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LOCAL δ-CLOSURE FUNCTIONS IN IDEAL TOPOLOGICAL SPACES

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ABSTRACT. In this paper we introduce and investigate a local function called local δ-closure function in an ideal topological space $(X, \mathscr{J}, \tau)$ denoted by $A_{\delta*}(\mathscr{J}, \tau)$ for any subset A of X with respect to the ideal $\mathscr{J}$ and topology τ which is defined by: $A_{\delta*}(\mathscr{J}, \tau) = \{x \in X : Int(cl_\delta(U)) \cap A \notin \mathscr{J}$ for every $U \in \tau(x)\}$. Also we investigate the basic properties and characterizations of $A_{\delta*}(\mathscr{J}, \tau)$. Also we discuss $\delta*$-local compatibility of τ with $\mathscr{J}$. Moreover, we introduce and investigate an operator $\Upsilon : P(X) \to \tau$ defined by $\Upsilon(A) = \{x \in X : Int(cl_\delta(U)) - A \in \mathscr{J}$ for every $U \in \tau(x)\}$, for each $A \in P(X)$. Also we proved the closure operator defined by $cl_{\delta*}(A) = A_{\delta*} \cup A$ is a Kuratowski closure operator and the topology obtained is $\tau_{\delta*} = \{U \subseteq X / cl_{\delta*}(X - U) = X - U\}$.

1. INTRODUCTION AND PRELEMINARIES

Kuratowski in [3] and Vaidhyanathaswamy in [8] was studied the notion of ideal topological spaces. Dontchev and Ganster in [1], Navaneethakrishnan and Joseph in [6], Jankovic and Hamlett in [2], Mukherjee, et al. in [4], Nasef and Mahmond in [5] were investigated applications of the ideal topology in various fields.

In a topological space (X, τ) with no separation properties assumed, for a subset A of X, $cl(A)$ the closure of A denotes intersection of all closed set containing A and $Int(A)$ denotes the union of all open set contained in A of (X, τ). An ideal $\mathscr{J}$ is a non empty collection of subsets of X which satisfies:

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(1) $A \in \mathscr{J}$ and $B \subseteq A$ implies $B \in \mathscr{J}$, and
(2) $A \in \mathscr{J}$ and $B \in \mathscr{J}$ implies $A \cup B \in \mathscr{J}$.

Given a topological space (X, τ) with an ideal $\mathscr{J}$ on X called ideal topological space denoted by $(X, \tau, \mathscr{J})$ and if $P(X)$ is the collection of all subsets of X a set operator $(.)^* : P(X) \to P(X)$ called a local function, [2,3], for any subset A of X with respect to $\mathscr{J}$ and τ is defined as:
$A^*(\mathscr{J}, \tau) = \{x \in X : U \cap A \notin \mathscr{J}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau / x \in U\}$.

A Kuratowski closure operator $cl^*(A)$ for a topology $\tau^*(\mathscr{J}, \tau)$ called $*$-topology finer than τ is defined by $cl^*(A) = A \cup (A)^*(\mathscr{J}, \tau)$. A subset A of X is said to be δ-closed set if $cl_\delta(A) = A$, where $cl_\delta(A) = \{x \in X : Int(cl(U)) \cap A \neq \phi,$ for every $U \in \tau(x)\}$, [9]. The complement of δ-closed set is δ-open set.

In this paper we introduce and investigate an operator $A_{\delta*}(\mathscr{J}, \tau)$ called local δ-closure function of A with respect to $\mathscr{J}$ and τ. Also, investigate a Kuratowski closure operator $cl_{\delta*}(A)$ and an operator $\Upsilon : P(X) \to \tau$ using $A_{\delta*}(\mathscr{J}, \tau)$.

2. LOCAL δ-CLOSURE FUNCTIONS

Definition 2.1. *Let $(X, \tau, \mathscr{J})$ be an ideal topological space and A a subset of X. Then $A_{\delta*}(\mathscr{J}, \tau) = \{x \in X : Int(cl_\delta(U)) \cap A \notin \mathscr{J}$ for every $U \in \tau(x)\}$ is called local δ-closure function of A with respect to the ideal $\mathscr{J}$ and topology τ, where $\tau(x) = \{U \in \tau / x \in U\}$.*

$A_{\delta*}(\mathscr{J}, \tau)$ is simply denoted by $A_{\delta*}$.

Remark 2.1. *The following Example 1 shows that, in general neither $A \subseteq A_{\delta*}$ nor $A_{\delta*} \subseteq A$.*

Example 1. *Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $\mathscr{J} = \{\phi, \{b\}\}$. Then $\{a, b\}_{\delta*} = \{a, d\}$.*

Theorem 2.1. *Let $(X, \tau, \mathscr{J})$ be an ideal topological space and A, B subsets of X. Then for local δ-closure functions the following properties hold.*

(i) $\phi_{\delta*} = \phi$.
(ii) *If $A \in \mathscr{J}$ then $A_{\delta*} = \phi$.*
(iii) $A \subseteq B$ *then* $A_{\delta*} \subseteq B_{\delta*}$.
(iv) $A_{\delta*} = cl(A_{\delta*})$, $A_{\delta*}$ *is closed.*

(v) $(A_{\delta *})_{\delta *} \subseteq A_{\delta *}$.

Proof. (i) is obvious.

(ii) is obvious.

(iii) Let $A \subseteq B$ and $x \notin B_{\delta *}$. Then there exists $U \in \tau(x)$ such that $Int(cl_\delta(U)) \cap B \in \mathscr{J}$. Since $A \subseteq B$, $Int(cl_\delta(U)) \cap A \in \mathscr{J}$ and hence $x \notin A_{\delta *}$.

(iv) We have $A_{\delta *} \subseteq cl(A_{\delta *})$. Let $x \in cl(A_{\delta *})$. Then $U \cap A_{\delta *} \neq \phi$ for every $U \in \tau(x)$ and hence $A_{\delta *} \cap Int(cl_\delta(U)) \neq \phi$. Therefore, there exists $y \in A_{\delta *} \cap Int(cl_\delta(U))$. Since $y \in A_{\delta *}$, $A \cap Int(cl_\delta(U)) \notin \mathscr{J}$ and hence $x \in A_{\delta *}$.

(v) Let $x \in (A_{\delta *})_{\delta *}$. Then for every $U \in \tau(x)$, $Int(cl_\delta(U)) \cap A_{\delta *} \notin \mathscr{J}$ and hence $Int(cl_\delta(U)) \cap A_{\delta *} \neq \phi$. Let $y \in Int(cl_\delta(U)) \cap A_{\delta *}$. Then $y \in Int(cl_\delta(U))$ and $y \in A_{\delta *}$. Therefore, $Int(cl_\delta(U)) \notin \mathscr{J}$ and hence $x \in A_{\delta *}$.

□

Theorem 2.2. *Let $(X, \tau, \mathscr{J})$ be an ideal topological space and A, B subsets of X. Then,*

(i) $(A \cup B)_{\delta *} = A_{\delta *} \cup B_{\delta *}$.

(ii) $(A \cap B)_{\delta *} \subseteq A_{\delta *} \cap B_{\delta *}$.

Proof. (i) By Theorem 2.1 (iii), $A_{\delta *} \cup B_{\delta *} \subseteq (A \cup B)_{\delta *}$. Now, let $x \in (A \cup B)_{\delta *}$. Then for every $U \in \tau(x)$, $(Int(cl_\delta(U)) \cap A) \cup (Int(cl_\delta(U)) \cap B) = Int(cl_\delta(U)) \cap (A \cup B) \notin \mathscr{J}$. Therefore, $Int(cl_\delta(U)) \cap A \notin \mathscr{J}$ or $Int(cl_\delta(U)) \cap B \notin \mathscr{J}$. Hence, $x \in A_{\delta *} \cup B_{\delta *}$.

(ii) Proof is clear by Theorem 2.1 (iii).

□

Remark 2.2. *The following Example 2 shows that the reverse inclusion of Theorem 2.2 (ii) is not always hold.*

Example 2. *In Example 1, $\{c\}_{\delta *} = \{b, c, d\}$ and $\{d\}_{\delta *} = \{d\}$.*

Theorem 2.3. *Let $(X, \tau, \mathscr{J})$ be an ideal topological space and A, B subsets of X. Then,*

(i) $A_{\delta *} - B_{\delta *} = (A - B)_{\delta *} - B_{\delta *} \subseteq (A - B)_{\delta *}$.

(ii) *If $U \in \tau$ then $U \cap A_{\delta *} = U \cap (Int(cl_\delta(U) \cap A)_{\delta *} \subseteq (Int(cl_\delta(U) \cap A)_{\delta *}$.*

(iii) *If $U \in \mathscr{J}$, then $(A - U)_{\delta *} = A_{\delta *}$.*

Proof. (i) Since $A = (A - B) \cup (B \cap A)$, then by Theorem 2.2 (i), $A_{\delta *} = (A - B)_{\delta *} \cup (B \cap A)_{\delta *}$.

Therefore, $A_{\delta *} - B_{\delta *} = A_{\delta *} \cap (X - B_{\delta *}) = (((A - B)_{\delta *} \cup (B \cap A)_{\delta *}) \cap (X - B_{\delta *})) = ((A - B)_{\delta *} \cap (X - B_{\delta *})) \cup ((B \cap A)_{\delta *} \cap (X - B_{\delta *})) = ((A - B)_{\delta *} - B_{\delta *}) \cup \phi \subseteq (A - B)_{\delta *}$.

(ii) Let$U \in \tau$ and $x \in U \cap A_{\delta *}$. Then $x \in U$ and $x \in A_{\delta *}$. Since for every $V \in \tau(x)$, $U \cap V = G \in \tau(x)$ and thus $Int(cl_\delta(V)) \cap (Int(cl_\delta(U) \cap A) \supset Int(cl_\delta(G)) \cap A \notin \mathscr{I}$. Hence $x \in (Int(cl_\delta(U) \cap A)_{\delta *}$. Also $U \cap A_{\delta *} \subseteq U \cap (Int(cl_\delta(U) \cap A)_{\delta *}$ and by Theorem 2.1 (iii) $(Int(cl_\delta(U) \cap A)_{\delta *} \subseteq A_{\delta *}$ and $U \cap (Int(cl_\delta(U) \cap A)_{\delta *} \subseteq U \cap A_{\delta *}$.

(iii) Since $A \cap U \subseteq U \in \mathscr{I}$, $A \cap U \in \mathscr{I}$ by heredity of $\mathscr{I}$ and by (ii) $(A \cap U)_{\delta *} = \phi$. Since $A = (A - U) \cup (A \cap U)$, $A_{\delta *} = (A - U)_{\delta *} \cup (A \cap U)_{\delta *} = (A - U)_{\delta *}$ by Theorem 2.1 (vi).

□

Theorem 2.4. *Let $(X, \tau, \mathscr{I})$ be a topological space with ideals $\mathscr{I}_1$ and $\mathscr{I}_2$ of X and A a subset of X. Then the following properties hold.*

(i) *If $\mathscr{I}_1 \subseteq \mathscr{I}_2$ then $A_{\delta *}(\mathscr{I}_2) \subseteq A_{\delta *}(\mathscr{I}_1)$*

(ii) $A_{\delta *}(\mathscr{I}_1 \cap \mathscr{I}_2) \subseteq A_{\delta *}(\mathscr{I}_1) \cup A_{\delta *}(\mathscr{I}_2)$.

Proof. (i) Let $\mathscr{I}_1 \subseteq \mathscr{I}_2$ and $x \notin A_{\delta *}(\mathscr{I}_1)$. Then $A \cap Int(cl_\delta(U)) \in \mathscr{I}_1$ for every $U \in \tau(x)$ and hence $A \cap Int(cl_\delta(U)) \in \mathscr{I}_2$. That is, $x \notin A_{\delta *}(\mathscr{I}_2)$.

(ii) We have $A_{\delta *}(\mathscr{I}_1) \subseteq A_{\delta *}(\mathscr{I}_1 \cap \mathscr{I}_2)$ and $A_{\delta *}(\mathscr{I}_2) \subseteq A_{\delta *}(\mathscr{I}_1 \cap \mathscr{I}_2)$ by (i). Therefore, $A_{\delta *}(\mathscr{I}_1) \cup A_{\delta *}(\mathscr{I}_2) \subseteq A_{\delta *}(\mathscr{I}_1 \cap \mathscr{I}_2)$. Let $x \in A_{\delta *}(\mathscr{I}_1 \cap \mathscr{I}_2)$ then for each $U \in \tau(x), Int(cl_\delta(U)) \cap A \notin \mathscr{I}_1 \cap \mathscr{I}_2$ and hence $Int(cl_\delta(U)) \cap A \notin \mathscr{I}_1$ and $Int(cl_\delta(U)) \cap A \notin \mathscr{I}_2$. Therefore, $x \in A_{\delta *}(\mathscr{I}_1) \cup A_{\delta *}(\mathscr{I}_2)$.

□

Theorem 2.5. $A^* \subseteq A_{\delta *}$.

Proof. The proof is clear by Definition 2.1. □

Remark 2.3. *The following Example 3 shows that Theorem 2.5 is true and the reverse direction is not always hold.*

Example 3. *Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}\}$ and $\mathscr{I} = \{\phi, \{a\}\}$. Then $\{b\}^* = \{b, c\}$ and $\{b\}_{\delta *} = X$.*

3. $\tau_{\delta*}$- OPEN SETS

In this section we defined a closure operator $cl_{\delta*}(A) = A \cup A_{\delta*}$ and proved that it is a Kuratowski closure operator.

Theorem 3.1. *Let $(X, \tau, \mathscr{J})$ be an ideal topological space, $cl_{\delta*}(A) = A \cup A_{\delta*}$ and A, B subsets of X. Then*

(i) *If $A \subseteq B$ then $cl_{\delta*}(A) \subseteq cl_{\delta*}(B)$.*
(ii) *$cl_{\delta*}(A \cap B) \subseteq cl_{\delta*}(A) \cap cl_{\delta*}(B)$.*
(iii) *If $U \in \tau(x)$ then $U \cap cl_{\delta*}(A) \subseteq cl_{\delta*}(Int(cl_\delta(U) \cap A)$.*
(iv) *$cl*(A) \subseteq cl_{\delta*}(A)$.*

Proof. (i) Let $A \subseteq B$, $cl_{\delta*}(A) = A \cup A_{\delta*} \subseteq B \cup B_{\delta*} = cl_{\delta*}(B)$ by Theorem 2.1 (iii).
(ii) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$ then by (i), $cl_{\delta*}(A \cap B) \subseteq cl_{\delta*}(A)$ and $cl_{\delta*}(A \cap B) \subseteq cl_{\delta*}(B)$ and hence $cl_{\delta*}(A \cap B) \subseteq cl_{\delta*}(A) \cap cl_{\delta*}(B)$.
(iii) Since $U \in \tau(x)$. By Theorem 2.3 (ii) we have, $U \cap cl_{\delta*}(A) = U \cap (A_{\delta*} \cup A) = (U \cap A_{\delta*}) \cup (U \cap A) \subseteq (Int(cl_\delta(U) \cap A)_{\delta*} \cup (Int(cl_\delta(U) \cap A) = cl_{\delta*}(Int(cl_\delta(U) \cap A)$.
(iv) Proof is clear by Theorem 2.5 and Definition of $cl_{\delta*}(A)$.

$\square$

Remark 3.1. *The following Examples shows that the reverse inclusion of Theorem 3.1 (ii) and (iv) are not always hold.*

Example 4. *In Example 3, $cl_{\delta*}(\{b\}) = cl_{\delta*}(\{c\}) = X$ and $cl^*(\{b\}) = \{b, c\}$.*

Theorem 3.2. *Let $(X, \tau, \mathscr{J})$ be an ideal topological space, $cl_{\delta*}(A) = A \cup A_{\delta*}$ and A, B subsets of X. Then*

(i) *$cl_{\delta*}(\phi) = \phi$ and $cl_{\delta*}(X) = X$.*
(ii) *$A \subseteq cl_{\delta*}(A)$.*
(iii) *$cl_{\delta*}(A \cup B) = cl_{\delta*}(A) \cup cl_{\delta*}(B)$.*
(iv) *$cl_{\delta*}(cl_{\delta*}(A)) = cl_{\delta*}(A)$.*

Proof. (i) $cl_{\delta*}(\phi) = \phi\delta * \cup \phi = \phi$ and $cl_{\delta*}(X) = X\delta * \cup X = X$.
(ii) $A \subseteq A \cup A_{\delta*} = cl_{\delta*}(A)$.
(iii) By Theorem 2.1 (iii), $cl_{\delta*}(A \cup B) = (A \cup B)_{\delta*} \cup (A \cup B) =$
$(A_{\delta*} \cup B_{\delta*}) \cup (A \cup B) = (A_{\delta*} \cup A) \cup (B_{\delta*} \cup B) = cl_{\delta*}(A) \cup cl_{\delta*}(B)$.

(iv) $cl_{\delta*}(cl_{\delta*}(A)) = cl_{\delta*}(A_{\delta*} \cup A) = ((A_{\delta*} \cup A)_{\delta*} \cup (A_{\delta*} \cup A)) = ((A_{\delta*})_{\delta*} \cup A_{\delta*}) \cup (A_{\delta*} \cup A) = A_{\delta*} \cup (A_{\delta*} \cup A) = A_{\delta*} \cup A = cl_{\delta*}(A)$ by Theorem 2.2 (i) and Theorem 2.1 (v).

□

Remark 3.2. *By Theorem 3.2, $cl_{\delta*}(A) = A \cup A_{\delta*}$ is a Kuratowski closure operator. The topology generated by $cl_{\delta*}(A)$ is denoted and defined by $\tau\delta* = \{U \subseteq X : cl_{\delta*}(X-U) = X-U\}$ the open sets in $\tau_{\delta*}$ is called $\tau_{\delta*}$-open sets and its complement is called $\tau_{\delta*}$- closed sets.*

Lemma 3.1. *Let $(X, \tau, \mathscr{J})$ be an ideal topological space and A, B subsets of X. Then $A_{\delta*} - B_{\delta*} = (A-B)_{\delta*} - B_{\delta*}$.*

Proof. By Theorem 2.2, $A_{\delta*} = ((A-B) \cup (B \cap A))_{\delta*} = (A-B)_{\delta*} \cup (A \cap B)_{\delta*} \subseteq (A-B)_{\delta*} \cup B_{\delta*}$. Thus $A_{\delta*} - B_{\delta*} \subseteq (A-B)_{\delta*} - B_{\delta*}$. Also by Theorem 2.1, $(A-B)_{\delta*} \subseteq A_{\delta*}$ and hence $(A-B)_{\delta*} - B_{\delta*} \subseteq A_{\delta*} - B_{\delta*}$. □

Corollary 3.1. *$(X, \tau, \mathscr{J})$ be an ideal topological space and A, B subsets of X with$B \in \mathscr{J}$. Then $(A \cup B)_{\delta*} = A_{\delta*} = (A-B)_{\delta*}$.*

Proof. Since $B \in \mathscr{J}$, by Theorem 2.1 (ii), $B_{\delta*} = \phi$. By Lemma 3.1 and by Theorem 2.2, $(A \cup B)_{\delta*} = A_{\delta*} = (A-B)_{\delta*}$. □

4. δ^*-LOCAL COMPATIBILITY WITH IDEAL

Definition 4.1. *Let $(X, \tau, \mathscr{J})$ be an ideal topological space. We say that the topology τ is δ^*-local compatible with the ideal $\mathscr{J}$, denoted by $\tau \sim \mathscr{J}|_{\delta*}$, if the following condition holds for every subset A of X, if for every $x \in A$ there exists a $U \in \tau(x)$ such that $Int(cl_\delta(U)) \cap A \in \mathscr{J}$, then $A \in \mathscr{J}$.*

Theorem 4.1. *Let $(X, \tau, \mathscr{J})$ be an ideal topological space then the following properties are equivalent:*

(i) $\tau \sim \mathscr{J}|_{\delta*}$.
(ii) *If a subset A of X has a cover of open sets each of whose interior δ-closure intersection with A is in $\mathscr{J}$, then $A \in \mathscr{J}$.*
(iii) *For every $A \subseteq X$, $A \cap A_{\delta*} = \phi$ implies that $A \in \mathscr{J}$.*
(iv) *For every $A \subseteq X$, $A - A_{\delta*} \in \mathscr{J}$.*

(v) *For every $A \subseteq X$, if A contains no nonempty set B with $B \subset B_{\delta *}$ then $A \in \mathscr{J}$.*

Proof. (i) $\Rightarrow$ (ii).

The proof is obvious by Definition.

(ii) $\Rightarrow$ (iii).

Let $A \subseteq X$ and $x \in A$. Then $x \notin A_{\delta *}$ and there exist $G \in \tau(x)$ such that $Int(cl_\delta(G)) \cap A \in \mathscr{J}$. Therefore we have, $A \subseteq \cup\{G : x \in A\}$ and $G \in \tau(x)$ and by (ii), $A \in \mathscr{J}$.

(iii) $\Rightarrow$ (iv).

For any $A \subseteq X, A - A_{\delta *} \subseteq A$. Then $((A - A_{\delta *}) \cap (A - A_{\delta *})_{\delta *} \subseteq (A - A_{\delta *}) \cap A_{\delta *} = \phi$. Therefore by (iii), $A - A_{\delta *} \in \mathscr{J}$.

(iv) $\Rightarrow$ (v).

By (iv), for every $A \subseteq X$, $A - A_{\delta *} \in \mathscr{J}$. Let $A - A_{\delta *} = \mathscr{J}_1 \in \mathscr{J}$. Then $A = \mathscr{J}_1 \cup (A \cap A_{\delta *})$, by Theorem 2.2 (i) and by Theorem 2.1 (ii), $A_{\delta *} = \mathscr{J}_{1\delta *} \cup (A \cap A_{\delta *})_{\delta *} = (A \cap A_{\delta *})_{\delta *}$. Therefore, $A \cap A_{\delta *} = A \cap (A \cap A_{\delta *})_{\delta *} \subseteq (A \cap A_{\delta *})_{\delta *}$ and $A \cap A_{\delta *} \subseteq A$. By the assumption, $A \cap A_{\delta *} = \phi$ and hence $A = A - A_{\delta *} \in \mathscr{J}$.

(v) $\Rightarrow$ (i).

Let $A \subseteq X$ and assume that for every $x \in A$, there exist $G \in \tau(x)$ such that $Int(cl_\delta(G)) \cap A \in \mathscr{J}$. Then $A \cap A_{\delta *} = \phi$. Suppose that $B \subseteq A$ such that $B \subseteq B_{\delta *}$. Then $B = B \cap B_{\delta *} \subseteq A \cap A_{\delta *} = \phi$. Therefore, A contains no nonempty set B such that $B \subseteq B_{\delta *}$. Hence $A \in \mathscr{J}$. □

Theorem 4.2. *Let $(X, \tau, \mathscr{J})$ be an ideal topological space. If τ is δ^*-local compatible with $\mathscr{J}$, then the following equivalent properties hold:*

(i) *For every $A \subseteq X$, $A \cap A_{\delta *} = \phi$ implies that $A_{\delta *} = \phi$.*
(ii) *For every $A \subseteq X$, $(A - A_{\delta *})_{\delta *} = \phi$.*
(iii) *For every $A \subseteq X$, $(A \cap A_{\delta *})_{\delta *} = A_{\delta *}$.*

Proof. First we prove that (i) holds if τ is δ^*-local compatible with $\mathscr{J}$. Let A be any subset of X and $A \cap A_{\delta *} = \phi$, Then by Theorem 4.1, $A \in \mathscr{J}$ and hence by Theorem 2.1, $A_{\delta *} = \phi$.

(i) $\Rightarrow$ (ii).

Assume that for every $A \subseteq X$, $A \cap A_{\delta *} = \phi$ implies that $A_{\delta *} = \phi$. Let $B = A - A_{\delta *} B \cap B_{\delta *} = (A - A_{\delta *}) \cap (A - A_{\delta *})_{\delta *}$. Now $B \cap B_{\delta *} = (A \cap (X - A_{\delta *})) \cap$

$(A \cap (X - A_{\delta*}))_{\delta*} \subseteq (A \cap (X - A_{\delta*})) \cap (A_{\delta*} \cap (X - A_{\delta*})_{\delta*}) = \phi$. Therefore by (i), $B_{\delta*} = \phi$.

(ii) $\Rightarrow$ (iii).

Assume that for every $A \subseteq X$, $(A - A_{\delta*})_{\delta*} = \phi$. $A = (A - A_{\delta*}) \cup (A \cap A_{\delta*})$. Therefore, $A_{\delta*} = (A - A_{\delta*})_{\delta*} \cup (A \cap A_{\delta*})_{\delta*} = (A \cap A_{\delta*})_{\delta*}$.

(iii) $\Rightarrow$ (i).

Assume that for every $A \subseteq X$, $A \cap A_{\delta*} = \phi$ and $(A \cap A_{\delta*})_{\delta*} = A_{\delta*}$. Hence, $A_{\delta*} = \phi$. □

Corollary 4.1. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A \subseteq X$. If τ is δ^*-local compatible with $\mathscr{I}$, then $A_{\delta*} = (A_{\delta*})_{\delta*}$.*

Proof. Let $A \subseteq X$, $A_{\delta*} = (A \cap A_{\delta*})_{\delta*} \subseteq A_{\delta*} \cap (A_{\delta*})_{\delta*} = (A_{\delta*})_{\delta*}$ by Theorem 4.2 and by Theorem 2.1. Therefore, $A_{\delta*} = (A_{\delta*})_{\delta*}$ again by Theorem 2.1. □

Theorem 4.3. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and τ is δ^*-local compatible with $\mathscr{I}$, A a $\tau_{\delta*}$-closed subset of X. Then $A = B \cup \mathscr{I}_1$, where B is closed and $\mathscr{I}_1 \in \mathscr{I}$.*

Proof. Let $A \subseteq X$ and A is $\tau_{\delta*}$-closed. Then $A_{\delta*} \subseteq A$ implies $A = (A - A_{\delta*}) \cup A_{\delta*}$ and hence by Theorem 4.1 and by Theorem 2.1, proof completes. □

Theorem 4.4. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space then the following properties are equivalent:*

(i) $\tau \sim \mathscr{I}|_{\delta*}$,
(ii) *For every $\tau_{\delta*}$-closed subset A, $A - A_{\delta*} \in \mathscr{I}$.*

Proof. (i) $\Rightarrow$ (ii).

The proof is clear by Theorem 4.1

(ii) $\Rightarrow$ (i)

Let $A \subseteq X$ and assume that for every $x \in A$, there exists an open set $U \in \tau(x)$ such that, $A \cap Int(cl_\delta(U)) \in \mathscr{I}$, then $A \cap A_{\delta*} = \phi$. Since $cl_{\delta*}(A)$ is $\tau_{\delta*}$-closed, $(A \cup A_{\delta*}) - (A \cup A_{\delta*})_{\delta*} \in \mathscr{I}$ and $(A \cup A_{\delta*}) - (A \cup A_{\delta*})_{\delta*} = (A \cup A_{\delta*}) - (A_{\delta*} \cup ((A_{\delta*})_{\delta*}) = (A \cup A_{\delta*}) - A_{\delta*} = A$. □

5. Υ-OPERATOR

Definition 5.1. *Let $(X, \tau, \mathscr{J})$ be an ideal topological space. An operator $\Upsilon(A) : P(X) \to \tau$ is defined as follows: for every subset A of X, $\Upsilon(A) = \{x \in X :$ there exist $G \in \tau(x)$ such that $Int(cl_\delta(G)) - A \in \mathscr{J}\}$. Clearly, $\Upsilon(A) = X - (X - A)_{\delta *}$.*

Theorem 5.1. *Let $(X, \tau, \mathscr{J})$ be an ideal topological space. Then the following properties hold:*

(i) *For every $A \subseteq X$, $\Upsilon(A)$ is open.*
(ii) *If $A \subseteq B$ then $\Upsilon(A) \subseteq \Upsilon(B)$.*
(iii) *If $A, B \in P(X)$, then $\Upsilon(A \cap B) = \Upsilon(A) \cap \Upsilon(B)$.*
(iv) *If $A \subseteq X$, then $\Upsilon(A) = \Upsilon(\Upsilon(A))$ iff $(X - A)_{\delta *} = ((X - A)_{\delta *})_{\delta *}$.*
(v) *If $A \in \mathscr{J}$, then $\Upsilon(A) = X - X_{\delta *}$.*
(vi) *If $A \subseteq X$, $\mathscr{J}_1 \in \mathscr{J}$, then $\Upsilon(A - \mathscr{J}_1) = \Upsilon(A)$.*
(vii) *If $A \subseteq X$, $\mathscr{J}_1 \in \mathscr{J}$, then $\Upsilon(A \cup \mathscr{J}_1) = \Upsilon(A)$.*
(viii) *If $(A - B) \cup (B - A) \in \mathscr{J}$, then $\Upsilon(A) = \Upsilon(B)$.*

Proof. (i) Proof is clear from Theorem 2.1 (iv).
(ii) Proof follows from Theorem 2.1 (iii).
(iii) $\Upsilon(A \cap B) = X - (X - (A \cap B))_{\delta *} = X - ((X - A) \cup (X - B))_{\delta *} = X - ((X - A)_{\delta *} \cup (X - B)_{\delta *}) = (X - (X - A)_{\delta *}) \cap (X - (X - B)_{\delta *}) = \Upsilon(A) \cap \Upsilon(B)$.
(iv) Let $A \subseteq X$. Since $\Upsilon(A) = X - (X - A)_{\delta *}$ and $\Upsilon(\Upsilon(A)) = X - (X - \Upsilon(A))_{\delta *} = X - (X - (X - (X - A)_{\delta *}))_{\delta *} = X - ((X - A)_{\delta *})_{\delta *}$ proof follows.
(v) By Corollary 3.1, if $A \in \mathscr{J}$, $(X - A)_{\delta *} = (X)_{\delta *}$.
(vi) By Corollary 3.1, $\Upsilon(A - \mathscr{J}_1) = X - (X - (A - \mathscr{J}_1))_{\delta *} = X - ((X - A) \cup \mathscr{J}_1))_{\delta *} = X - (X - A)_{\delta *} = \Upsilon(A)$.
(vii) $\Upsilon(A \cup \mathscr{J}_1) = X - (X - (A \cup \mathscr{J}_1))_{\delta *} = X - ((X - A) - \mathscr{J}_1)_{\delta *} = X - (X - A)_{\delta *} = \Upsilon(A)$.
(viii) Assume that, $(A - B) \cup (B - A) \in \mathscr{J}$. Let $A - B = \mathscr{J}_1$ and $B - A = \mathscr{J}_2$. Therefore, $\mathscr{J}_1, \mathscr{J}_2 \in \mathscr{J}$ by heredity. Also observe that $B = (A - \mathscr{J}_1) \cup \mathscr{J}_2$. Thus $\Upsilon(A) = \Upsilon(A - \mathscr{J}_1) = \Upsilon((A - \mathscr{J}_1) \cup \mathscr{J}_2] = \Upsilon(B)$, by (vi) and (vii).

□

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