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SPECIAL ANALYSIS ON A PYTHAGOREAN TRIANGLE WHICH SATISFIES α ((HYPOTONUSE$\times$ PERIMETER)-4(AREA))=($A^2 - B^2$)(PERIMETER) FOR SOME PARTICULAR DIFFERENT VALUES OF α

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ABSTRACT. We obtain non-trivial integer solutions for the sides of the Pythagorean triangle, for some particular values of α which satisfies ((Hypotonuse$\times$ Perimeter)-4(Area))=$(a^2 - b^2)$(Perimeter) . A few interesting relations between the sides of the Pythagorean triangle are presented.

1. INTRODUCTION

One well known set of solutions of the Pythagorean equation $x^2 + y^2 = z^2$ are $x = 2uv$, $y = u^2 - v^2$ and $z = u^2 + v^2$. Many mathematicians has used this set of solutions to obtain the non-zero integral values for x, y and z [1–3]. As a new approach, in this paper we introduce another set of solutions $x = 2U + 1$, $y = 2U^2 + 2U$ and $z = 2U^2 + 2U + 1$ for the equation $x^2 + y^2 = z^2$. By using this solution we obtain three non-zero integers x, y and z under certain relations satisfying the equation $x^2 + y^2 = z^2$ [4–6]. In this communication we present yet another interest Pythagorean triangles, such that in each of them, α ((Hypotonuse$\times$ Perimeter)-4(Area))=$(a^2 - b^2)$(Perimeter). A few interesting relations are also given. In addition, the recurrence relations for the sides of the triangle are presented.

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2. Methods of analysis

Taking $U > 0$, to be the generators of the Pythagorean triangle (x, y, z), the assumption that α ((Hypotonuse× Perimeter)-4(Area))=$(a^2 - b^2)$ (Perimeter) becomes:

$$\alpha(((2U^2+2U+1)\times(2U+1+2U^2+2U+2U^2+2U+1))-4(\frac{1}{2}(2U+1)(2U^2+2U)) =$$

$$= (a^2 - b^2)(2U + 1 + 2U^2 + 2U + 2U^2 + 2U + 1)$$

which leads to the equation

$$\alpha(2U^2 + 1) = (a^2 - b^2)\,. \tag{2.1}$$

For the understanding we consider the cases, $\alpha = 1$ and $\alpha = 3$.

Choice I:
If we consider $\alpha = 1$ then the equation (2.1) becomes $a^2 = 2U^2 + b^2 + 1$, which leads to the Pellian equation

$$X^2 = DY^2 + K \tag{2.2}$$

where $D = 2, X = a, Y = U$ and $K = b^2 + 1$. Clearly K is not a Perfect square. For the sake of clear understanding, we present below forms of integral solutions and thus the following choice of b:

(i) b=1
(ii) b=4
(iii) b=7

Case (1): Setting $b = 1$, so that $K = 2$. The equation (2.2) becomes

$$X^2 = 2Y^2 + 2 \tag{2.3}$$

$(X_0, Y_0) = (10, 7)$ will be the initial solution of (2.3). Consider the Pellian

$$X^2 = 2Y^2 + 1 \tag{2.4}$$

Let $((\bar{x_0}), (\bar{y_0})) = (3, 2)$ be the initial solution of (2.4). Using Brahmagupta lemma, the general solution $(\bar{x}_n, \bar{y}_n)$ of the equation (2.4) is given by:

$$\bar{x}_n + \sqrt{2}\bar{y}_n = (3 + 2\sqrt{2})^{n+1}, n = 0, 1, 2, ...$$

Since irrational roots occur in pairs,

$$\bar{x}_n + \sqrt{2}\bar{y}_n = (3-2\sqrt{2})^{n+1}, n = 0, 1, 2, ...$$

From the previous equations, we obtain

$$\bar{x}_n = \frac{1}{2}[(3+2\sqrt{2})^{n+1} + (3-2\sqrt{2})^{n+1}], n = 0, 1, 2, 3, ... \tag{2.5}$$

$$\bar{y}_n = \frac{1}{2\sqrt{2}}[(3+2\sqrt{2})^{n+1} + (3-2\sqrt{2})^{n+1}], n = 0, 1, 2, 3, ... \tag{2.6}$$

Using the equations (2.5) and (2.6), the solutions of equation (2.3) are given by:

$$X_{n+1} = X_0\bar{x}_n + DY_0\bar{y}_n, n = 0, 1, 2, 3, ...$$

$$Y_{n+1} = X_0\bar{y}_n + Y_0\bar{x}_n, n = 0, 1, 2, 3, ...$$

So that,

$$X_{n+1} = [(3+2\sqrt{2})^{n+1}(10+7\sqrt{2}) + (3-2\sqrt{2})^{n+1}(10-7\sqrt{2})], n = -1, 0, 1, 2, 3, ...$$

$$U_{n+1} = Y_{n+1} = \frac{1}{\sqrt{2}}[(3+2\sqrt{2})^{n+1}(10+7\sqrt{2}) - (3-2\sqrt{2})^{n+1}(10-7\sqrt{2})], n = -1, 0, 1, 2, 3, ...$$

n	X_{n+1}	$U_{n+1} = Y_{n+1}$	$x = 2U+1$	$y = 2U^2+2U$	$z = 2U^2+2U+1$
-1	10	7	15	112	113
0	58	41	83	3444	3445
1	338	239	479	114720	114721
2	1970	1393	2787	3883684	3883685
3	11482	8119	16239	131852560	131852561
4	66922	47321	94643	4478648724	4478648725

TABLE 1. Numerical examples

Observations

(1) Recurrence relations for X and Y are

$$X_{n+3} - 6X_{n+2} + X_{n+1} = 0$$

$$Y_{n+3} - 6Y_{n+2} + Y_{n+1} = 0$$

(2) For all values of n, $X_{n+3} + X_{n+1} \equiv 0(mod 6)$

(3) For all values of n, $Y_{n+3} + Y_{n+1} \equiv 0(mod 6)$

(4) For all values of n, X is even and Y is odd.
(5) For all values of n , X is divisible by 2.

Case (2): Setting $b = 4$, so that $K = 17$. The equation $X^2 = DY^2 + K$ becomes

$$X^2 = 2Y^2 + 17 \tag{2.7}$$

$(X_0, Y_0) = (5, 2)$ will be the initial solution of (2.7). Consider the Pellian

$$X^2 = 2Y^2 + 1 \tag{2.8}$$

Let $((\bar{x_0}), (\bar{y_0})) = (3, 2)$ be the initial solution of (2.8). Using Brahmagupta lemma, the general solution $(\bar{x}_n, \bar{y}_n)$ of equation (2.8) is given by:

$$\bar{x}_n + \sqrt{2}\bar{y}_n = (3 + 2\sqrt{2})^{n+1}, n = 0, 1, 2, ... \tag{2.9}$$

Since irrational roots occur in pairs,

$$\bar{x}_n - \sqrt{2}\bar{y}_n = (3 - 2\sqrt{2})^{n+1}, n = 0, 1, 2, ... \tag{2.10}$$

Using equations (2.9) and (2.10), we obtain:

$$\bar{x}_n = \frac{1}{2}[(3 + 2\sqrt{2})^{n+1} + (3 - 2\sqrt{2})^{n+1}], n = 0, 1, 2, 3, ... \tag{2.11}$$

$$\bar{y}_n = \frac{1}{2\sqrt{2}}[(3 + 2\sqrt{2})^{n+1} - (3 - 2\sqrt{2})^{n+1}], n = 0, 1, 2, 3, ... \tag{2.12}$$

Using the equations (2.11) and (2.12), the solutions of the equation (2.7) is given by:

$$X_{n+1} = \frac{1}{2}[(3 + 2\sqrt{2})^{n+1}(5 + 2\sqrt{2}) + (3 - 2\sqrt{2})^{n+1}(5 - 2\sqrt{2})], n = -1, 0, 1, 2, 3, ...$$

$$U_{n+1} = Y_{n+1} = \frac{1}{2\sqrt{2}}[(3+2\sqrt{2})^{n+1}(5+2\sqrt{2})-(3-2\sqrt{2})^{n+1}(5-2\sqrt{2})], n = -1, 0, 1, 2, 3, ...$$

Observations

(1) Recurrence relations for X and Y are:

$$X_{n+3} - 6X_{n+2} + X_{n+1} = 0$$

$$Y_{n+3} - 6Y_{n+2} + Y_{n+1} = 0$$

(2) For all values of n, $X_{n+3} + X_{n+1} \equiv 0 (mod 6)$
(3) For all values of n, $Y_{n+3} + Y_{n+1} \equiv 0 (mod 6)$

n	X_{n+1}	$U_{n+1} = Y_{n+1}$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
-1	5	2	5	12	13
0	23	16	33	544	545
1	133	94	189	17860	17861
2	775	548	1097	601704	601705
3	4517	3194	6389	20409660	20409661

TABLE 2. Numerical examples

(4) For all values of n, X is odd and Y is even.
(5) For all values of n, Y is divisible by 2.

Case (3): Setting $b = 7$, so that $K = 50$. The equation $X^2 = DY^2 + K$ becomes

$$X^2 = 2Y^2 + 50 \tag{2.13}$$

$(X_0, Y_0) = (10, 5)$ will be the initial solution of (2.13). Consider the Pellian

$$X^2 = 2Y^2 + 1 \tag{2.14}$$

Let $(\bar{x_0}, \bar{y_0}) = (3, 2)$ be the initial solution of (2.14). Using Brahmagupta lemma, the general solution $(\bar{x}_n, \bar{y}_n)$ of equation (2.14) is given by:

$$\bar{x}_n + \sqrt{2}\bar{y}_n = (3 + 2\sqrt{2})^{n+1}, n = 0, 1, 2, ... \tag{2.15}$$

Since irrational roots occur in pairs

$$\bar{x}_n - \sqrt{2}\bar{y}_n = (3 - 2\sqrt{2})^{n+1}, n = 0, 1, 2, ... \tag{2.16}$$

Using equations (2.15) and (2.16), we obtain:

$$\bar{x}_n = \frac{1}{2}[(3 + 2\sqrt{2})^{n+1} + (3 - 2\sqrt{2})^{n+1}], n = 0, 1, 2, 3, ... \tag{2.17}$$

$$\bar{y}_n = \frac{1}{2\sqrt{2}}[(3 + 2\sqrt{2})^{n+1} - (3 - 2\sqrt{2})^{n+1}], n = 0, 1, 2, 3, ... \tag{2.18}$$

Using the equations (2.17) and (2.18), the solutions of equation (2.13) is given by:

$$X_{n+1} = \frac{1}{2}[(3+2\sqrt{2})^{n+1}(10+5\sqrt{2}) + (3-2\sqrt{2})^{n+1}(10-5\sqrt{2})], n = -1, 0, 1, 2, 3, ...$$

$$U_{n+1} = Y_{n+1} = \frac{1}{2\sqrt{2}}[(3+2\sqrt{2})^{n+1}(10+5\sqrt{2}) - (3-2\sqrt{2})^{n+1}(10-5\sqrt{2})],$$

where $n = -1, 0, 1, 2, 3, ...$

n	X_{n+1}	$U_{n+1} = Y_{n+1}$	$x = 2U+1$	$y = 2U^2+2U$	$z = 2U^2+2U+1$
-1	10	5	11	60	61
0	50	35	71	2520	2521
1	290	205	411	84460	84461
2	1690	1195	2391	2858440	2858441
3	9850	6965	13931	97036380	97036381

TABLE 3. Numerical examples

Observations:

(1) Recurrence relations for X and Y are:

$$X_{n+3} - 6X_{n+2} + X_{n+1} = 0$$

$$Y_{n+3} - 6Y_{n+2} + Y_{n+1} = 0$$

(2) For all values of n, $X_{n+3} + X_{n+1} \equiv 0(mod 6)$

(3) For all values of n, $Y_{n+3} + Y_{n+1} \equiv 0(mod 6)$

(4) For all values of n, X is even and Y is odd.

(5) For all values of n, X is divisible by 5 and 10, and Y is divisible by 5.

Choice II:

Consider the, $\alpha = 3$ so that equation (2.1) becomes $a^2 = 6U^2 + b^2 + 3$, which leads to the Pellian equation

$$X^2 = DY^2 + K \tag{2.19}$$

where $D = 6, X = a, Y = U$ and $K = b^2 + 3$. Clearly K is not a Perfect square. For the sake of clear understanding, we present below forms of integral solutions of (2.19) and thus the following choices of b:

(i) $b = 1$

(ii) $b = 3$

Case (1): Setting $b = 1$, so that $K = 4$ (Perfect Square). The equation $X^2 = DY^2 + K$ becomes

(2.20) $$X^2 = 6Y^2 + 4$$

$(X_0, Y_0) = (10, 4)$ will be the initial solution of (2.20). Consider the Pellian

(2.21) $$X^2 = 6Y^2 + 1$$

Let $(\bar{x}_0, \bar{y}_0)$=(5,2) be the initial solution of (2.21). Using Brahmagupta lemma, the general solution $(\bar{x}_n, \bar{y}_n)$ of equation (2.21) is given by:

(2.22) $$\bar{x}_n + \sqrt{2}\bar{y}_n = (5 + 2\sqrt{6})^{n+1}, n = 0, 1, 2, ...$$

Since irrational roots occur in pairs,

(2.23) $$\bar{x}_n - \sqrt{2}\bar{y}_n = (5 - 2\sqrt{6})^{n+1}, n = 0, 1, 2, ...$$

From equation (2.22) and (2.23), we obtain:

(2.24) $$\bar{x}_n = \frac{1}{2}[(5 + 2\sqrt{6})^{n+1} + (5 - 2\sqrt{6})^{n+1}], n = 0, 1, 2, 3, ...$$

(2.25) $$\bar{y}_n = \frac{1}{2\sqrt{6}}[(5 + 2\sqrt{6})^{n+1} - (5 - 2\sqrt{6})^{n+1}], n = 0, 1, 2, 3, ...$$

Using the equations (2.24) and (2.25), the solutions of equation (2.20) is given by:

$$X_n = \sqrt{K}\bar{x}_n, n = 0, 1, 2, 3, ...$$
$$Y_n = \sqrt{K}\bar{y}_n, n = 0, 1, 2, 3, ...$$

So that,

$$X_n = [(5 + 2\sqrt{6})^{n+1} + (5 - 2\sqrt{6})^{n+1}], n = 0, 1, 2, 3, ...$$
$$U_n = Y_n = \frac{1}{\sqrt{6}}[(5 + 2\sqrt{6})^{n+1} - (5 - 2\sqrt{6})^{n+1}], n = 0, 1, 2, 3, ...$$

Observations

(1) Recurrence relations for X and Y are

$$X_{n+2} - 10X_{n+1} + X_n = 0$$

$$Y_{n+2} - 10Y_{n+1} + Y_n = 0$$

(2) For all values of n, $X_{n+2} + X_n \equiv 0(mod 10)$

n	X_{n+1}	$U_{n+1} = Y_{n+1}$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
-1	10	4	9	40	41
0	98	40	81	3280	3281
1	970	396	793	314424	314425
2	9602	3920	7841	30740640	30740641
3	95050	38804	77609	3011578440	3011578441

TABLE 4. Numerical examples

(3) For all values of n, $Y_{n+2} + Y_n \equiv 0(mod 10)$
(4) For all values of n, both X and Y are even.
(5) For all values of n, X is divisible by 2, and Y is divisible by 4.

Case (2): Setting $b = 3$, so that $K = 12$ (Non Perfect Square). The equation $X^2 = DY^2 + K$ becomes

$$X^2 = 6Y^2 + 12 \tag{2.26}$$

$(X_0, Y_0) = (6, 2)$ will be the initial solution of (2.26) Consider the Pellian

$$X^2 = 6Y^2 + 1 \tag{2.27}$$

Let $(\bar{x}_0, \bar{y}_0) = (5, 2)$ be the initial solution of (2.27). Using Brahmagupta lemma, the general solution $(\bar{x}_n, \bar{y}_n)$ of equation (2.27) is given by:

$$\bar{x}_n + \sqrt{2}\bar{y}_n = (5 + 2\sqrt{6})^{n+1}, n = 0, 1, 2, ... \tag{2.28}$$

Since irrational roots occur in pairs,

$$\bar{x}_n - \sqrt{2}\bar{y}_n = (5 - 2\sqrt{6})^{n+1}, n = 0, 1, 2, ... \tag{2.29}$$

From equation (2.28) and (2.29), we obtain:

$$\bar{x}_n = \frac{1}{2}[(5 + 2\sqrt{6})^{n+1} + (5 - 2\sqrt{6})^{n+1}], n = 0, 1, 2, 3, ... \tag{2.30}$$

$$\bar{y}_n = \frac{1}{2\sqrt{6}}[(5 + 2\sqrt{6})^{n+1} - (5 - 26)^{n+1}], n = 0, 1, 2, 3, ... \tag{2.31}$$

Using the equations ((2.30) and (2.31), the solutions of equation (2.26) are given by:

$$X_{n+1} = [(5 + 2\sqrt{6})^{n+1}(3 + \sqrt{6}) + (5 - 2\sqrt{6})^{n+1}(3 - \sqrt{6})], n = -1, 0, 1, 2, 3, ...$$

$$U_{n+1} = Y_{n+1} = \frac{1}{2\sqrt{2}}[(3 + 2\sqrt{2})^{n+1}(5 + 2\sqrt{2}) - (3 - 2\sqrt{2})^{n+1}(5 - 2\sqrt{2})],$$

where $n = -1, 0, 1, 2, 3, ...$

n	X_{n+1}	$U_{n+1} = Y_{n+1}$	$x = 2U + 1$	$y = 2U^2 + 2U$	$z = 2U^2 + 2U + 1$
-1	6	2	5	12	13
0	54	22	45	1012	1013
1	534	218	437	95484	95485
2	5286	2158	4317	9318244	9318245
3	52326	21362	42725	912712812	912712813

TABLE 5. Numerical examples

Observations

(1) Recurrence relations for $X$and Y are:

$$X_{n+3} - 10X_{n+2} + X_{n+1} = 0$$

$$Y_{n+3} - 10Y_{n+2} + Y_{n+1} = 0$$

(2) For all values of n, $X_{n+3} + X_{n+1} \equiv 0(mod 10)$
(3) For all values of n, $Y_{n+3} + Y_{n+1} \equiv 0(mod 10)$
(4) For all values of n, both X and Y are even.
(5) For all values of n, X is divisible by 2, 3 and 6, and Y is divisible by 2.

3. CONCLUSION

One may search for other patterns of solutions and their corresponding properties.

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