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LEXICOGRAPHIC ORDER BASED RANKING FOR Z-NUMBERS

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ABSTRACT. Zadeh's Z - numbers have the potential for use in several optimisation and decision-making problems. For applying Z - numbers in optimisation problems, ranking methods for Z - numbers are essential. A new approach to ordering / ranking Z - numbers is proposed. This novel method is based on the Lexicographic ordering. It is simple to apply. At the same time, it is very apt and suitable in many applications.

1. INTRODUCTION

Decision-making really belongs to an uncertainty, imprecise and partially reliable concept. By relating the Fuzzy logic and "Computing with Words", the new idea Z-number is introduced by Zadeh [1]. Using Z-numbers and Z-valuations, Yager [2] gave some important idea about uncertainity.

Shahila Bhanu M. and Velammal G. [3–5] have researched into computations with Z-numbers. Computation with discrete Z-numbers approach was done by R. A. Aliev, O. H. Huseynov and others [6–9]. P. Rani, G. Velammal [10–12] using fuzzy if-then rules, solved the problems involving 'interpolation' and 'applying modifier' in terms of Z-numbers.

Bingyi, Yong and Rehan [13] have discussed some applications of ordering of Z-numbers. B. Kang, D. Wei, Y. Li and Y. Deng [14] gave methods of converting classical fuzzy number from Z-number.

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Z-number attracts many researchers from mathematics and other disciplines, but there are very few papers regarding ranking of Z-numbers. By using generalized fuzzy numbers, Weng Jiang, Chunhe Xie, Yu Luo and Youngchuan Tang [15] described principles of ranking Z-numbers.

Adopting Z-number is very simple in the point of representing human knowledge. Zadeh had pointed that real-life problems where the data available is not only imprecise but also reliable only to certain extent can be represented by Z-numbers. Hence several classical problems like Sequencing problem, Assignment problem and so many other problems in operations research can be recast in terms of Z-numbers. Indeed, ranking or ordering of Z-numbers is an essential task in solving optimisation problems and decision-making problems.

So, this novel approach of Ordering of Z-numbers based on Lexicographic Ordering is very essential part of real-life problems.

2. Preliminaries

Definition 2.1. *Two ways of defining the order relation on the cartesian product of given ordered sets are as follows :*

(a) *Product Order: For the two ordered sets* $(S, R_1)\&(T, R_2)$*, the order relation (*R *as* $'\preceq'$*) on the product set* $S \times T$*,* $(a, b)\, R\, (a', b')$*, if a* $R_1\{a\}'\&\ b\ R_2 b'$*, called the product order.*

(b) *Lexicographical Order: For the linearly ordered sets* $(S, R_1)\&(T, R_2)$*, the order relation (*R *as* $' <'$*),* $(a, b)\ R\ \ (a', b')$*, if a* $R_1 a'$ *or if* $a = a'\& b R_2 b'$ *on the product set* $S \times T$*, called the Lexicographical or dictionary order.*

Definition 2.2 (Fuzzy Subset)**.** *A Fuzzy Subset, is the set of all ordered pairs of x,mx, Where x is in the universal set* X *and* $m : X \to [0, 1]$ *is its membership function, and is denoted by*

$$FS = \{x, mx/ \textit{ for all } x \in X\}.$$

Definition 2.3 (Fuzzy Number)**.** *If the membership function of a fuzzy set has the following characteristics, it is called a Fuzzy Number.*

(i) $m : X \to [0, 1]$ *is continuous,*

(ii) $m\,(x) = 0$ *for all* $x \in (-\infty, a] \cup [d, \infty)$*,*

(iii) m *is strictly increasing on* $[a, b]$ *and strictly decreasing on* $[c, d]$*,*

(iv) $m\,(x) = 1$ *for all* $x \in [b,\ c]$*, where* $a < b < c < d$*.*

Definition 2.4. *The membership function of the Trapezoidal Fuzzy Numbers -* $A(a, b, c, d)$ *is defined by*

$$m\,(x) = \begin{cases} \frac{x-a}{b-a},\ if\ x \in [a, b] \\ 1,\ if\ x \in [b, c] \\ \frac{d-x}{d-c},\ if\ x \in [c, d] \\ 0,\ otherwise\,. \end{cases}$$

Definition 2.5 (Ranking Function)**.** *Let* F *be a set of fuzzy numbers. A ranking function* r_k *on* F *is a real valued function defined on* F*. Given a ranking function* r_k *on* F *we can order the elements in* F *as follows:*
$A_1 \preceq A_2$ *if and only if* $r_k(A_1) \leq r_k(A_2)$*.*

3. Some well-known ranking functions

Consider the Trapezoidal fuzzy number $C(a, b, c, d)$.

(1) Centre of Gravity
The defuzzified value $d_{CA}(C)$ [8] is defined as the area under the graph of membership function C is

$$r_1(C) = \frac{(c^2 + \ d^2 + cd - \ a^2 \ \ - \ b^2 \ - ab)}{3\,(c + d - a - b)}$$

(2) Centre of Maxima
The defuzzified value $d_{CM}(C)$ is defined by $r_2\,(C) = d_{CM}(C) = \frac{infM + supM}{2} = \frac{b+c}{2}$, where $M = \{z \in [-c, c]/c(z) = h(C)\}$, (Continuous Case),

$$r_2\,(C) = \frac{b + c}{2}$$

(3) Median
Minimum value of $\left| \int_{-\infty}^{Med(A)} C\,(x)\,dx \ - \ \int_{Med(A)}^{\infty} C\,(x)\,dx \right|$ is called Median. Here,

$$r_3\,(C) = Med(C) = \frac{a + b + c + d}{4}.$$

Definition 3.1 (Z-number). *The ordered pair of fuzzy number indicated as $Z = (A, B)$, is called Z-number, if A is a restriction of real-valued uncertain variable X, and B is a measure of reliability (confidence) of the first component A. Mainly, Z-number gives the information about the uncertain variable and also the reliability of the information.*

Example 1. *Consider the sentence "Ram will surely get 80 to 90 marks in his final exam."*
By converting this to Z-number with the 1st component as 80 to 90 marks and the 2^{nd} component is reliability of 1^{st} as surely.
So, the given sentence is in the form of Z-number is (80 to 90 marks, surely)

Definition 3.2 (Z-valuation). *"Zadeh [1] refers, The ordered triple (V, A, B) is a Z-valuation" is equivalent to the assignment statement "V is (A, B)", where V is an uncertain variable and that takes the value A with probability equal B.*

Example 2. *(Weight of a Man, about 50, likely) and (Fall of Rain in a city at a particular time, almost 2cm, not likely).*

Definition 3.3 (Trapezoidal Z-number). *If the two components A, B in "$Z = (A, B)$", are trapezoidal fuzzy numbers, then the corresponding Z-number is called a Trapezoidal Z-number.*

Example 3. *"A man do a piece of work by almost 3 to 4 hours very likely"-*
This Z-number (almost 3 to 4 hours, very likely) is equivalent to the trapezoidal Z-number as ((2,3,4,5),(0.7,0.8,0.9,0.95))

4. Ordering of Z-numbers - proposed approach

Definition 4.1 (Lexicographic Order $\mathcal{L}(\mathcal{R}_\mathbf{1},\ \mathcal{R}_\mathbf{2})$ for Z-numbers). *Let $R_1 \& R_2$ be any two ranking function and let $Z_1 = (A_1, B_1) \& Z_2 = (A_2,\ B_2)$ be any two Z-numbers. Define $Z_1 \preceq Z_2$ under the Lexicographic order $L\,(R_1,\ R_2)$ if and only if*

$$\text{(i) } R_1(A_1) < R_1(A_2) \quad \text{(or)} \quad (ii)\, R_1(A_1) = R_1(A_2) \& R_2(B_1) \geq R_2(B_2).$$

Definition 4.2. *Flipped Lexicographic Ordering $\mathcal{FL}(\mathcal{R}_\mathbf{1},\ \mathcal{R}_\mathbf{2})$ for Z-numbers Let $R_1 \& R_2$ be two ranking function. Let $Z_1 = (A_1, B_1) \& Z_2 = (A_2, B_2)$. We say*

$Z_1 \preceq Z_2$ under $FL\,(R_1,\ R_2)$ if and only if $(i)\,R_2(B_1) > R_2(B_2)$ or $(ii)\,R_2(B_1) = R_2(B_2) \& R_1(A_1) \leq R_1(A_2)$

Remark 4.1. *Depending on the area of application, R_1 and R_2 can be suitably chosen. So, this Lexicographic approach is highly flexible. Again, the decision maker can decide which is critical - the first component or the second component. According he or she can use the $L(R_1,\ R_2)$ or $FL(R_1,\ R_2)$ to rank or order.*

Example 4. *Suppose a manager has three routes from "city A to city" to travel. The information regarding the journey time on these routes is as follows:*
On the first route it is likely it will take six to seven hours.
On the second route it is very likely it will take seven to eight hours.
On the third route it is quite likely it will take seven to eight hours.
Journey times on the three routes are represented by the z-numbers Z_1, Z_2 and Z_3 given below: $Z_1 =$ (6 to 7 hours, likely)
$Z_2 =$ (7 to 8 hours, very likely)
$Z_3 =$ (7 to 8 hours, quite likely)
Now it is very clear that Z_1 is preferable to Z_3. The first component in Z_1 is smaller than the first component in Z_3 and the second component which is the reliability component in Z_1 is higher than the second component in Z_3. Hence, we may conclude $Z_1 < Z_3$.
Similarly, we may conclude $Z_2 < Z_3$.
However, in comparing Z_1 and Z_2, it is not so easy to arrive at a conclusion. The first component in Z_1 is smaller than the first component in Z_2 but the reliability component in Z_2 is better than the second component in Z_1.
In these situations, the decision maker has to take a call- should he give more importance to the first component or the second component? If he gives more importance to first component, he has to use $L\,(R_1,\ R_2)$, in which case he will conclude $Z_1 < Z_2$.
On the other hand, if he feels reliability is more essential, he will use $FL(R_1,\ R_2)$ to rank the numbers. So, he will conclude $Z_1 > Z_2$.

5. Numerical Computation

Lexicographic and Flipped Lexicographic Order for the given Z-numbers by using $L\left(R_1,\ R_2\right)$ and $FL\left(R_1,\ R_2\right)$

Consider the following Z-numbers

$Z_1 = (A_1, B_1) = ((55, 70, 80, 95), (.8, .85, .9, .95))$,
$Z_2 = (A_2, B_2) = ((65, 70, 80, 92), (.8, .85, .9, .95))$,
$Z_3 = (A_3, B_3) = ((50, 70, 80, 100), (.8, .85, .9, .95))$
$Z_4 = (A_4, B_4) = ((50, 70, 80, 100), (.85, .9, .95, 1))$,
$Z_5 = (A_5, B_5) = ((70, 75, 95, 100), (.7, .8, .9, 1))$
$Z_6 = (A_6, B_6) = ((70, 75, 95, 100), (.8, .85, .95, 1))$.

Let $r_1(C) = \dfrac{(c^2 +\ d^2 + cd -\ a^2\ -\ b^2\ - ab)}{3\left(c + d - a - b\right)}, r_2\left(C\right) = \dfrac{b + c}{2}$, and $r_3\left(C\right) = \dfrac{a +\ b + c + d}{4}$.

Let us find the rank for all above trapezoidal Z- number and then order. Here, take r_1 is ranking from Centre of Gravity, r_2 is from Centre of Maxima and r_3 is from Median method.

$Z_i = (A_i, B_i)$ (1)	r_1 (A_i) (2)	r_1 (B_i) (3)	Rank using L (r_1, r_1) (2,3)	r_2 (A_i) (4)	r_2 (B_i) (5)	Rank using L (r_2, r_2) (4,5)	r_3 (A_i) (6)	r_3 (B_i) (7)	Rank using L (r_3, r_3) (6,7)	Rank using L (r_1, r_2) (2,5)	Rank using L $(r_1 r_3)$ (2,7)	Rank using L (r_2, r_3) (4,7)
Z_1=((55,70,80,95), (.8,.85,.9, .95))	75	0.88	2	75	0.88	2	75	0.88	2	2	2	2
Z_2=((65,70,80,92), (.8,.85,.9,.95))	77.02	0.88	3	75	0.88	2	76.75	0.88	3	3	3	2
Z_3=((50,70,80,100), (.8,.85,.9,.95))	75	0.88	2	75	0.88	2	75	0.88	2	2	2	2
Z_4=((50,70,80,100), (.85,.9,.95, 1))	75	0.93	1	75	0.93	1	75	0.93	1	1	1	1
Z_5=((70,75,95,100), (.7, .8, .9, 1))	85	0.85	5	85	0.85	4	85	0.85	5	5	5	4
Z_6=((70,75,95,100), (.8, .85, .95, 1))	85	0.9	4	85	0.9	3	85	0.9	4	4	4	3

Let us use FL(r_i, r_j) to order the above Z – numbers

Z_i= (A_i,B_i) (1)	r_1 (A_i) (2)	r_1 (B_i) (3)	Rank using FL (r_1,r_1) (3,2)	r_2 (A_i) (4)	r_2 (B_i) (5)	Rank using FL (r_2,r_2) (5,4)	r_3 (A_i) (6)	r_3 (B_i) (7)	Rank using FL (r_3,r_3) (7,6)	Rank using FL (r_1,r_2) (3,4)	Rank using FL (r_1,r_3) (3,6)	Rank using FL (r_2,r_3) (5,6)
Z_1=((55,70,80,95), (.8,.85,.9,.95))	75	0.88	3	75	0.88	3	75	0.88	3	3	3	3
Z_2= (65,70,80,92), (.8,.85,.9,.95))	77.02	0.88	4	75	0.88	3	76.75	0.88	4	3	4	4
Z_3=((50,70,80,100), (.8,.85,.9,.95))	75	0.88	3	75	0.88	3	75	0.88	3	3	3	3
Z_4=((50,70,80,100), (.85,.9,.95,1))	75	0.93	1	75	0.93	1	75	0.93	1	1	1	1
Z_5=((70,75,95,100), (.7,.8,.9,1))	85	0.85	5	85	0.85	4	85	0.85	5	4	5	5
Z_6=((70,75,95,100), (.8, .85, .95, 1))	85	0.9	2	85	0.9	2	85	0.9	2	2	2	2

From the above two tables, for the given Z numbers, the comparison for $L(r_i, r_j)$ and $FL(r_i, r_j)$ as follows:

(r_i, r_j)	$L(r_i, r_j)$	$FL(r_i, r_j)$
(r_1, r_1)	$Z_4 \preccurlyeq Z_1 \preccurlyeq \boldsymbol{Z_3} \preccurlyeq \boldsymbol{Z_2} \preccurlyeq Z_6 \preccurlyeq Z_5$	$Z_4 \preccurlyeq Z_6 \preccurlyeq \boldsymbol{Z_1} \preccurlyeq \boldsymbol{Z_3} \preccurlyeq Z_2 \preccurlyeq Z_5$
(r_2, r_2)	$Z_4 \preccurlyeq Z_1 \preccurlyeq Z_2 \preccurlyeq Z_3 \preccurlyeq Z_6 \preccurlyeq Z_5$	$Z_4 \preccurlyeq Z_6 \preccurlyeq Z_1 \preccurlyeq Z_2 \preccurlyeq Z_3 \preccurlyeq Z_5$
(r_3, r_3)	$Z_4 \preccurlyeq Z_1 \preccurlyeq \boldsymbol{Z_3} \preccurlyeq \boldsymbol{Z_2} \preccurlyeq Z_6 \preccurlyeq Z_5$	$Z_4 \preccurlyeq Z_6 \preccurlyeq \boldsymbol{Z_1} \preccurlyeq \boldsymbol{Z_3} \preccurlyeq Z_2 \preccurlyeq Z_5$
(r_1, r_2)	$Z_4 \preccurlyeq Z_1 \preccurlyeq \boldsymbol{Z_3} \preccurlyeq \boldsymbol{Z_2} \preccurlyeq Z_6 \preccurlyeq Z_5$	$Z_4 \preccurlyeq Z_6 \preccurlyeq Z_1 \preccurlyeq Z_2 \preccurlyeq Z_3 \preccurlyeq Z_5$
(r_1, r_3)	$Z_4 \preccurlyeq Z_1 \preccurlyeq \boldsymbol{Z_3} \preccurlyeq \boldsymbol{Z_2} \preccurlyeq Z_6 \preccurlyeq Z_5$	$Z_4 \preccurlyeq Z_6 \preccurlyeq \boldsymbol{Z_1} \preccurlyeq \boldsymbol{Z_3} \preccurlyeq Z_2 \preccurlyeq Z_5$
(r_2, r_3)	$Z_4 \preccurlyeq Z_1 \preccurlyeq Z_2 \preccurlyeq Z_3 \preccurlyeq Z_6 \preccurlyeq Z_5$	$Z_4 \preccurlyeq Z_6 \preccurlyeq \boldsymbol{Z_1} \preccurlyeq \boldsymbol{Z_3} \preccurlyeq Z_2 \preccurlyeq Z_5$

6. CONCLUSION

Recently Z-numbers are being increasingly used to build realistic mathematical models of real-life problems. Solving these models frequently requires a procedure for comparing or ordering Z-numbers. Hence developing good ranking methods for Z-numbers is vital.

This suggested idea is a novel approach to ranking of Z-numbers. The method is simple as well as flexible. It will be a very advantage tool in optimisation problems involving Z-numbers.

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