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TETRA DUAL MAGIC GRAPHS

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ABSTRACT. A Tetra Dual Magic graph is a dual graph for which the primal graph is a Magic Pyramidal graph. A Magic Pyramidal graph is said to be a Tetra Dual Magic graph if the magic constants are all tetrahedral numbers. These graphs are got from either a Vertex Magic Pyramidal graph or an Edge Magic Pyramidal graph by replacement of the magic pyramidal constants by tetra magic constants. In this paper we prove that all Paths, Stars, Complete bipartite graphs are Tetra Dual Magic graphs and investigate the behaviour of a Peterson graph to be a Tetra Dual Magic graph.

1. INTRODUCTION

A labeling of a graph is an assignment of integers as labels to the vertices or edges or to both the vertices and edges of a graph. Rosa introduced graph labeling in 1967 and the concept of Magic labeling in graphs was first introduced by Sedlacek. Sedlacek defined a Magic labeling as a function f from the set of edges of a graph G into non-negative real numbers so that the sums of the labels of the edges incident at any vertex in G is constant for all $v \in V$.

The concept of Magic Pyramidal graphs was introduced in our research work with pyramidal numbers as magic constants, see [1–6]. If a graph satisfies both vertex magic pyramidal labeling and edge magic pyramidal labeling then it is termed as a Magic Pyramidal graph.

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In this paper we introduce Tetra Dual Magic graphs for which the primal graph is a Magic Pyramidal graph and analyse the range of the Tetra vertex magic constants T_v and Tetra edge magic constants T_e in relation with the range of the vertex magic constants λ_f and the edge magic constants μ_f respectively. Also we prove that some classes of graphs satisfy Tetra Dual Magic labeling and investigate classes of Non-Tetra Dual Magic graphs.

2. Tetra Dual Magic graphs

Definition 2.1. *A Triangular number is a number got by adding all positive integers less than or equal to a given positive integer n. If n^{th} Triangular number is denoted by T_n then $T_n = \dfrac{n(n+1)}{2}$. Triangular numbers are found in the third diagonal of Pascal's Triangle. They are* $1, 3, 6, 10, 15, 21, \ldots$

Definition 2.2. *Tetrahedral numbers are the sum of Consecutive triangular numbers. They are found in the fourth diagonal of Pascal's Triangle. These numbers are* $1, 4, 10, 20, 35, \ldots$ *The n^{th} tetrahedral number is denoted by the notation t_n where* $t_n = \dfrac{n(n+1)(n+2)}{6}$.

Remark 2.1. *The Pyramidal numbers or Square Pyramidal numbers are the sums of consecutive pairs of tetrahedral numbers. They are 1, 5, 14, 30, 55, ... If p_n is the n^{th} pyramidal number then $p_n = \dfrac{n(n+1)(2n+1)}{6}$.*

Remark 2.2. *The following is Pascal's Triangle in which the third and fourth rows contain respectively the triangular and tetrahedral numbers.*

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Definition 2.3. *Let $G = (V, E)$ be a graph with $|V| = p$ vertices and $|E| = q$ edges. The Vertex Magic Pyramidal labeling of G is a one-to-one function f from $V(G) \cup E(G)$ onto the integers $\{1, 2, 3, ..., p_q\}$ with the property that there is a constant λ_f such that $f(u) + \sum f(uv) = \lambda_f$ where the sum runs over all vertices v adjacent to u and uv is the edge joining the vertices u and v and the constant λ_f must be a Pyramidal number.*

Here p_q denotes the q^{th} Pyramidal number. The constant λ_f is called the Vertex Magic constant of the given graph. A graph G which satisfies the above labeling is called a Vertex Magic Pyramidal graph.

Definition 2.4. *The Edge Magic Pyramidal labeling of a Graph G is defined as a one-to-one function f from $V(G) \cup E(G)$ onto the integers $\{1, 2, 3, ..., p_q\}$ with the property that there is a constant μ_f such that $f(u) + f(v) + f(uv) = \mu_f$ where $uv \in E(G)$ and the constant μ_f must be a Pyramidal number.*

Here p_q denotes the q_{th} Pyramidal number. The constant μ_f is called the Edge Magic constant of the given graph. A graph G which satisfies the above labeling is called an Edge Magic Pyramidal graph.

Definition 2.5. *If a graph G admits both Vertex Magic Pyramidal labeling and Edge Magic Pyramidal labeling then it is termed as a Magic Pyramidal graph.*

Definition 2.6. *A Tetra Dual Vertex Magic labeling for a graph G is a dual labeling derived from a Vertex Magic Pyramidal graph by replacing the vertex magic constant λ_f with the tetra vertex magic constant T_v and the q^{th} Pyramidal number p_q with the q^{th} tetrahedral number t_q. Similarly a Tetra Dual Edge Magic labeling is derived from a Edge Magic Pyramidal graph by replacing the edge magic constant μ_f with the tetra edge magic constant T_e and the q^{th} Pyramidal number p_q with the q^{th} tetrahedral number t_q.*

If a Magic pyramidal graph admits both Tetra Dual Vertex Magic labeling and Tetra Dual Edge Magic labeling then it is termed as a Tetra Dual Magic graph. Here the tetra vertex magic constants T_v and the tetra edge magic constants T_e are the tetrahedral numbers.

Theorem 2.1. *All Paths P_n are Tetra Dual Magic graphs for $n \geq 4$ where the tetra vertex magic constants T_v range from $\left\lceil \frac{5n-1}{2} \right\rceil \leq T_v \leq t_{n-1}$ for $4 \leq n \leq 9$ and*

$\left[\dfrac{5n+7}{2}\right] \leq T_v \leq t_{n-1}$ *for all* $n \geq 10$ *and the tetra edge magic constants* T_e *range from* $3n \leq T_e \leq t_{n-1}$ *where* t_{n-1} *is the* $(n-1)^{th}$ *tetrahedral number.*

Proof. Let the Path P_n be a Vertex Magic Pyramidal graph with n vertices $v_1, v_2, ..., v_n$. Let e_i, $i = 1$ to $n-1$ be the edges of the Path.

Case 1. n is odd, $n \geq 5$.

Define $f : V \cup E \to \{1, 2, 3, ..., t_q\}$ as follows:

$f(v_1) = T_v - 1$ where T_v is the tetra vertex magic constant.

$$f(v_i) = \begin{cases} f(v_{i-1}) - 2 & for\ 2 \leq i \leq n-1 \\ T_v - (i-1) & for\ i = 1 \end{cases}$$

$$f(e_i) = i\ for\ 1 \leq i \leq n-1$$

Case 2. n is even, $n \geq 4$.

Define $f(v_1) = T_v - 2$ where T_v is the tetra vertex magic constant.

$$f(v_i) = \begin{cases} f(v_{i-1}) - 1 & for\ i = 2 \\ f(v_{i-1}) - 2 & for\ 3 \leq i \leq n-1 \\ T_v - i & for\ i = n \end{cases}$$

$$f(e_i) = \begin{cases} i+1 & for\ i = 1 \\ i-1 & for\ i = 2 \\ f(e_{i-2} + 2) & for\ 3 \leq i \leq n-1 \end{cases}$$

Hence all Paths P_n satisfy Tetra Dual Vertex Magic labeling for $n \geq 4$.

Now, let the Path P_n be an Edge Magic Pyramidal graph. We discuss this for any $n \geq 4$.

Case 3. For any $n \geq 4$, define $f : V \cup E \to \{1, 2, 3, ..., t_q\}$ as follows:

$$f(v_i) = i\ for\ 1 \leq i \leq n$$

$$f(e_i) = \begin{cases} T_e - 3 & for\ i = 1 \\ f(e_{i-1}) - 2 & for\ 2 \leq i \leq n-1 \end{cases}$$

Hence all Paths P_n satisfy Tetra Dual Edge Magic labeling for $n \geq 4$. Therefore all Paths P_n are Tetra Dual Magic graphs for $n \geq 4$. □

Example 1.

Theorem 2.2. *All Complete bipartite graphs* $K_{m,n}$ *are Tetra Dual Magic graphs where the tetra vertex magic constants* T_v *range from* $t_{m+3} \leq T_v \leq t_{mn}$ *for* $m \neq$

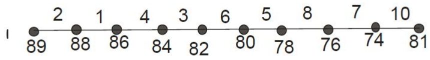

FIGURE 1. Primal graph: Vertex Magic Pyramidal labeling of $P_{10}(\lambda_f = 91 = p_6)$.

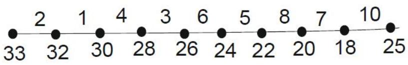

FIGURE 2. Dual graph: Tetra Dual Vertex Magic labeling of $P_{10}(T_v = 35 = t_5)$.

$n, m > n$ and $t_{n+3} \leq T_v \leq t_{mn}$ for $m \neq n,\ n > m$. For $m = n$, T_v range from $t_{m+4} \leq T_v \leq p_{mn}$ and the tetra edge magic constants T_e range from $t_{m+4} \leq T_e \leq t_{m^2+i}$ for $m = n$ where i takes the values 0,2,4,6,... for each m ranging from 2,3,4,... and for $m \neq n$, T_e approximately varies from $t_{m+4} \leq T_e \leq t_{mn+m\sim n}$ where t_n is the n^{th} tetrahedral number.

Proof. Case 1. Let the Complete bipartite graph $G = K_{m,n}$ be a Vertex Magic Pyramidal graph. Let $G = (V(G), E(G))$. Then V can be partitioned into two subsets V_1 and V_2 such that every line joins a point of V_1 to a point of V_2. Let $v_1, v_2, ..., v_m$ be the vertices of V_1 and $u_1, u_2, ..., u_n$ be the vertices of V_2. Let $e_1, e_2, ..., e_{mn}$ be the edges of $K_{m,n}$. Therefore we have $V = V_1 \cup V_2$. Let $|V_1(G)| = m$ and $|V_2(G)| = n$.

Hence $|V(G)| = m + n$, $|E(G)| = mn$.

Define $f : V(G) \cup E(G) \to \{1, 2, 3, ..., t_q\}$ as follows:

$$\begin{aligned}
f(e_1) &= m + n \\
f(e_i) &= f(e_{i-1}) + 1\ for\ 2 \leq i \leq mn \\
f(v_1) &= T_v - \sum_{i=1}^{n} f(e_i) \\
f(v_i) &= f(v_{i-1}) - n^2,\ \forall v_i \in V_1\ 2 \leq i \leq m \\
f(u_1) &= T_v - \sum f(e_i)\ where\ i = 1, n+1, 2n+1, 3n+1... \\
f(u_i) &= f(u_{i-1}) - m,\ \forall u_i \in V_2,\ 2 \leq i \leq n
\end{aligned}$$

Hence all Complete bipartite graphs $K_{m,n}$ satisfy Tetra Dual Vertex Magic labeling.

Case 2. Let the Complete bipartite graph $K_{m,n}$ be an Edge Magic Pyramidal graph.

Define $f : V(G) \cup E(G) \to \{1, 2, 3, ..., t_q\}$ as follows:

$$f(v_1) = \left[\frac{T_e}{2}\right]$$
$$f(v_i) = f(v_{i-1}) - (n+2) \; for \; 2 \leq i \leq m, \; \forall v_i \in V_1$$

Define $f(u_1) = \begin{cases} f(v_1) - 1 & if \; T_e \; is \; odd \\ f(v_1) - 2 & if \; T_e \; is \; even \end{cases}$

$$f(u_i) = f(u_{i-1}) - 1 \; for \; 2 \leq i \leq n, \; \forall u_i \in V_2$$
$$f(e_i) = \begin{cases} i+1 & for \; 1 \leq i \leq mn \; and \; i \neq n+1, 2n+1, 3n+1, ... \\ f(e_{i-1}) + 3 & for \; i = n+1, 2n+1, 3n+1, ... \end{cases}$$

Hence all Complete bipartite graphs $K_{m,n}$ satisfy Tetra Dual Edge Magic labeling. Therefore all graphs $K_{m,n}$ are Tetra Dual Magic graphs for any m, n. □

Example 2.

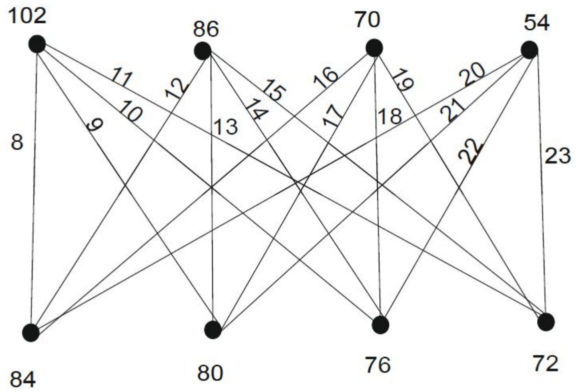

FIGURE 3. Primal graph: Vertex Magic Pyramidal labeling of $K_{4,4}(\lambda_f = 140)$.

Theorem 2.3. *All Stars $K_{1,n}$ are Tetra Dual Magic graphs for $n \geq 3$ where the tetra vertex magic constants T_v range from $\frac{n^2+3n}{2} < T_v \leq t_n$ and the tetra edge magic constants T_e range from $2n+3 \leq T_e \leq t_n$ where t_n is the n^{th} tetrahedral number.*

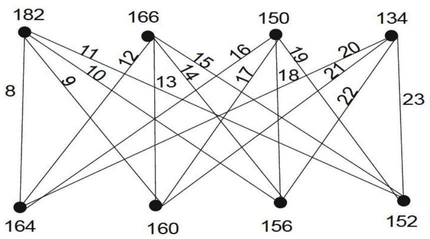

FIGURE 4. Dual graph: Tetra Dual Vertex Magic labeling of $K_{4,4}(T_v = 220 = t_{10})$.

Proof. Case 1. Let the Star $K_{1,n}$ be a Vertex Magic Pyramidal graph. Let v_0 be the root vertex of the Star $K_{1,n}$. Let $v_i,\ i = 1$ to n be the pendent vertices and $e_i,\ i = 1$ to n be the edges.

Define $f : V \cup E \to \{1, 2, 3, ..., t_q\}$ as follows:

Define $f(v_1) = T_v - 1$,

$$\begin{aligned} f(v_i) &= f(v_{i-1}) - 1 \; for \; 1 \leq i \leq n \\ f(e_i) &= i \; for \; 1 \leq i \leq n \\ f(v_0) &= T_v - \sum_{i=1}^{n} f(e_i) \end{aligned}$$

Hence all Stars $K_{1,n}$ satisfy Tetra Dual Vertex Magic labeling for $n \geq 3$.

Case 2. Let the Star $K_{1,n}$ be an Edge Magic Pyramidal graph.

Define $f : V \cup E \to \{1, 2, 3, ..., t_q\}$ as follows:

Define $f(v_0) = 1$

$$\begin{aligned} f(v_1) &= T_e - 3 \\ f(v_i) &= f(v_{i-1}) - 1 \; for \; 2 \leq i \leq n \\ f(e_i) &= i + 1 \; for \; 1 \leq i \leq n \end{aligned}$$

Hence all Stars $K_{1,n}$ satisfy Tetra Dual Edge Magic labeling for $n \geq 3$. Therefore all Stars $K_{1,n}$ are Tetra Dual Magic graphs for $n \geq 3$. $\square$

The following is an example:

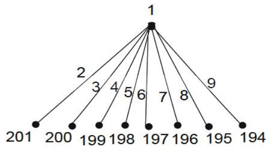

FIGURE 5. Primal graph: Edge Magic Pyramidal labeling of $K_{1,8}(\mu_f = 204)$.

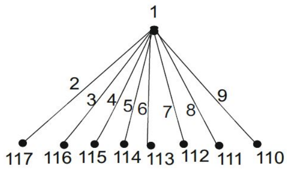

FIGURE 6. Dual graph: Tetra Dual Edge Magic labeling of $K_{1,8}(T_e = 120 = t_8)$.

3. NON - TETRA DUAL MAGIC GRAPHS

Graphs which do not satisfy Tetra Dual Magic labeling are termed as Non - Tetra Dual Magic graphs. Such graphs may not satisfy either the condition of Tetra Dual Vertex magic labeling or the condition of Tetra Dual Edge Magic labeling. The Peterson graph has 10 vertices, 15 edges and the degree of each vertex is three with two cycles and the graph is a Magic Pyramidal graph. As every edge in a graph is incident with two vertices the condition of Tetra Dual Edge magic labeling is satisfied in Peterson graph but it does not satisfy Tetra Dual Vertex magic labeling. Hence the Peterson graph is a Non - Tetra Dual Magic graph. If a graph has at least three cycles with at least four vertices of degree three then the graph fails to be a Tetra Magic Dual graph. The Hamiltonian graph Dodecahedron, the Sunflower graphs, all triangular crocodiles with at least four cycles, all Wheel graphs $W_n,\ n \geq 5$, Closed helms, Mongolian tent, Book graphs are all Non - Tetra Dual Magic graphs.

4. CONCLUSION

This work has brought into focus both pyramidal numbers and tetrahedral numbers. As the difference between the i^{th} Pyramidal number p_i and the i^{th} tetrahedral number t_i is sufficiently large for $i \geq 7$, the range of the magic constants does not coincide for certain graphs. The complete bipartite $K_{m,n}$

takes a particular range for its magic constants when it is considered as a Magic Pyramidal graph but it takes a different range while it behaves as a Tetra Dual Magic graph although the condition of duality is satisfied. All Cycles C_n are Magic Pyramidal graphs and also they satisfy the condition of Tetra Dual Magic labeling. Hence it is interesting to investigate Magic Pyramidal graphs satisfying the condition of Tetra Dual Magic labeling.

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