

MITTAG-LEFFLER-ULAM STABILITIES OF FRACTIONAL EVOLUTION EQUATIONS

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ABSTRACT. In this paper, we present and discuss four types of Mittag-Leffler-Ulam stability: Mittag-Leffler-Ulam-Hyers stability, generalized Mittag-Leffler-Ulam-Hyers stability, Mittag-Leffler-Ulam-Hyers-Rassias stability and generalized Mittag-Leffler-Ulam-Hyers-Rassias stability for fractional evolution equation in Banach spaces.

1. INTRODUCTION

Numerous research papers and monographs have appeared devoted to fractional differential and fractional integral equations. It draws a great application in nonlinear oscillations of earthquakes, many physicalphenomena such as see page flow in porous media and in fluid dynamic traffic model. For more detail on fractional calculus theory, one can see the monographs of Kilb et al. in [1], Miller and Ross in [2], Podlubny in [3]. Fractional differential equations and its optimal control problems have been studied in the papers, [4–6]. On the other hand, in the theory of functional equations there are some special kind of data dependence: Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Hyers-Bourgin and Aoki-Rassias, [7–9]. Motivated by [5, 10], we present four types of Mittag-Leffler-Ulam stability for the following fractional evolution equation in a Banach space

$${}^c D^q u(t) = -Au(t) + f(t, u(t)), q \in (0, 1) t \in I \subset \mathbb{R},$$

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where ${}^c D^q$ is the Caputo fractional derivative of order q and

- (i) $I := [0, b] \text{ or } [0, +\infty)$;
- (ii) $(\mathbb{B}, |\cdot|)$ is a Banach space;
- (iii) $-A : D(A) \rightarrow \mathbb{B}$ is the infinitesimal generator of a C_0 -semigroup $\{S(t), t \geq 0\}$ or an analytic semigroup of uniformly bounded linear operators $\{S(t), t \geq 0\}$;
- (iv) $f \in C(I \times \mathbb{B}, \mathbb{B})$ or $f \in C(I \times \mathbb{B}_\alpha, \mathbb{B})$, where $\mathbb{B}_\alpha = D(A^\alpha)$ ($0 < \alpha < 1$) is a Banach space with the norm $|x|_\alpha = |A^\alpha x|$ for $x \in \mathbb{B}_\alpha$.

2. PRELIMINARIES

Let $(\mathbb{B}, |\cdot|)$ and $(\mathbb{Y}, |\cdot|)$ be two Banach spaces, $L_b(\mathbb{B}, \mathbb{Y})$ denote the space of bounded linear operators from \mathbb{B} to \mathbb{Y} . We denote $C(I, \mathbb{B})$ the Banach space of all continuous functions from I into \mathbb{B} with the norm $\|X\|_C := \sup\{|x(t)| : t \in I\}$.

For the main operator $-A$, we consider the following two cases.

Case1. Let $-A : D(A) \rightarrow \mathbb{B}$, be the infinite simalgenerator of aC_0 -semigroup $\{S(t), t \geq 0\}$. That is, for some fixed $T > 0$, we can denote

$M := \sup_{t \in [0, T]} \|S(t)\|_{L_b(\mathbb{B}, \mathbb{B})}$, which is a finite number.

Case2. Let $-A : D(A) \rightarrow \mathbb{B}$, be the in finite simalgenerator of an analytic semigroup of uniformly bounded linear operators $\{S(t), t \geq 0\}$. This means that there exists $M > 1$ such that $\|S(t)\| \leq M$. Let us recall the following known definitions of fractional calculus. For more details, see Kilbas et al. in [1].

Definition 2.1. The fractional integral of order γ with the lower limit zero for a function f is defined as

$$I^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \quad t > 0, \gamma > 0,$$

provided the right side is point-w is defined on $[0, \infty)$, where $\Gamma()$ is the gamma function.

Definition 2.2. The Riemann Liouville derivative of order γ with the lower limit zero for a function $f : [0, \infty) \rightarrow \mathbb{R}$ can be written as

$${}^L D^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} ds, \quad t > 0, -1 < \gamma < n.$$

Definition 2.3. The Caputo derivative of order γ for a function $f : [0, \infty) \rightarrow \mathbb{R}$ can be written as

$${}^c D^\gamma f(t) = {}^L D^\gamma (f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)), t > 0, n-1 < \gamma < n$$

Remark 2.1.

- (1) If $f(t) \in C^n[0, \infty)$, then ${}^c D^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}} ds$
 $= I^{n-\gamma} f^{(n)}(t) \quad t > 0, -1 < \gamma < n.$
- (2) The Caputo derivative of a constant is equal to zero.
- (3) If f is an abstract function with values in B , then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

We recall the following definition of a mild solution for the problem below. For more details, one can see our earlier work, [5, 6].

Definition 2.4.

$$\begin{cases} {}^c D^q u(t) = -Au(t) + f(t, u(t)), & t \in I, q \in (0, 1), \\ u(0) = u_0 \end{cases}$$

we mean that the function $u \in C(I, \mathbb{B})$ which satisfies

$$u(t) = \mathcal{T}(t)u_0 + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) f(s, u(s)) ds, t \in I,$$

where

$$\begin{aligned} \mathcal{T}(t) &= \int_0^\infty \xi_q(\theta) S(t^q \theta) d\theta, \mathcal{S}(t) = q \int_0^\infty \theta \xi_q(\theta) S(t^q \theta) d\theta, \\ \xi_q(\theta) &= \frac{1}{q} \theta^{-1-\frac{1}{q}} \varpi_q\left(\theta^{-\frac{1}{q}}\right) \geq 0, \varpi_q(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-qn-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \\ \theta &\in (0, \infty), \xi_q \text{ is a probability density function defined on } (0, \infty), \text{ i.e.,} \\ \xi_q(\theta) &\geq 0, \theta \in (0, \infty) \text{ and } \int_0^\infty \xi_q(\theta) d\theta = 1. \end{aligned}$$

The following results will be used throughout this paper.

Lemma 2.1. (Lemma 2.9, [5]). The operators \mathcal{T} and \mathcal{S} have the following properties:

- (1) For any fixed $t \geq 0$, $\mathcal{T}(t)$ and $\mathcal{S}(t)$ are linear and bounded operators, i.e., for any $x \in \mathbb{B}$, $|\mathcal{T}(t)(x)| \leq M|x|$ and $|\mathcal{S}(t)x| \leq \frac{qM}{\Gamma(1+q)}|x|$.
- (2) $\{\mathcal{T}(t), t \geq 0\}$ and $\{\mathcal{S}(t), t \geq 0\}$ are strongly continuous.
- (3) For every $\epsilon > 0$, $\mathcal{T}(t)$ and $\mathcal{S}(t)$ are also compact operators if $S(t)$ is compact.

(4) For any $x \in \mathbb{B}$, $\beta \in (0, 1)$ and $\alpha \in (0, 1)$, $A\mathcal{S}(t)x = A^{1-\beta}\mathcal{S}(t)A^\beta x$, $t \in [0, T]$ and

$$|A^\alpha \mathcal{S}(t)| \leq \frac{M_\alpha q \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} t^{-\alpha q}, \quad 0 < t \leq T.$$

(5) For any fixed $t \leq 0$ and $x \in \mathbb{B}_\alpha$, $|\mathcal{T}(t)x|_\alpha \leq M|x|_\alpha$ and $|\mathcal{S}(t)x|_\alpha \leq \frac{qM}{\Gamma(1+q)}|x|_\alpha$.

To end this section, we collect an important singular type Gronwall inequality which is introduced by Yeetal in [11] and can be used in fractional differential equations.

Theorem 2.1. Suppose $\tilde{a}(t)$ is a nonnegative function locally integrable on $[0, +\infty)$ and $\tilde{g}(t)$ is a nonnegative, nondecreasing continuous function defined on $\tilde{g}(t) \leq M$, $t \in [0, +\infty)$, and suppose $u(t)$ is nonnegative and locally integrable on $[0, +\infty)$ with

$$u(t) \leq \tilde{a}(t) + \tilde{g}(t) \int_0^t (t-s)^{q-1} u(s) ds, \quad t \in [0, +\infty).$$

Then $u(t) \leq \tilde{a}(t) + \int_0^t [\sum_{n=1}^\infty \frac{(\tilde{g}(t)\Gamma(q)^n}{\Gamma(nq)} (t-s)^{nq-1} \tilde{a}(s)] ds$, $t \in [0, +\infty)$.

Remark 2.2. Under the hypothesis of Theorem 2.1, let $\tilde{a}(t)$ be a nondecreasing function on $[0, +\infty)$. Then we have $u(t) \leq \tilde{a}(t) E_q[\tilde{g}(t) \Gamma(q) t^q]$, where E_q is the Mittag-Leffler function defined by $E_q[z] = \sum_{k=0}^\infty \frac{z^k}{\Gamma(kq+1)}$, $z \in \mathbb{C}$.

3. DEFINITIONS OF MITTAG - LEFFLER - ULAM STABILITIES

For $f \in (I \times \mathbb{B}, \mathbb{B})$ (or $f \in (I \times \mathbb{B}_\alpha, \mathbb{B})$), $\epsilon > 0$ and $\varphi \in C(I, R_+)$ we consider the following equation

$$(3.1) \quad {}^c D^q u(t) = -Au(t) + f(t, u(t)), \quad t \in I,$$

and the following inequations

$$(3.2) \quad |{}^c D^q v(t) + Av(t) - f(t, v(t))| \leq \epsilon, \quad t \in I,$$

$$(3.3) \quad |{}^c D^q v(t) + Av(t) - f(t, v(t))| \leq \varphi(t), \quad t \in I,$$

$$(3.4) \quad |{}^c D^q v(t) + Av(t) - f(t, v(t))| \leq \epsilon \varphi(t), \quad t \in I.$$

Definition 3.1. The equation (3.1) is Mittag-Leffler-Ulam-Hyers stable, with respect to E_q , if there exists a real number $c > 0$ such that for each $\epsilon > 0$ and for each solution $v \in C^1(I, \mathbb{B})(\text{or } C^1(I, \mathbb{B}_\alpha))$ of inequation (3.2) there exists a mild solution $u \in C(I, \mathbb{B})(\text{or } C(I, \mathbb{B}_\alpha))$ of equation (3.1) with $|v(t) - u(t)| \leq c\epsilon E_q[t]$, for all $t \in I$.

Definition 3.2. Equation (3.1) is generalized Mittag-Leffler-Ulam-Hyers stable, with respect to E_q , if there exists $\theta \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta(0) = 0$, such that for each solution $v \in C^1(I, \mathbb{B})(\text{or } C^1(I, \mathbb{B}_\alpha))$ of inequation (3.2) there exists a mild solution $u \in C(I, \mathbb{B})(\text{or } C(I, \mathbb{B}_\alpha))$ of equation 3.1 with $|v(t) - u(t)| \leq \theta(\epsilon) E_q[t]$, for all $t \in I$.

Definition 3.3. Equation (3.1) is Mittag-Leffler-Ulam-Hyers-Rassias stable, with respect to ϕE_q , if there exists $c\phi > 0$ such that for each $\epsilon > 0$ and for each solution $v \in C^1(I, \mathbb{B})(\text{or } C^1(I, \mathbb{B}_\alpha))$ of inequation (3.4) there exists a mild solution $u \in C(I, \mathbb{B})(\text{or } C(I, \mathbb{B}_\alpha))$ of equation (3.1) with $|v(t) - u(t)| \leq c\phi\epsilon\phi(t)E_q[t]$, for all $t \in I$.

Definition 3.4. Equation (3.1) is generalized Mittag-Leffler-Ulam-Hyers-Rassias stable, with respect to ϕE_q , if there exists $c\phi > 0$ such that for each solution $v \in C^1(I, \mathbb{B})(\text{or } C^1(I, \mathbb{B}_\alpha))$ of (3.3) there exists a mild solution

$$u \in C(I, \mathbb{B})(\text{or } C(I, \mathbb{B}_\alpha))$$

of equation (3.1) with $|v(t) - u(t)| \leq c\phi\phi(t)E_q[t]$, for all $t \in I$.

Remark 3.1. It is clear that:

- (i) Definition 3.1 \Rightarrow Definition 3.2;
- (ii) Definition 3.3 \Rightarrow Definition 3.4.

Remark 3.2. A function $v \in C^1(I, \mathbb{B})(\text{or } C^1(I, \mathbb{B}_\alpha))$ is a solution of inequation (3.2) if and only if there exists a function $g \in C(I, \mathbb{B})(\text{or } C(I, \mathbb{B}_\alpha))$ (which depend on v) such that:

- (i) $|g(t)| \leq \epsilon$, for all $t \in I$;
- (ii) ${}^c D^q v(t) = -Av(t) + f(t, v(t)) + g(t)$, $t \in I$.

We have similar remarks for in equations (3.3) and (3.4).

Remark 3.3. (1) If $v \in C^1(I, \mathbb{B})$ is a solution of in equation (3.2), then v is a solution of the following integral in equation

$$\left| v(t) - \mathcal{T}(t)v(0) - \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) f(s, v(s)) ds \right| \\ \leq \epsilon \int_0^t (t-s)^{q-1} \|\mathcal{S}(t-s)\| ds.$$

(2) If $v \in C^1(I, \mathbb{B}_\alpha)$ is a solution of in equation (3.2), then v is a solution of the following integral in equation

$$\left| v(t) - \mathcal{T}(t)v(0) - \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) f(s, v(s)) ds \right|_\alpha \\ \leq \epsilon \int_0^t t(t-s)^{q-1} \|\mathcal{S}(t-s)\|_\alpha ds.$$

We have similar remarks for the solutions of in equations (3.3) and (3.4).

4. MITTAG-LEFFLER-ULAM-HYERS STABILITY ON A COMPACT INTERVAL

Let us consider equation (3.1) and in equation (3.2) in the case $I := [0, b]$.

Case 1. $\{S(t), t \geq 0\}$ is C_0 -semigroup.

Theorem 4.1. We suppose that:

- (1) $f \in C([0, b] \times \mathbb{B}, \mathbb{B})$
- (2) there exists $L_f > 0$ such that $|f(t, w_1) - f(t, w_2)| \leq L_f |w_1 - w_2|$, for all $t \in [0, b], w_1, w_2 \in \mathbb{B}$. Then equation (3.1) is Mittag-Leffler-Ulam-Hyers stable.

Proof. Let $v \in C^1(I, \mathbb{B})$ be a solution of in equation (3.2). Let us denote by $u \in C([0, b], \mathbb{B})$ the unique mild solution of the Cauchy problem

$$(4.1) \quad \begin{cases} {}^c D^q u(t) = -Au(t) + f(t, u(t)), t \in [0, b], \\ u(0) = v(0) \end{cases}.$$

We have

$$u(t) = \mathcal{T}(t)v(0) + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) f(s, u(s)) ds, t \in [0, b],$$

And applying Lemma 2.1, one can get

$$\begin{aligned}
 & \left| v(t) - \mathcal{T}(t)v(0) - \int_0^t (t-s)^{q-1} \mathcal{S}(t-s)f(s, v(s)) ds \right| \\
 & \leq \epsilon \int_0^t t(t-s)^{q-1} \|\mathcal{S}(t-s)\| ds \\
 & \leq \frac{qM\epsilon}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} ds \\
 & \leq \frac{Mb^q}{\Gamma(1+q)} \epsilon.
 \end{aligned}$$

From these relations, we have

$$\begin{aligned}
 |v(t) - u(t)| & \leq \left| v(t) - \mathcal{T}(t)v(0) - \int_0^t (t-s)^{q-1} \mathcal{S}(t-s)f(s, v(s)) ds \right| \\
 & + \int_0^t t(t-s)^{q-1} \|\mathcal{S}(t-s)\| |f(s, v(s)) - f(s, u(s))| ds \\
 & \leq \frac{Mb^q}{\Gamma(1+q)} \epsilon + \frac{L_f q M}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} |v(s) - u(s)| ds.
 \end{aligned}$$

It comes from Theorem 2.1 that

$$|v(t) - u(t)| \leq \frac{Mb^q}{\Gamma(1+q)} E_q \left[\frac{L_f q M}{\Gamma(1+q)} \Gamma(q) t^q \right] \epsilon.$$

Thus, the conclusion of our theorem holds.

Case 2. $\{S(t), t \geq 0\}$ is an analytic semigroup.

□

Theorem 4.2. *We suppose that*

- (1) $f \in C([0, b] \times \mathbb{B}_\alpha, \mathbb{B})$
- (2) *there exists $L_f > 0$ such that $|f(t, w_1) - f(t, w_2)| \leq L_f |w_1 - w_2|_\alpha$, for all $t \in [0, b], w_1, w_2 \in \mathbb{B}_\alpha$. Then (3.1) is Mittag-Leffler-Ulam-Hyers stable.*

Proof. Let $v \in C^1([0, b], \mathbb{B}_\alpha)$ be a solution of in equation (3.2). Let us denote by $u \in C([0, b], \mathbb{B}_\alpha)$ the unique mild solution of the Cauchy problem and apply

Lemma 2.1, one can get

$$\begin{aligned} & \left| v(t) - \mathcal{T}(t)v(0) - \int_0^t (t-s)^{q-1} \mathcal{S}(t-s)f(s, v(s)) ds \right|_\alpha \\ & \leq \epsilon \int_0^t t(t-s)^{q-1} \|\mathcal{S}(t-s)\|_\alpha ds \\ & \leq \frac{b^{q(1-\alpha)} M_\alpha \Gamma(2-\alpha)}{(1-\alpha) \Gamma(1+q(1-\alpha))} \epsilon. \end{aligned}$$

From these relations, we have

$$\begin{aligned} & |v(t) - u(t)|_\alpha \\ & \leq \frac{b^{q(1-\alpha)} M_\alpha \Gamma(2-\alpha)}{(1-\alpha) \Gamma(1+q(1-\alpha))} \epsilon \\ & + \frac{L_f M_\alpha q \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} \int_0^t (t-s)^{q-\alpha q-1} |v(s) - u(s)|_\alpha ds. \end{aligned}$$

It comes from Theorem 2.1 that

$$\begin{aligned} |v(t) - u(t)|_\alpha & \leq \frac{b^{q(1-\alpha)} M_\alpha \Gamma(2-\alpha)}{(1-\alpha) \Gamma(1+q(1-\alpha))} E_{q(1-\alpha)} \\ & \left[\frac{L_f M_\alpha q \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} \Gamma(q(1-\alpha)) t^{q(1-\alpha)} \right] \epsilon. \end{aligned}$$

□

5. GENERALIZED MITTAG-LEFFLER-ULAM-HYERS STABILITY ON $[0, +\infty)$

Let us consider equation (3.1) and in equation (3.3) in the case $I := [0, +\infty)$.

Case 1. $\{S(t), t \geq 0\}$ is a C_0 -semigroup.

Theorem 5.1. *We suppose that*

- (1) $f \in C([0, +\infty) \times \mathbb{B}, \mathbb{B})$
- (2) $L(t)$ is a nonnegative, nondecreasing continuous function defined on $L(t) \leq M, t \in [0, +\infty)$ and $|f(t, w_1) - f(t, w_2)| \leq L(t)|w_1 - w_2|$ for all $t \in [0, +\infty), w_1, w_2 \in \mathbb{B}$

- (3) the function $\varphi \in C([0, +\infty), \mathbb{R}_+)$ is increasing and there exists $\lambda > 0$ such that $\int_0^t t(t-s)^{q-1} \|\mathcal{S}(t-s)\| \varphi(s) ds \leq \lambda \varphi(t)$ for all $t \in [0, +\infty)$. Then equation (3.1) is generalized Mittag-Leffler-Ulam-Hyers-Rassias stable with respect to ϕE .

Proof. Let $v \in C^1([0, +\infty), \mathbb{B})$ be a solution of equation (3.3). By Remark 3.2, we have that

$$\begin{aligned} & \left| v(t) - \mathcal{T}(t)v(0) - \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) f(s, v(s)) ds \right| \\ & \leq \int_0^t t(t-s)^{q-1} \|\mathcal{S}(t-s)\| \varphi(s) ds \leq \lambda \varphi(t), \end{aligned}$$

for all $t \in [0, +\infty)$.

Let us denote by $u \in C([0, +\infty), \mathbb{B})$ the unique mild solution of the Cauchy problem

$$(5.1) \quad \begin{cases} {}^c D^q u(t) = -Au(t) + f(t, u(t)), & t \in [0, +\infty), \\ u(0) = v(0) \end{cases}$$

we have that

$$u(t) = \mathcal{T}(t)v(0) + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) f(s, u(s)) ds, t \in [0, +\infty).$$

It follows from

$$|v(t) - u(t)| \leq \lambda \varphi(t) + \frac{qML(t)}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} |v(s) - u(s)| ds$$

and Theorem 2.1 that equation (3.1) is generalized Mittag-Leffler-Ulam-Hyers-Rassias stable.

Case 2. $\{S(t), t \geq 0\}$ is an analytic semigroup.

□

Theorem 5.2. We suppose that

- (1) $f \in C([0, +\infty) \times \mathbb{B}_\alpha, \mathbb{B})$
- (2) $L(t)$ is a nonnegative, nondecreasing continuous function defined on $L(t) \leq M_L, t \in [0, +\infty)$ and $|f(t, w_1) - f(t, w_2)| \leq L(t)|w_1 - w_2|_\alpha$ for all $t \in [0, +\infty), w_1, w_2 \in \mathbb{B}_\alpha$

- (3) the function $\varphi \in C([0, +\infty), \mathbb{R}_+)$ is increasing and there exists $\lambda > 0$ such that $\int_0^t t(t-s)^{q-1} \|\mathcal{S}(t-s)\| \varphi(s) ds \leq \lambda \varphi(t)$ for all $t \in [0, +\infty)$. Then equation (3.1) is generalized Mittag-Leffler-Ulam-Hyers-Rassias stable with respect to ϕEq .

Proof. Let $v \in C^1([0, +\infty), \mathbb{B}_\alpha)$ be a solution of in equation (3.3). By Remark 3.2, we have that

$$\left| v(t) - \mathcal{T}(t)v(0) - \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) f(s, v(s)) ds \right|_\alpha \leq \lambda \varphi(t),$$

for all $t \in [0, +\infty)$.

Let us denote by $u \in C([0, +\infty), \mathbb{B}_\alpha)$ the unique mild solution of the Cauchy problem (5.1) and by applying Lemma 2.1, one can get

$$|v(t) - u(t)|_\alpha \leq \lambda \varphi(t) + \frac{L(t)M_\alpha q \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} \cdot \int_0^t (t-s)^{q-\alpha q-1} |v(s) - u(s)|_\alpha ds.$$

Using Theorem 2.1 again, one can obtain the results. \square

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