}
(\sum_{i=0}^{k-1} D^i C A^{k-1-i})XA =& D^0 C A^{k-1} XA + \cdots + D^{k-1} C A^0 XA \\
=& D^0 (X_1 A^1) A^{k-1} XA + D^1 (X_2 A^2) A^{k-2} XA + \cdots \\
& + D^{k-1} (X_k A^k) A^0 XA \\
=& D^0 X_1 A^k XA + D^1 X_2 A^k + \cdots + D^{k-1} X_k A^k \\
=& D^0 C A^{k-1} + D^1 C A^{k-2} + \cdots + D^{k-1} C
\end{aligned}$$

Now by using (3.5) and (3.3), the last term $D^k YC$ in the L.H.S of (3.7) can be written as

$$D^k Y C = D^k Y(DV) = D^k V = D^{k-1}(DV) = D^{k-1} C.$$

Therefore,

$$\begin{aligned}
(\sum_{i=0}^{k-1} D^i C A^{k-1-i})XA + D^k Y C =& (\sum_{i=0}^{k-1} D^i C A^{k-1-i})XA + D^{k-1} C \\
=& \sum_{i=0}^{k-1} D^i C A^{k-1-i}.
\end{aligned}$$

Thus $M^k Z M = M^k$ and M is right k-regular. Hence the theorem is proved. □

Theorem 3.5. *Let M be of the form* (3.1) *with A left k_1-regular and D left k_2-regular. If $R(C) \subseteq R(A)$ and $C(C) \subseteq C(D^k)$, then M is left k-regular IFM, where $k = max\,\{k_1, k_2\}$.*

Proof. This can be proved along the same lines as that of Theorem 3.4 and hence omitted. ☐

Remark 3.1. *If $k_1 = k_2 = k$ in Theorem 3.4 and Theorem 3.5, then by using Lemma 2.3, it can be verified that $U = \begin{bmatrix} X & 0 \\ YCX & Y \end{bmatrix}$ is a right(left) k-g inverse of M for each right(left) k-g inverse X of A and Y of D.*

Theorem 3.6. *Let M be of the form* (3.1)*, $M = UL$ where $U = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$ and $L = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ satisfying $C = XA = DX$, then M is k-regular and M has a lower block triangular k-g inverse $\Leftrightarrow$ the blocks A and D are k-regular.*

Proof.

$$M = UL = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ XA & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$$

and

$$M = LU = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ DX & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}.$$

Therefore, $LU = UL$.
Now,

$$M^2 = (UL)^2 = ULUL = ULLU = UL^2U\,,$$

and

$$M^3 = (UL)^3 = ULULUL = UL^3U\,.$$

Therefore, in general,

$$M^k = (UL)^k = UL^kU$$

$U^2 = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} = U$ and hence U is idempotent. Since A and D are k-regular, $A^kYA = A^k$ for some k-g inverse Y of A and $D^kZD = D^k$ for some k-g inverse

Z of D.

$$L^kL^-L = \begin{bmatrix} A^k & 0 \\ 0 & D^k \end{bmatrix} \begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$
$$= \begin{bmatrix} A^kYA & 0 \\ 0 & D^kZD \end{bmatrix} = \begin{bmatrix} A^k & 0 \\ 0 & D^k \end{bmatrix} = L^k .$$

Similarly, $LL^-L^k = L^k$. Thus L is k-regular and L^- is a k-g inverse of L. We claim that L^-U is a k-g inverse of M.

$$M^k(L^-U)M = (UL^kU)(L^-U)(UL) .$$

Since $UL = LU$, L is k-regular and U is idempotent,

$$M^k(L^-U)M = UL^kU = M^k$$

Similarly, $M(L^-U)M^k = M^k$. Thus M is k-regular and L^-U is a lower block k-g inverse of M.

Conversely, by Theorem 3.4, if M is k-regular and M has a lower block triangular k-g inverse $M^- = \begin{bmatrix} Y & 0 \\ ZX & Z \end{bmatrix}$, then the blocks A and D are k-regular. $\square$

Theorem 3.7. *Let M be of the form $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $R(C) \subseteq R(A^k)$, $C(C) \subseteq C(D^k)$, $C(B) \subseteq C(A^k)$ and $R(B) \subseteq R(D^k)$, the Schur complements M/A and M/D are intuitionistic fuzzy matrices, then M is k-regular and*

$$m = \begin{bmatrix} X + XBYCX & XBY \\ YCX & Y \end{bmatrix} \tag{3.8}$$

is a k-g inverse of M for some k-g inverse X of A and Y of D, respectively.

Proof. Since A is k-regular with $R(C) \subseteq R(A^k)$ and $C(B) \subseteq C(A^k)$ by Lemma 2.3, $C = CXA, B = AXB$ for each k-g inverse X of A. Since M/A is a IFM, it follows that CXB is invariant for all choices of k-g inverses of A and $D = D + CXB$. Now, under the given conditions, M can be expressed as $M = ULV$ where

$$U = \begin{bmatrix} I & 0 \\ CX & I \end{bmatrix}, L = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, V = \begin{bmatrix} I & XB \\ 0 & I \end{bmatrix}.$$

Let us define $L^- = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$ where X is a k-g inverse of A and Y is a k-g inverse of D. On computation, we see that $VL^-U = V \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} U = m$ defined in (3.7). By using induction on k, let us prove that M is k-regular.

For $k = 1$, the result is directly follows from Theorem 2.3.

For $k = 2$, $M^2mM = (ULV)^2(VL^-U)(ULV)$. Since U and V are idempotent matrices, $M^2mM = (ULV)^2L^-(ULV) = M^2L^-M$.
Let $M^2 = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ then $\begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} A^2 + BC & AB + BD \\ CA + DC & CB + D^2 \end{bmatrix}$. Hence

$$P = A^2 + BC \; and \; Q = AB + BD. \tag{3.9}$$

$$\begin{aligned} M^2mM = M^2L^-M &= \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ &= \begin{bmatrix} PXA + QYC & PXB + QYD \\ RXA + SYC & RXB + SYD \end{bmatrix}. \end{aligned}$$

Now, we prove that $M^2mM = M^2L^-M = M^2$. First we prove that the $(1,1)^{th}$ block of M^2L^-M and that of M^2 are equal. For this, it is enough to prove that $A^2 + BC = PXA + QYC$. By induction hypothesis, the given conditions reduce to the following:

$$A \text{ is 2-regular} \Rightarrow A^2XA = A^2.$$

By Lemma 2.1,

$$C(C) \subseteq C(D^2) \Rightarrow C = DYC, \tag{3.10}$$

$$R(C) \subseteq R(A^2) \Rightarrow C = CXA, \tag{3.11}$$

$$M/D \; is \; an \; IFM \Rightarrow A = A + BYC. \tag{3.12}$$

Using equations (3.8) to (3.12) yields:

$$\begin{aligned}PXA+QYC=&A^2XA+BCXA+ABYC+BDYC\\=&A^2+BC+ABYC+BC\\=&A^2+BC+ABYC\\=&A(A+BYC)+BC\\=&AA+BC=A^2+BC\,.\end{aligned}$$

Thus, the $(1,1)^{th}$ block of M^2L^-M and the $(1,1)^{th}$ block of M^2 are equal. Similarly, it can be verified that the remaining blocks of M^2L^-M and M^2 are equal. Hence M is 2-regular.

Assume that $M^{k-1}L^-M=M^{k-1}$, then:

$$M^kL^-M=M(M^{k-1}L^-M)=MM^{k-1}=M^k\,.$$

Hence,

$$M^kmM=M^kL^-M=M^k\,.$$

Thus M is k-regular and m is a k-g inverse of M.
Thus the Theorem is proved. □

Example 2. *Let* $M=\begin{bmatrix}\langle 0.3,0\rangle & \langle 0,1\rangle & \vdots & \langle 0.2,0.4\rangle & \langle 0.1,0.4\rangle\\ \langle 0.5,0\rangle & \langle 0.2,0\rangle & \vdots & \langle 0.2,0.3\rangle & \langle 0.2,0.3\rangle\\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots\cdots\\ \langle 0.2,0.2\rangle & \langle 0,1\rangle & \vdots & \langle 0.2,0.3\rangle & \langle 0.1,0\rangle\\ \langle 0.2,0.2\rangle & \langle 0,1\rangle & \vdots & \langle 0.4,0\rangle & \langle 0.2,0\rangle\end{bmatrix}$,
where $A=\begin{bmatrix}\langle 0.3,0\rangle & \langle 0,1\rangle\\ \langle 0.5,0\rangle & \langle 0.2,0\rangle\end{bmatrix}, B=\begin{bmatrix}\langle 0.2,0.4\rangle & \langle 0.1,0.4\rangle\\ \langle 0.2,0.3\rangle & \langle 0.2,0.3\rangle\end{bmatrix}$,
$C=\begin{bmatrix}\langle 0.2,0.2\rangle & \langle 0,1\rangle\\ \langle 0.2,0.2\rangle & \langle 0,1\rangle\end{bmatrix}$ *and* $D=\begin{bmatrix}\langle 0.2,0.3\rangle & \langle 0.1,0\rangle\\ \langle 0.4,0\rangle & \langle 0.2,0\rangle\end{bmatrix}$.

$A=\langle A_\mu,A_\vartheta\rangle\,,B=\langle B_\mu,B_\vartheta\rangle\,,C=\langle C_\mu,C_\vartheta\rangle$ *and* $D=\langle D_\mu,D_\vartheta\rangle$.

$A=\begin{bmatrix}\langle 0.3,0\rangle & \langle 0,1\rangle\\ \langle 0.5,0\rangle & \langle 0.2,0\rangle\end{bmatrix}\in(IF)_2$, *where* $A_\mu=\begin{bmatrix}0.3 & 0\\ 0.5 & 0.2\end{bmatrix}\in F_2^M$ *and* $A_\vartheta=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\in F_2^N$. *Since each row of* A_μ *cannot be expressed as linear combination* *of the other row, by Definition 2.8 the rows are linearly independent. By Definition 2.8, they form a standard basis for the row space of* A_μ*. For both permutation matrices* $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$A_\mu P_1 A_\mu = \begin{bmatrix} 0.3 & 0 \\ 0.3 & 0.2 \end{bmatrix} \neq A_\mu$$

and

$$A_\mu P_2 A_\mu = \begin{bmatrix} 0.3 & 0.2 \\ 0.5 & 0.2 \end{bmatrix} \neq A_\mu \, .$$

Hence A_μ *is not regular. Namely,* A_μ *is regular iff* $A_\mu P A_\mu = A_\mu$ *for some permutation matrix* P*. Since* A_ϑ *is idempotent,* A_ϑ *itself is a* g*-inverse of* A_ϑ*, therefore* A_ϑ *is regular under min max composition. Hence by Theorem 2.1,* A *is not regular.*
For this A*,* $A^2 = \begin{bmatrix} \langle 0.3, 0\rangle & \langle 0, 1\rangle \\ \langle 0.3, 0\rangle & \langle 0.2, 0\rangle \end{bmatrix}$*. For* $X = \begin{bmatrix} \langle 1, 0\rangle & \langle 0, 1\rangle \\ \langle 0, 0\rangle & \langle 0.2, 0\rangle \end{bmatrix}$, $A^2XA = A^2 = AXA^2$ *holds. Hence* A *is 2-regular.*

Similarly we can prove that D *is not regular.*
For $D = \begin{bmatrix} \langle 0.2, 0.3\rangle & \langle 0.1, 0\rangle \\ \langle 0.4, 0\rangle & \langle 0.2, 0\rangle \end{bmatrix}$*,* $D^2 = \begin{bmatrix} \langle 0.2, 0.3\rangle & \langle 0.1, 0\rangle \\ \langle 0.2, 0\rangle & \langle 0.2, 0\rangle \end{bmatrix}$.
For $Y = \begin{bmatrix} \langle 0.2, 0.3\rangle & \langle 0.1, 0\rangle \\ \langle 0, 0\rangle & \langle 0.2, 0\rangle \end{bmatrix}$, $D^2YD = D^2 = DYD^2$ *holds.*
Hence D *is 2-regular.*

Since A *and* D *are 2-regular with* $R(C) \subseteq R(A^2), C(C) \subseteq C(D^2), C(B) \subseteq C(A^2)$ *and* $R(B) \subseteq R(D^2)$*, by Lemma 2.3,*

$$C = CXA, C = DYC, B = AXB \text{ and } B = BYD. \tag{3.13}$$

For the above A, B, C, D, X *and* Y*, the set of equations* (3.12) *hold. Also,* $D = D + CXB$ *and* $A = A + BYC$.
Now,

$$X + XBYCX = \begin{bmatrix} \langle 1, 0\rangle & \langle 0, 1\rangle \\ \langle 0.2, 0\rangle & \langle 0.2, 0\rangle \end{bmatrix}$$

$$XBY = \begin{bmatrix} \langle 0.2, 0.4\rangle & \langle 0.1, 0.4\rangle \\ \langle 0.2, 0.3\rangle & \langle 0.2, 0.3\rangle \end{bmatrix}$$

$$YCX = \begin{bmatrix} \langle 0.2, 0.2\rangle & \langle 0, 1\rangle \\ \langle 0.2, 0.2\rangle & \langle 0, 1\rangle \end{bmatrix}$$

$$Y = \begin{bmatrix} \langle 0.2, 0.3\rangle & \langle 0.1, 0\rangle \\ \langle 0, 0\rangle & \langle 0.2, 0\rangle \end{bmatrix}$$

Take,

$$m = \begin{bmatrix} X + XBYCX & XBY \\ YCX & Y \end{bmatrix} = \begin{bmatrix} \langle 1, 0\rangle & \langle 0, 1\rangle & \vdots & \langle 0.2, 0.4\rangle & \langle 0.1, 0.4\rangle \\ \langle 0.2, 0\rangle & \langle 0.2, 0\rangle & \vdots & \langle 0.2, 0.3\rangle & \langle 0.2, 0.3\rangle \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots\cdots \\ \langle 0.2, 0.2\rangle & \langle 0, 1\rangle & \vdots & \langle 0.2, 0.3\rangle & \langle 0.1, 0\rangle \\ \langle 0.2, 0.2\rangle & \langle 0, 1\rangle & \vdots & \langle 0, 0\rangle & \langle 0.2, 0\rangle \end{bmatrix}.$$

Here, $M^2mM = \begin{bmatrix} \langle 0.3, 0\rangle & \langle 0, 0.4\rangle & \vdots & \langle 0.2, 0.4\rangle & \langle 0.1, 0.4\rangle \\ \langle 0.3, 0\rangle & \langle 0.2, 0\rangle & \vdots & \langle 0.2, 0.3\rangle & \langle 0.2, 0.3\rangle \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots\cdots \\ \langle 0.2, 0.2\rangle & \langle 0, 0.2\rangle & \vdots & \langle 0.2, 0\rangle & \langle 0.1, 0\rangle \\ \langle 0.2, 0.2\rangle & \langle 0, 0.2\rangle & \vdots & \langle 0.2, 0\rangle & \langle 0.2, 0\rangle \end{bmatrix} = M^2.$

Hence M is 2-regular and m is a 2-g inverse of M.

Theorem 3.8. *Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a block intuitionistic fuzzy matrix with $R(C) \subseteq R(A), C(B^k) \subseteq C(A^k), R(B) \subseteq R(D)$ and $C(C^k) \subseteq C(D^k)$. If M is right k-regular BIFM then A and D are right k-regular IFM, where $M^k = \begin{bmatrix} A^k & B^k \\ C^k & D^k \end{bmatrix}$.*

Proof. Let $M^k = \begin{bmatrix} A^k & B^k \\ C^k & D^k \end{bmatrix}$. Let M be right k-regular BIFM and $X = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ be a right k-g inverse of M. Hence $M^kXM = M^k$.

$$M^kXM = M^k$$
$$\Rightarrow \begin{bmatrix} A^k & B^k \\ C^k & D^k \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A^k & B^k \\ C^k & D^k \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} A^kP + B^kR & A^kQ + B^kS \\ C^kP + D^kR & C^kQ + D^kS \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A^k & B^k \\ C^k & D^k \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A^kPA + B^kRA + A^kQC + B^kSC & A^kPB + B^kRB + A^kQD + B^kSD \\ C^kPA + D^kRA + C^kQC + D^kSC & C^kPB + D^kRB + C^kQD + D^kSD \end{bmatrix} = \begin{bmatrix} A^k & B^k \\ C^k & D^k \end{bmatrix}.$$

By comparing the corresponding diagonal blocks, we get

$$A^kPA + B^kRA + A^kQC + B^kSC = A^k\,, \tag{3.14}$$

and

$$C^kPB + D^kRB + C^kQD + D^kSD = D^k\,. \tag{3.15}$$

By Lemma 2.1,

$$R(C) \subseteq R(A) \Rightarrow C = UA \text{ for some } U \in (IF)_n$$

$$C(B^k) \subseteq C(A^k) \Rightarrow B^k = A^kV \text{ for some } V \in (IF)_n$$

$$R(B) \subseteq R(D) \Rightarrow B = V_1D \text{ for some } V_1 \in (IF)_n$$

$$C(C^k) \subseteq C(D^k) \Rightarrow C^k = D^kU_1 \text{ for some } U_1 \in (IF)_n\,.$$

By substituing $B^k = A^kV$ and $C = UA$ in (3.14), we get

$$A^k[P + VR + QU + VSU]A = A^k\,.$$

Hence A is right k-regular.

Similarly, by substituting $B = V_1D$ and $C^k = D^kU_1$ in (3.15), we can prove that D is right k-regular. □

Theorem 3.9. *Let* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *be a block intuitionistic fuzzy matrix with* $R(C^k) \subseteq R(A^k), C(B) \subseteq C(A), R(B^k) \subseteq R(D^k)$ *and* $C(C) \subseteq C(D)$. *If* M *is left* k*-regular BIFM then* A *and* D *are left* k*-regular IFM, where* $M^k = \begin{bmatrix} A^k & B^k \\ C^k & D^k \end{bmatrix}$.

Proof. This is similar to that of Theorem 3.8 and hence omitted. □

Remark 3.2. *For* $k = 1$*, Theorem 3.8 and Theorem 3.9 reduces to the following Theorem.*

Theorem 3.10. *[14] Let* $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *be a BIFM with* $R(C) \subseteq R(A), C(B) \subseteq C(A), R(B) \subseteq R(D)$ *and* $C(C) \subseteq C(D)$. *If* X *is regular then* A *and* D *are regular.*

Lemma 3.1. *For $A, B, C \in (IF)_n$, the following statements hold:*

(i) *If $R(C) \subseteq R(A^k)$, then A is right k-regular $\Leftrightarrow$ $\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$ is right k-regular.*

(ii) *If $C(B) \subseteq C(A^k)$, then A is left k-regular $\Leftrightarrow$ $\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ is left k-regular.*

Proof.

(i) Let $M = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$, then it can be easily verified that $M^k = \begin{bmatrix} A^k & 0 \\ CA^{k-1} & 0 \end{bmatrix}$.
From Lemma 2.3, if $R(C) \subseteq R(A^k)$ and A is right k-regular, then $C = CXA$, for each right k-g inverse X of A. We claim that $m = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$ is a right k-g inverse of M.

$$M^k mM = \begin{bmatrix} A^k & 0 \\ CA^{k-1} & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} = \begin{bmatrix} A^kXA & 0 \\ CA^{k-1}XA & 0 \end{bmatrix}.$$

Since A is right k-regular, we have, $A^kXA = A^k$. Furthermore, $C = CXA$. Therefore,

$$\begin{aligned} M^k mM &= \begin{bmatrix} A^kXA & 0 \\ CA^{k-1}XA & 0 \end{bmatrix} = \begin{bmatrix} A^kXA & 0 \\ CXAA^{k-1}XA & 0 \end{bmatrix} \\ &= \begin{bmatrix} A^kXA & 0 \\ CXA^kXA & 0 \end{bmatrix} = \begin{bmatrix} A^k & 0 \\ CXA^k & 0 \end{bmatrix} \\ &= \begin{bmatrix} A^k & 0 \\ CXAA^{k-1} & 0 \end{bmatrix} = \begin{bmatrix} A^k & 0 \\ CA^{k-1} & 0 \end{bmatrix} \\ &= M^k. \end{aligned}$$

Hence M is right k-regular.

Conversely, $M = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$ is right k-regular and by Lemma 2.1 $R(C) \subseteq R(A^k) \Rightarrow C = XA^k$ for some $X \in (IF)_n$.
Hence

$$M = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ XA^k & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ XA^{k-1} & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = UA'$$

where $U = \begin{bmatrix} I & 0 \\ XA^{k-1} & 0 \end{bmatrix}$ and $A' = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$.

Since M is right k-regular, UA is right k-regular. Hence $(UA')^k M^- UA' = (UA')^k$ where M^- is a right k-g inverse of (UA'). Since

$$(UA')^k = U(A')^k, U(A')^k M^- UA' = U(A')^k\,, \tag{3.16}$$

for $U^- = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $U^-U = I$ and U^- is a right k-g inverse of U, pre multiplying (3.16) by U^-, we obtain

$$U^-U(A')^k M^- UA' = U^-U(A')^k \Rightarrow (A')^k M^- UA' = (A')^k\,.$$

Thus A' is right k-regular. Hence A is right k-regular.

(ii) Can be Proved in the same manner.

□

Remark 3.3. *For $k = 1$, the above Lemma reduces to the following result.*

Lemma 3.2. *[11] For the IFMs, A, B, C of order $m \times n$ the following statement hold:*

(i) *If $R(C) \subseteq R(A)$, then A is regular $\Leftrightarrow \begin{bmatrix} A & C \end{bmatrix}^T$ is regular.*
(ii) *If $C(B) \subseteq C(A)$, then A is regular $\Leftrightarrow \begin{bmatrix} A & B \end{bmatrix}$ is regular.*

Theorem 3.11. *Let $A \in (IF)_n$ and k be a positive integer, then the following statement are equivalent:*

(i) *A is k-regular.*
(ii) *λA is k-regular for $\lambda \neq 0 \in F$.*
(iii) *PAP^T is k-regular for some permutation matrix $P \in (IF)_n$.*

Proof.

(i)$\Leftrightarrow$ (ii) Let $A = \langle A_\mu, A_\vartheta \rangle = \langle (a_{ij\mu}), (a_{ij\vartheta}) \rangle$. Then

$$\lambda A = \lambda \langle A_\mu, A_\vartheta \rangle = \langle min(\lambda, a_{ij\mu}), max(\lambda, a_{ij\vartheta}) \rangle = A\lambda$$

and $\lambda.\lambda = \lambda$. A is k-regular $\Rightarrow A^k XA = A^k \Rightarrow (\lambda A)^k X(\lambda A) = (\lambda A)^k$ for $\lambda \neq 0 \in F \Rightarrow \lambda A$ is k-regular. If λA is k-regular, then for $\lambda = 1$, A is k-regular.

(i)$\Leftrightarrow$ (iii) A is k-regular $\Leftrightarrow$ $(PA^kP^T)(PXP^T)(PAP^T) = PA^kP^T$ for some permutation matrix P and $X \in (IF)_n$ $\Leftrightarrow$ $(PAP^T)^k(PXP^T)(PAP^T) = (PAP^T)^k$ $\Leftrightarrow$ PAP^T is k-regular.

Hence the proof. $\square$

Lemma 3.3. *For $A, B \in (IF)_n$, the following statements hold:*

(i) $\begin{bmatrix} A & B \end{bmatrix}$ *is k-regular* $\Leftrightarrow$ $\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ *is k-regular, where* $\begin{bmatrix} A & B \end{bmatrix}^k = \begin{bmatrix} A^k & B^k \end{bmatrix}$.

(ii) $\begin{bmatrix} A \\ B \end{bmatrix}$ *is k-regular* $\Leftrightarrow$ $\begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix}$ *is k-regular, where* $\begin{bmatrix} A \\ B \end{bmatrix}^k = \begin{bmatrix} A^k \\ B^k \end{bmatrix}$.

Proof. (i) Let $\begin{bmatrix} A & B \end{bmatrix}$ be right k-regular and $\begin{bmatrix} X \\ Y \end{bmatrix}$ be a right k-g inverse.

Therefore,

$$\begin{bmatrix} A & B \end{bmatrix}^k \begin{bmatrix} X \\ Y \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix}^k$$

$$\Rightarrow \begin{bmatrix} A^k & B^k \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} A^k & B^k \end{bmatrix}.$$

By equating the corresponding blocks on both sides, $(A^kX+B^kY)A = A^k$ and $(A^kX + B^kY)B = B^k$.

$$\begin{aligned}
\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}^k \begin{bmatrix} X & 0 \\ Y & 0 \end{bmatrix} \begin{bmatrix} A & B \\ B & 0 \end{bmatrix} &= \begin{bmatrix} A^k & B^k \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ Y & 0 \end{bmatrix} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} A^kX + B^kY & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} (A^kX + B^kY)A & (A^kX + B^kY)B \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} A^k & B^k \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}^k .
\end{aligned}$$

Hence $\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ is right k-regular.

Conversely, if $\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ is right k-regular, let $\begin{bmatrix} X & U \\ Y & V \end{bmatrix}$ be a right k-g inverse of $\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$. Hence,

$$\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}^k \begin{bmatrix} X & U \\ Y & V \end{bmatrix} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}^k.$$

Take $\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}^k = \begin{bmatrix} A^k & B^k \\ 0 & 0 \end{bmatrix}$. On computation we get, $(A^kX+B^kY)A = A^k$ and $(A^kX + B^kY)B = B^k$. This can be written as:

$$\begin{bmatrix} A & B \end{bmatrix}^k \begin{bmatrix} X \\ Y \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix}^k.$$

Thus A is right k-regular.

Similarly, we can prove that $\begin{bmatrix} A & B \end{bmatrix}$ is left k-regular $\Leftrightarrow \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ is left k-regular.

Hence the proof follows.

(ii) This can be proved in the same manner.

□

Lemma 3.4. *For $A, B \in (IF)_n$, the following statements hold:*

(i) $\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ *is k-regular* $\Leftrightarrow \begin{bmatrix} 0 & 0 \\ B & A \end{bmatrix}$ *is k-regular.*

(ii) $\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$ *is k-regular* $\Leftrightarrow \begin{bmatrix} 0 & C \\ 0 & A \end{bmatrix}$ *is k-regular.*

Proof. From Theorem 3.11, M is k-regular $\Leftrightarrow PMP^T$ is k-regular for some permutation matrix $P \in (IF)_n$.

(i) $\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ is k-regular $\Leftrightarrow P \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} P^T$ is k-regular for some permutation matrix $P \Leftrightarrow \begin{bmatrix} 0 & 0 \\ B & A \end{bmatrix}$ is k-regular.

(ii) Can be proved in the same manner.

□

Example 3. $P\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}P^T = \begin{bmatrix} 0 & 0 \\ B & A \end{bmatrix}$ *for some permutation matrix* $P \in (IF)_n$. *It is shown in this example.*

Let $P = \langle P_\mu, P_\vartheta \rangle = \begin{bmatrix} \langle 0,1\rangle & \langle 1,0\rangle \\ \langle 1,0\rangle & \langle 0,1\rangle \end{bmatrix} \in (IF)_2$. *Let* $M = \langle M_\mu, M_\vartheta\rangle = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \langle a_{ij\mu}, a_{ij\vartheta}\rangle & \langle b_{ij\mu}, b_{ij\vartheta}\rangle \\ \langle 0,0\rangle & \langle 0,0\rangle \end{bmatrix} \in (IF)_2$, *satisfying the condition* $0 \leq a_{ij\mu} + a_{ij\vartheta} \leq 1$ *and* $0 \leq b_{ij\mu} + b_{ij\vartheta} \leq 1$.

Now,

$$P_\mu M_\mu P_\mu^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_{ij\mu} & b_{ij\mu} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ b_{ij\mu} & a_{ij\mu} \end{bmatrix}$$

and

$$P_\vartheta M_\vartheta P_\vartheta^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a_{ij\vartheta} & b_{ij\vartheta} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ b_{ij\vartheta} & a_{ij\vartheta} \end{bmatrix}$$

$$\begin{aligned} P\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}P^T =& PMP^T = \langle P_\mu, P_\vartheta\rangle \langle M_\mu, M_\vartheta\rangle \left\langle P_\mu^T, P_\vartheta^T\right\rangle \\ =& \left\langle P_\mu M_\mu P_\mu^T, P_\vartheta M_\vartheta P_\vartheta^T\right\rangle \\ =& \begin{bmatrix} \langle 0,0\rangle & \langle 0,0\rangle \\ \langle b_{ij\mu}, b_{ij\vartheta}\rangle & \langle a_{ij\mu}, a_{ij\vartheta}\rangle \end{bmatrix} \\ =& \begin{bmatrix} 0 & 0 \\ B & A \end{bmatrix} \end{aligned}$$

.

4. Conclusion

In this work, we introduce the concept of generalized regular block intuitionistic fuzzy matrices as a generalization of regular block intuitionistic fuzzy matrices. A formula for k-g inverse of a block and triangular block intuitionistic fuzzy matrices are obtained.

Conflicts of Interest. The authors declare that there is no conflicts interest for publication of this paper.

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