

NOVEL BINARY OPERATIONS ON Z-NUMBERS AND THEIR APPLICATION IN FUZZY CRITICAL PATH METHOD

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ABSTRACT. A human brain has got enormous capacity to take decisions even when the data available for scrutiny is imprecise and vague. The Aristotelian binary logic was found to be ineffective when dealing with such imprecise and vague data which abounds in real life situation. Ali Askar Zadeh introduced fuzzy sets and consequently fuzzy numbers to overcome this difficulty. Extending the notion of fuzzy numbers Zadeh introduced Z- numbers in 2011. He had also defined binary operations and outlined methods of computing sum of z-numbers etc. In this paper novel operations on Z-numbers are introduced. An application to finding the critical path in a network model where duration times of activities are z-numbers is also given

1. INTRODUCTION

In the real world, information and communication are crucial. However, in most of the situations only imprecise or vague information are available. Human beings and their languages possess remarkable flair in processing and communicating vague or uncertain data. They deal with imprecise data by analyzing and computing with 'words' rather than making cumbersome and lengthy mathematical calculations.

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Lotfi Askar Zadeh refers to this phenomenon as "computing with words" (CWW). Further he introduced the concepts of Z-numbers and z-valuations to assist in the process of mathematically modeling the concept of CWW.

In his 2011 introductory paper [1] Zadeh had stated that "An important issue relates to computation with Z-numbers. Examples: What is the sum of (about 45 MIN, very sure) and (about 30 MIN, sure)? What is the square root of (approximately 100, likely)? Computation with Z-numbers falls within the province of Computing with Words (CW or CWW)."

Computations with Z-numbers is a fertile area of research. Aliev, O. H. Huseynov and others [4–7] suggested a general and computationally effective approach for computation with discrete Z-numbers. Shahila Bhanu and Velammal [3] have made extensive study on computations with Z-numbers. P. Rani and G. Velammal [8] developed the basic ideas which will be instrumental to solve the problem of interpolation of fuzzy if-then rules in terms of Zadeh's Z-numbers.

Considering the complexity and intricacy of the method suggested by Zadeh [1], Novel R type binary operations on Z-numbers are introduced without losing the essence of the characteristics of Z-numbers treating the components separately as it is the case in most of the linguistic expressions.

2. PRELIMINARY DEFINITIONS

2.1. Fuzzy sets and fuzzy numbers.

Definition 2.1 (Fuzzy subset). *A fuzzy subset $\bar{A}$ of a universe X may be given as $\bar{A} = \{ \langle x, \mu_{\bar{A}}(x) \rangle / x \in X \}$, where $\mu_{\bar{A}} : X \rightarrow [0, 1]$ is its membership function. The membership value $\mu_{\bar{A}}(x)$ describes the degree of membership value of x in $\bar{A}$.*

Definition 2.2 (Fuzzy number). *A fuzzy set $\bar{A}$, defined on the universal set of real numbers $\mathbb{R}$, is said to be a fuzzy number if its membership function has the following characteristics:*

- (1) $\mu_{\bar{A}} : X \rightarrow [0, 1]$ is continuous,
- (2) $\mu_{\bar{A}}(x) = 0$ for all $x \in (-\infty, a] \cup [d, \infty)$,
- (3) $\mu_{\bar{A}}(x)$ is strictly increasing on $[a, b]$ and strictly decreasing on $[c, d]$,
- (4) $\mu_{\bar{A}}(x) = 1$ for all $x \in [b, c]$, where $a < b < c < d$.

Definition 2.3 (Triangular fuzzy number). *The membership function of the triangular fuzzy number $A(a, b, c)$ is defined by*

$$\mu_A(x) = \begin{cases} \frac{x-a}{b-a}, & \text{if } x \in [a, b] \\ 1, & \text{if } x = b \\ \frac{c-x}{c-b}, & \text{if } x \in [b, c] \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.4. [Comparing fuzzy numbers using ranking functions] *Let F be a set of fuzzy numbers. A ranking function r_k on F is a real valued function defined on F . Given a ranking function r_k on F we can order the elements in F as follows: $A_1 \preceq A_2$ if and only if $r_k(A_1) \leq r_k(A_2)$.*

Though there are many procedures to rank fuzzy numbers, the ranking function we had taken up for solving the succeeding problem is the center of the area method.

If $F = F(a, b, c)$ is a given triangular fuzzy number the rank of A is,

$$r_k(A) = \frac{a+b+c}{3}.$$

Define $MAX(A, B) = A$, and $MIN(A, B) = B$, if $r_k(A) \geq r_k(B)$.

We recall the definition of usual Z-numbers.

2.2. Z-numbers.

Definition 2.5 (Zadeh's definition of Z-number). *Zadeh defines Z-number as follows [1]: A Z-number is an ordered pair of fuzzy numbers, $Z = (A, B)$, associated with a real-valued uncertain variable X , with the first component A , a restriction on the values which X can take and the second component B , a measure of reliability of the first component.*

In the above definition the term "fuzzy numbers" is used somewhat vaguely. So a more precise definition:

Definition 2.6 (Formal definition of Z-number). *A Z-number is an ordered pair of fuzzy sets (A, B) where A is a fuzzy set defined on the real line and B is a fuzzy number whose support is contained in the segment $[0, 1]$.*

Definition 2.7 (Fuzzy event probability). *If X is a random variable, ' X is A ' represents a fuzzy event in $\mathbf{R}$, the real line. The probability of this event is expressed*

as $FEP(X \text{ is } A) = \int \mu_A(u)p_X(u)du$, where μ_A is the membership function of A , p_X is the underlying probability density of X and u is a generic value of X .

Here FEP stands for fuzzy event probability.

Note: Fuzzy event probability is a generalization of ordinary probability.

Definition 2.8 (Formal definition of Z-valuation). A Z-valuation is an ordered triple (X, A, B) where X is an uncertain variable, A is a fuzzy set defined on the real line and B is a fuzzy number whose support is contained in $[0, 1]$.

The Z-valuation (X, A, B) may also be denoted as ' $X \text{ is } z(A, B)$ '. A Z-valuation is equivalent to an assignment statement " $X \text{ is } (A, B)$ ". It may be viewed as a restriction on X defined by $FEP(X \text{ is } A) \text{ is } B$ More explicitly.

Possibility, $(FEP(x \in A) = u) = \mu_B(u)$. Since 2011 there has been a lot of interest in the area of z-numbers. The computations with z-numbers have been studied in both continuous and discrete cases [2, 3].

3. R TYPE OPERATIONS ON Z-NUMBERS

We introduce several R type binary operations. Let $\star$ be any one of the basic arithmetic operations like addition, subtraction, multiplication or division.

Definition 3.1 (Product R Type Operation:).

$$(A_1, B_1) (\star, \cdot) (A_2, B_2) = (A_1 \star A_2, B_1 \cdot B_2)$$

$A_1 \star A_2$ and $B_1 \cdot B_2$ are calculated using extension principle.

MAX, MIN R Operation:

$$(A_1, B_1) (MAX, MIN) (A_2, B_2) = (MAX(A_1, A_2), MIN(B_1, B_2))$$

MIN, MIN R Operation:

$$(A_1, B_1) (MIN, MIN) (A_2, B_2) = (MIN(A_1, A_2), MIN(B_1, B_2))$$

where MIN and MAX are operations on fuzzy number as in Definition 2.4 above.

4. APPLICATION IN CPM MODEL

Project scheduling is the process of chronologically sequencing a collection of predetermined activities of a project depending on the constraints so that the project is completed within the optimal time limit.

Critical Path Method is a procedure developed to identify the activities which are critical for timely completion of the project. The estimated completion time of an activity and related parameters are often vague and imprecise thus fuzzy in nature. In addition the data may not be 100% reliable. Z-numbers are the best fit to model such situations. While the first component gives the information, the second component refers to the reliability of this information.

In the proposed method we are using the novel R type arithmetic operations and a ranking method for z-numbers for solving project scheduling problems with z-numbers representing the activities of the project. Solving project scheduling under fuzzy environment had been a fascination for many and they have applied different methods of ranking fuzzy numbers. M. Shanmugasundari and K. Ganesan [9] devised a method to solve a project scheduling problem under fuzzy environment where the fuzzy duration of the activities are trapezoidal fuzzy numbers. D. Stephen Dinagar and N. Ramerasian [10] analyzed the critical path method where the processing time of all activities are octagonal fuzzy numbers.

4.1. Algorithm to find fuzzy critical path with activities as triangular Z-number.

Step 1: Identify the fuzzy activities of the project with a suitable triangular Z-numbers.

Step 2: Find the precedence relation of each activity and draw the network diagram.

Step 3: Compute the fuzzy earliest start time at each node as below. Let there be n nodes vertices in the network diagram. Let $\bar{EST}_1 = ((0, 0, 0), (1, 1, 1))$. We estimate the $\bar{EST}_j$ at succeeding vertices as follows:

$$\bar{EST}_j = (MAX, MIN)_{i \in N_j} \{ \bar{EST}_i (+, MIN) D_{ij} \}$$

where N_j is the set of all vertices preceding i^{th} vertex and D_{ij} is the duration of the activity $i \rightarrow j$. Let $\bar{LCT}_n = \bar{EST}_n$,

$$\bar{LCT}_i = (MIN, MIN)_{j \in M_i} \{ \bar{EST}_j (-, MIN) D_{ij} \}$$

where M_i is the set of all vertices succeeding the i^{th} vertex and D_{ij} is the fuzzy duration of the activity $i \rightarrow j$.

Step 4: Identify the fuzzy critical path and fuzzy float time as triangular Z-numbers. Fuzzy Critical path is identified as in the crisp case by identifying the nodes where the earliest starting time and the latest completion time are nearly equal. Fuzzy float time is calculated as in the formula below:

$$\bar{F}L_{i,j} = \{[L\bar{C}T_j(-, MIN)E\bar{S}T_i](-, MIN)D_{i,j}\}$$

Step 5: Convert the conclusions as expressions in words for public consumptions.

5. NUMERICAL EXAMPLE

Consider the following project scheduling problem with precedence relating given by diagram below

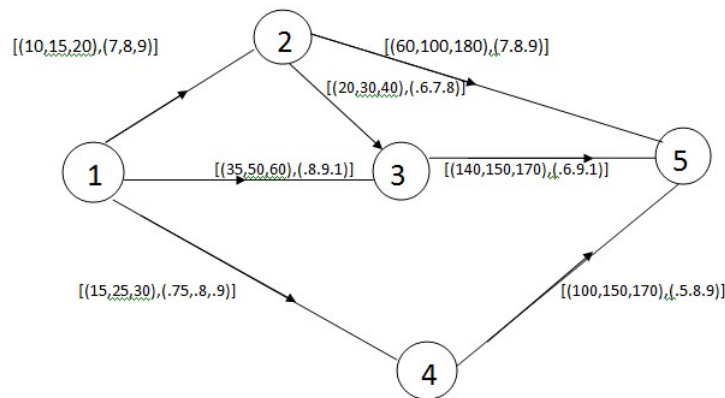


FIGURE 1. Project scheduling problem

Activity	Duration as Z-numbers
$1 \rightarrow 2$	$[(10,15,20),(.7,.8,.9)]$
$1 \rightarrow 3$	$[(35,50,60),(.8,.9,.1)]$
$1 \rightarrow 4$	$[(15,25,30),(.75,.8,.9)]$
$2 \rightarrow 3$	$[(20,30,40),(.6,.7,.8)]$
$2 \rightarrow 5$	$[(60,100,180),(.7,.8,.9)]$
$3 \rightarrow 5$	$[(140,150,170),(.6,.9,.1)]$
$4 \rightarrow 5$	$[(100,150,170),(.5,.8,.9)]$

TABLE 1. Project scheduling problem

6. CALCULATIONS

6.1. Forward iterations.

$$E\bar{S}T_1 = [(0, 0, 0), (1, 1, 1)]$$

$$\begin{aligned} E\bar{S}T_2 &= [(0, 0, 0), (1, 1, 1)] (+, MIN) [(10, 15, 20), (.7, .8, .9)] \\ &= [(10, 15, 20), (.7, .8, .9)] \end{aligned}$$

$$\begin{aligned} E\bar{S}T_3 &= (MAX, MIN) \{[(35, 50, 60), (.8, .9, .1)], \{[(10, 15, 20), (.7, .8, .9)] \\ &\quad (+, MIN) [(20, 30, 40), (.6, .7, .8)]\}\} \\ &= (MAX, MIN) \{[(35, 50, 60), (.8, .9, .1)], [(30, 45, 60), (.6, .7, .8)]\} \end{aligned}$$

$$r_k(35, 50, 60) = 48.3, r_k(30, 45, 60) = 45$$

$$r_k(.8, .9, .1) = .9, r_k(.6, .7, .8) = .7$$

hence

$$\begin{aligned} r_k(35, 50, 60) &\geq r_k(30, 45, 60), r_k(.8, .9, .1) \geq r_k(.6, .7, .8) \\ E\bar{S}T_3 &= [(35, 50, 60), (.6, .7, .8)] \end{aligned}$$

Like wise,

$$E\bar{S}T_4 = [(15, 25, 30), (.75, .8, .9)]$$

$$E\bar{S}T_5 = [(175, 200, 230), (.5, .8, .9)]$$

6.2. Backward iterations.

Let

$$\overline{LCT}_5 = E\bar{S}T_5 = [(175, 200, 230), (.5, .8, .9)]$$

$$\begin{aligned}\overline{LCT}_4 &= \{[(175, 200, 230), (.5, .7, .9)](-, MIN)[100, 150, 170), (.5, .8, .9)]\} \\ &= [(5, 50, 130), (.5, .7, .9)]\end{aligned}$$

$$\begin{aligned}\overline{LCT}_3 &= \{[(175, 200, 230), (.5, .7, .9)](-, MIN)[(140, 150, 170), (.6, .9, 1)]\} \\ &= [(5, 50, 90), (.5, .7, .9)]\end{aligned}$$

$$\begin{aligned}\overline{LCT}_2 &= (MIN, MIN)\{\{\overline{LCT}_3(-, MIN)[(20, 30, 40), (.6, .7, .8)]\}, \{\overline{LCT}_5 \\ &\quad (-, MIN)[(60, 100, 180), (.7, .8, .9)]\} \\ &= (MIN, MIN)\{[(5, 50, 90), (.5, .7, .9)](-, MIN)[(20, 30, 40), (.6, .7, .8)], \\ &\quad [(175, 200, 230), (.5, .7, .9)](-, MIN)[(60, 100, 180), (.7, .8, .9)] \\ &= (MIN, MIN)\{[(-35, 20, 70), (.5, .7, .8)], [(-5, 100, 170), (.5, .7, .9)]\}\end{aligned}$$

$$r_k(-35, 20, 70) = 18.3, r_k(-5, 100, 170) = 88.3$$

$$r_k(.5, .7, .8) = .7, r_k(.5, .7, .9) = .7$$

$$\overline{LCT}_2 = [(-35, 20, 70), (.6, .7, .8)]$$

Like wise, $\overline{LCT}_1 = [((-55, , 0, 55), (.5, .7, .8))]$

6.3. Different parameters of the project scheduling.

Activity	1-2	1-3	1-4
Duration	[(10,15,20),(.7,.8,.9)]	[(35,50,60),(.8,.9,1)]	[(15,25,30),(.75,.8,.9)]
Earliest starting time	[(0,0,0),(1,1,1)]	[(0,0,0),(1,1,1)]	[(0,0,0),(1,1,1)]
Earliest finishing time	[(10,15,20),(.7,.8,.9)]	[(35,50,60),(.6,.7,.8)]	[(15,25,30),(.75,.8,.9)]
Latest starting time	[((-55,-5,50),(.5,.7,.8)]	[(-55,-0,55),(.5,.7,.8)]	[(-25,25,115),(.5,.7,.8)]
Latest finishing time	[(-35,20,70),(.5,.7,.8)]	[(5,50,90),(.5,.7,.9)]	[(5,50,130),(.5,.7,.9)]
Float	[(-55,5,60),(.6,.7,.8)]	[(-55,0,65),(.6,.7,.8)]	[(-25,25,115),(.5,.7,.8)]

TABLE 2

Critical Path $1 \rightarrow 3 \rightarrow 5$

Activity	2-3	2-5	3-5
Duration	$[(20,30,40),(.6,.7,.8)]$	$[(60,100,180),(.7,.8,.9)]$	$[(140,150,170),(.6,.9,1)]$
Earliest starting time	$[(10,15,20),(.7,.8,.9)]$	$[(10,15,20),(.7,.8,.9)]$	$[(35,50,60),(.6,.7,.8)]$
Earliest finishing time	$[(35,45,60),(.6,.7,.8)]$	$[(70,115,200),(.5,.7,.9)]$	$[(175,200,230),(.5,.7,.9)]$
Latest starting time	$[(-35,20,70),(.5,.7,.8)]$	$[(-5,100,170),(.5,.7,.8)]$	$[(5,50,90),(.5,.7,.9)]$
Latest finishing time	$[(5,50,90),(.5,.7,.9)]$	$[(175,200,230),(.5,.7,.9)]$	$[(175,200,230),(.5,.7,.9)]$
Float	$[(-55,5,60),(.6,.7,.8)]$	$[(-25,80,160),(.6,.7,.8)]$	$[(0,0,0),(.6,.7,.8)]$

TABLE 3

Activity	4-5
Duration	$[(100,150,170),(.5,.8,.9)]$
Earliest starting time	$[(15,25,30),(.75,.8,.9)]$
Earliest finishing time	$(115,175,200),(.5,.7,.9)$
Latest starting time	$[(5,50,130),(.5,.7,.9)]$
Latest finishing time	$(175,200,230),(.5,.7,.9)$
Float	$[(-25,25,115),(.5,.7,.8)]$

TABLE 4

7. CONCLUSION

The paper attempts to solve project scheduling in fuzzy environments with the parameters as Z-numbers. There are parameters where we have Z-numbers with negative values. This need not be taken as an annoyance since we look at it only as a Z-numbers, not as a real numbers.

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REFERENCES

- [1] L. A. ZADEH: *A note on Z-numbers*, Information Science, **181** (2011), 2923–2932.
- [2] R. R. YAGER: *On a view of Zadeh's Z-numbers*, Advances in Computational Intelligence, **299** (2012), 90–101.
- [3] M. S. BHANU, G. VELAMMAL: *Operations on Zadeh's Z-numbers*, IOSR Journal of Mathematics, **11** (2015), 88–94.
- [4] R. A. ALIEV, A. V. ALIZAEDH, O. H. HUSEYNOV: *The Arithmetic of discrete Z-numbers*, Information Sciences, **290** (2015), 134–155.
- [5] R. A. ALIEV, O. H. HUSEYNOV, L. M. ZEINLOVA: *The Arithmetic of Continuous Z-numbers*, Information Sciences, **373** (2016), 441–460.
- [6] R. A. ALIEV, O. H. HUSEYNOV, R. R. ALIYEV: *A sum of a large number of Z-numbers*, Procedia Computer Science, **120** (2017), 16–22.
- [7] R. A. ALIEV, A. V. ALIZAEDH, O. H. HUSEYNOV: *An Introduction to the arithmetic of Z-numbers by using horizontal membership functions*, Procedia Computer Science, **120** (2017), 349–356.
- [8] P. RANI, G. VELAMMAL: *Interpolation of Fuzzy if-then rules in context of Zadeh's Z-numbers*, American International Journal of Research in Science, Technology, Engineering & Mathematics, **15**(1) (2016), 30–34.
- [9] M. SHANMUGASUNDARI, K. GANESAN: *Project Scheduling Problems Under Fuzzy Environment: A New Solution Approach*, International Journal of Pure and Applied Mathematics, **95**(3) (2014), 387–399.
- [10] D. S. DINAGAR, N. RAMESHAN: *A new approach for fuzzy critical path method using octagonal fuzzy numbers*, International Journal of Pure and Applied Mathematics, **119**(13) (2018), 357–364.

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