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HERMITE-HADAMARD TYPE OF INEQUALITIES FOR HARMONICALLY CONVEX FUNCTION USING FOURIER INTEGRAL TRANSFORM

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ABSTRACT. The aim of this paper is to establish some new Hermite-Hadamard form of inequalities of harmonic convex characteristic in reference to Fourier transform integral. Some results are obtained related to Hermite-Hadamard type of inequalities of these magnificence of characteristic.

1. INTRODUCTION

The inequalities plays a significant part in the world of mathematics. In fact, most of the mathematical inequalities act as a basic tool for constructing proof of important theorems and estimate several known definite integrals. The theory of convexity has been subject so far-reaching studies throughout the past few years because of its performance in numerous branches of pure and applied mathematics. The standards of convexity has been unlimited and regularly occurring in numerous directions. The classical convexity is defined as follows: s function $\mathcal{K} : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$\mathcal{K}\left(\xi x + (1-\xi)y\right) \leq \xi\mathcal{K}(x) + (1-\xi)\mathcal{K}(y)$$

holds for all $x, y \in \mathbb{I}$ and $\xi \in [0, 1]$. The convexity in connection with inequalities has also influenced several mathematicians from all over the world. For the

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past three decades, many researchers have provided special attention to study various aspects of convex functions. The most fascinating inequality associated with convex function is called as Hermite-Hadamard inequality. In literatures, the Hermite-Hadamard inequality is named by Charles Hermite $(1822 - 1901)$ and Jacques Hadamard $(1865 - 1963)$. This states that if a function $\mathcal{K} : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ is convex, then the following chain of inequalities holds

$$\frac{\mathcal{K}(c) + \mathcal{K}(d)}{2} \leq \frac{1}{d-c} \int_c^d \mathcal{K}(x)dx \leq \frac{\mathcal{K}(c) + \mathcal{K}(d)}{2}.$$

For more results which provide calculations of the mean value of convex functions, extension, generalizations, refinements and numerous application, see [2, 6]. In line with the above mentioned definition, the considerable convex is harmonically convex functions has been introduced and studied by Iscan [1].

Definition 1.1. *A mapping $\mathcal{K} : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be harmonic convex if*

$$\mathcal{K}\left(\frac{xy}{\xi x + (1-\xi)y}\right) \leq \xi\mathcal{K}(x) + (1-\xi)\mathcal{K}(y), \quad \forall x, y \in \mathbb{I}.$$

The Hermite-Hadamard inequality for harmonically convex function is as the following:

Theorem 1.1. *[1] Let $\mathcal{K} : \mathbb{I} \subseteq \mathbb{R}/0 \to \mathbb{R}$ be harmonically convex and $a < b$. If $f \in L[a, b]$, then following inequalities hold*

$$\mathcal{K}\left(\frac{2cd}{c+d}\right) \leq \frac{cd}{d-c} \int_c^d \frac{\mathcal{K}(x)}{x^2} dx \leq \frac{\mathcal{K}(c) + \mathcal{K}(d)}{2}.$$

For some results to harmonic convex function and its generalization, the researcher refer the readers to visit [3–5]. The convexity of function and their generalized forms play an immodest role in many fields such as optimization, economic science, biology.

Theorem 1.2. *[2] If a function $\mathcal{K} : \mathbb{I} \subseteq (-\infty, \infty) \to \mathbb{R}$ is harmonic convex function and $g : (-\infty, \infty) \to \mathbb{R}$ is a linear function, then $f \circ g$ is a harmonic convex.*

Now we take into account some primary ideas of Fourier transform [7]. If a function $\mathcal{K} : \mathbb{R} \to F$ is piecewise continuous in each finite interval and is absolutely integrable on $\mathbb{R}$, then the Fourier transform is given by the integral

$$\widehat{F}(\mathcal{K}(\xi)) = \int_{-\infty}^{\infty} \mathcal{K}(x)e^{i\xi x}dx\,.$$

The inverse Fourier transform is defined by

$$\mathcal{K}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{F}(\mathcal{K}(\xi))e^{-i\xi x}d\xi\,.$$

Definition 1.2. *The Fourier transform of composition of two function $\mathcal{K}(x)$ and $\mathcal{K}(x)$ is defined as*

$$\widehat{F}(\mathcal{K}\circ\mathcal{G})(x) = \int_{-\infty}^{\infty} \mathcal{K}(\mathcal{G}(x))e^{-i\xi x}dx\,.$$

Motivated and considered by ongoing research in this field, the aim of this paper is to establish new Hermite-Hadamard type of inequalities for harmonic convex function in terms of Fourier transform.

2. Main results

In this section, we are going to derive our results related to Fourier integral transform of harmonically convex function.

Theorem 2.1. Let $\mathcal{K} : \mathbb{I} \subseteq (-\infty,\infty) \to \mathbb{R}$ be a convex function with $c < d$ and $c, d \in \mathbb{I}$. Then, the following inequalities for Fourier integral transform hold:

$$\mathcal{K}\left(\frac{c+d}{2}\right) \le \frac{i\xi}{2(1-e^{-i\rho})}\left(\widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi+\frac{1}{d}) + \widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi-\frac{1}{c})\right) \le \mathcal{K}(c)+\mathcal{K}(d)\,.$$

Proof. Consider the function $\mathcal{K}$ is harmonically convex function on $[u,v] \subseteq (-\infty,\infty)$, for all $u, v \in [c,d]$, we have

$$\mathcal{K}\left(\frac{2uv}{u+v}\right) \le \frac{\mathcal{K}(u)+\mathcal{K}(v)}{2}\,.$$

Setting $u = \dfrac{cd}{td+(1-t)c}$ and $v = \dfrac{cd}{(1-t)d+tc}$, then we get

$$\mathcal{K}\left(\frac{2cd}{c+d}\right) \le \frac{\mathcal{K}\left(\frac{cd}{tc+(1-t)d}\right) + \mathcal{K}\left(\frac{cd}{td+(1-t)c}\right)}{2}\,. \tag{2.1}$$

Let $\rho = \dfrac{\xi(d-c)}{cd}$. Multiplying both sides of (2.1) by $e^{-i\rho t}$ and then integrating the resulting inequality with respect to t over $[0,1]$, then we obtain

$$\int_0^1 \mathcal{K}\left(\frac{2cd}{c+d}\right) e^{-i\rho t}dt \leq \frac{1}{2}\left\{\int_0^1 \mathcal{K}\left(\frac{cd}{td+(1-t)c}\right) e^{-i\rho t}dt + \int_0^1 \mathcal{K}\left(\frac{cd}{(1-t)d+tc}\right) e^{-i\rho t}dt\right\}$$

$$\mathcal{K}\left(\frac{2cd}{c+d}\right)\left(\frac{1-e^{-i\rho}}{i\rho}\right) \leq \frac{cd}{2(d-c)}\left\{\left(\left(\int_{\frac{1}{b}}^{\frac{1}{c}} \mathcal{K}(\frac{1}{s}) e^{-i\xi(s-\frac{1}{b})}ds + \int_{\frac{1}{d}}^{\frac{1}{c}} \mathcal{K}(\frac{1}{t}) e^{-i\xi(\frac{1}{a}-t)}dt\right)\right\}$$

$$\mathcal{K}\left(\frac{2cd}{c+d}\right) \leq \frac{i\xi}{2\left(1-e^{-i\rho}\right)}\left\{\left(e^{i\frac{1}{d}\xi}\int_{\frac{1}{d}}^{\frac{1}{c}} \mathcal{K}\left(\frac{1}{s}\right) e^{-i\xi s}ds + e^{-i\frac{1}{c}\xi}\int_{\frac{1}{d}}^{\frac{1}{c}} \mathcal{K}\left(\frac{1}{t}\right) e^{i\xi t}dt\right)\right\}.$$

Let $\mathcal{G}(s) = \dfrac{1}{s}$, $\forall s \in \mathbb{I}$ and by convolution property of Fourier transform, then we get

$$\mathcal{K}\left(\frac{2cd}{c+d}\right) \leq \frac{i\xi}{2\left(1-e^{-i\rho}\right)}\left\{\widehat{F}(\mathcal{K}\circ\mathcal{G})\left(\xi+\frac{1}{b}\right) + \widehat{F}(\mathcal{K}\circ\mathcal{G})\left(\xi-\frac{1}{a}\right)\right\}.$$

Thus the first part of the inequality is established. For the proof of the second inequality, since $\mathcal{K}$ is harmonically convex function, then for $t \in [0,1]$ it gives

$$\mathcal{K}\left(\frac{cd}{td+(1-t)c}\right) \leq t\mathcal{K}(c) + (1-t)\mathcal{K}(d)\,, \tag{2.2}$$

and

$$\mathcal{K}\left(\frac{cd}{tc+(1-t)d}\right) \leq t\mathcal{K}(d) + (1-t)\mathcal{K}(c)\,. \tag{2.3}$$

By addition of the inequalities (2.2) and (2.3), we get

$$\mathcal{K}\left(\frac{cd}{td+(1-t)c}\right) + \mathcal{K}\left(\frac{cd}{tc+(1-t)d}\right) \leq \mathcal{K}(c) + \mathcal{K}(d)\,. \tag{2.4}$$

Then multiplying both sides of (2.4) by $e^{-i\rho t}$ and integrating the resulting inequality with respect to t over $[0,1]$, we get,

$$\int_0^1 \mathcal{K}\left(\frac{cd}{td+(1-t)c}\right) e^{-i\rho t}dt + \int_0^1 \mathcal{K}\left(\frac{cd}{tc+(1-t)d}\right) e^{-i\rho t}dt \leq \int_0^1 (\mathcal{K}(c)+\mathcal{K}(d))e^{-i\rho t}dt$$

$$\left(e^{i\frac{1}{d}\xi}\int_{\frac{1}{d}}^{\frac{1}{c}} \mathcal{K}\left(\frac{1}{s}\right) e^{-i\xi s}ds + e^{-i\frac{1}{c}\xi}\int_{\frac{1}{d}}^{\frac{1}{c}} \mathcal{K}\left(\frac{1}{t}\right) e^{i\xi t}dt\right) \leq (\mathcal{K}(c)+\mathcal{K}(d))\left(\frac{1-e^{-i\rho}}{i\xi}\right)$$

$$\frac{i\xi}{(1-e^{-i\rho})}\left(\widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi+\frac{1}{d}) + \widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi-\frac{1}{c})\right) \leq \mathcal{K}(c)+\mathcal{K}(d)\,.$$

This completes the proof. □

Theorem 2.2. Let $\mathcal{K} : \mathbb{I} \subseteq (-\infty, \infty) \to \mathbb{R}$ be a differentiable function over an interval I such that $f' \in L[c,d]$, where $c, d \in \mathcal{I}$ with $a < b$. Then we have:

$$\mathcal{K}\left(\frac{c+d}{2}\right) - \frac{i\xi}{2(1-e^{-i\rho})}\left(\widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi+\frac{1}{d}) + \widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi-\frac{1}{c})\right)$$
$$=\frac{cd(d-c)}{2(e^{-i\rho-1})}\int_0^1 \frac{e^{-i\rho t}-e^{-i\rho(1-t)}}{(tc+(1-t)d)^2} f'\left(\frac{cd}{tc+(1-t)d}\right) dt\,.$$

Proof. Consider

$$\begin{aligned}
&\frac{cd(d-c)}{2}\int_0^1 \frac{e^{-i\rho t}-e^{-i\rho(1-t)}}{(tc+(1-t)d)^2}\mathcal{K}'\left(\frac{cd}{tc+(1-t)d}\right) dt\\
=&\frac{cd(d-c)}{2}\int_0^1 \frac{e^{-i\rho t}}{(tc+(1-t)d)^2}\mathcal{K}'\left(\frac{cd}{tc+(1-t)d}\right) dt\\
-&\frac{cd(d-c)}{2}\int_0^1 \frac{e^{-i\rho(1-t)}}{(tc+(1-t)d)^2}\mathcal{K}'\left(\frac{cd}{tc+(1-t)d}\right) dt\\
&= I_1 - I_2\,.
\end{aligned}$$

By integration by parts,

$$\begin{aligned}
I_1 =&\frac{1}{2}\left(e^{-i\rho t}\mathcal{K}\left(\frac{cd}{tc+(1-t)d}\right)\right)_0^1 + \frac{i\rho}{2}\int_{\frac{1}{d}}^{\frac{1}{c}} e^{-i\rho t}\mathcal{K}\left(\frac{cd}{tc+(1-t)d}\right) dt\\
=&\frac{1}{2}\left(e^{-i\rho}\mathcal{K}(d) - \mathcal{K}(c)\right) + \frac{i\xi}{2}\int_{\frac{1}{d}}^{\frac{1}{c}} e^{-i\xi(s-\frac{1}{d})}\mathcal{K}(\frac{1}{s})ds\\
=&\frac{1}{2}\left(e^{-i\rho}\mathcal{K}(d) - \mathcal{K}(c)\right) + \frac{i\xi}{2}e^{\frac{i\xi}{d}}\widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi+\frac{1}{b})\,,
\end{aligned}$$

and

$$\begin{aligned}
I_2 =&\frac{1}{2}\left(e^{-i\rho(1-t)}\mathcal{K}\left(\frac{cd}{tc+(1-t)d}\right)\right)_0^1 - \frac{i\rho}{2}\int_{\frac{1}{d}}^{\frac{1}{c}} e^{-i\rho(1-t)}\mathcal{K}\left(\frac{cd}{tc+(1-t)d}\right) dt\\
=&\frac{1}{2}\left(\mathcal{K}(d) - e^{-i\rho}\mathcal{K}(c)\right) - \frac{i\xi}{2}\int_{\frac{1}{d}}^{\frac{1}{c}} e^{-i\xi(s-\frac{1}{d})}\mathcal{K}(\frac{1}{s})ds\\
=&\frac{1}{2}\left(\mathcal{K}(d) - e^{-i\rho}\mathcal{K}(c)\right) - \frac{i\xi}{2}e^{\frac{-i\xi}{c}}\widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi-\frac{1}{a})\,.
\end{aligned}$$

In view of the previous equation, we have:

$$\begin{aligned}
&\frac{cd(d-c)}{2}\int_0^1 \frac{e^{-i\rho t}-e^{-i\rho(1-t)}}{(tc+(1-t)d)^2}\mathcal{K}'\left(\frac{cd}{tc+(1-t)d}\right)dt \\
&=\frac{1}{2}\left(e^{-i\rho}\mathcal{K}(d)-\mathcal{K}(c)\right)+\frac{i\xi}{2}e^{\frac{i\xi}{d}}\widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi+\frac{1}{b}) \\
&+\frac{1}{2}\left(\mathcal{K}(d)-e^{-i\rho}\mathcal{K}(c)\right)-\frac{i\xi}{2}e^{\frac{-i\xi}{c}}\widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi-\frac{1}{a}) \\
&=\frac{e^{-i\rho}}{2}\left(f(a)+f(b)\right)-\frac{1}{2}\left(f(a)+f(b)\right) \\
&+\frac{i\xi}{2}\left(\widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi+\frac{1}{b})+\widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi-\frac{1}{a})\right).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\frac{cd(d-c)}{2(e^{-i\rho}-1)}\int_0^1 \frac{e^{-i\rho t}-e^{-i\rho(1-t)}}{(tc+(1-t)d)^2}\mathcal{K}'\left(\frac{cd}{tc+(1-t)d}\right)dt=\frac{(f(a)+f(b))}{2} \\
&+\frac{i\xi}{2(e^{-i\rho}-1)}\left(\widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi+\frac{1}{b})+\widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi-\frac{1}{a})\right).
\end{aligned}$$

The proof is complete. □

Lemma 2.1. *For $0<\vartheta\leq 1$ and $0\leq c<d$, we have*

$$\left|c^{\vartheta}-d^{\vartheta}\right|\leq (d-c)^{\vartheta}\leq d^{\vartheta}+c^{\vartheta}.$$

Theorem 2.3. *Let $\mathcal{K}:\mathbb{I}(0,\infty)\to\mathbb{R}$ be a differentiable function on $\mathcal{I}$ such that $\mathcal{K}'\in L(c,d)$, where $c,d\in\mathbb{I}$ with $c<d$. If $|f'|$ is harmonically convex on $[a,b]$ for some fixed $q>1$, then the following inequality for Fourier integral transform holds:*

$$\begin{aligned}
&\mathcal{K}\left(\frac{c+d}{2}\right)-\frac{i\xi}{2(1-e^{-i\rho})}\left(\widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi+\frac{1}{d})+\widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi-\frac{1}{c})\right) \\
&\qquad\leq\frac{cd(d-c)}{2(e^{-i\rho}-1)}\left(\mathcal{B}_1^{\frac{1}{2}}+\mathcal{B}_2^{\frac{1}{2}}\right)\left(\frac{|\mathcal{K}'(d)|^p+|\mathcal{K}'(c)|^q}{2}\right)^{\frac{1}{q}},
\end{aligned}$$

where,

$$\mathcal{B}_1=\int_0^1\frac{e^{-i\rho(1-t)p}}{(tc+(1-t)d)^{2p}}dt \quad \text{and} \quad \mathcal{B}_2=\int_0^1\frac{e^{-i\rho tp}}{(tc+(1-t)d)^{2p}}dt.$$

Proof. Using Theorem 2.2 and Lemma 2.1, we have

$$\begin{aligned}
&\left|\mathcal{K}\left(\frac{c+d}{2}\right)-\frac{i\xi}{2(1-e^{-i\rho})}\left(\widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi+\frac{1}{d})+\widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi-\frac{1}{c})\right)\right|\\
&\leq\frac{cd(d-c)}{2(e^{-i\rho-1})}\int_0^1\frac{e^{-i\rho t}-e^{-i\rho(1-t)}}{(tc+(1-t)d)^2}\left|\mathcal{K}'\left(\frac{cd}{tc+(1-t)d}\right)\right|dt\\
&\leq\frac{cd(d-c)}{2(e^{-i\rho-1})}\left\{\int_0^1\frac{e^{-i\rho(1-t)}}{(tc+(1-t)d)^2}\left|\mathcal{K}'\left(\frac{cd}{tc+(1-t)d}\right)\right|dt\right.\\
&\quad\left.+\int_0^1\frac{e^{-i\rho t}}{(tc+(1-t)d)^2}\left|\mathcal{K}'\left(\frac{cd}{tc+(1-t)d)}\right)\right|dt\right\}
\end{aligned}$$

By Holders inequality and harmonic convexity of $|\mathcal{K}'|^q$, we get

$$\begin{aligned}
&\left|\mathcal{K}\left(\frac{c+d}{2}\right)-\frac{i\xi}{2(1-e^{-i\rho})}\left(\widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi+\frac{1}{d})+\widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi-\frac{1}{c})\right)\right|\\
&\leq\frac{cd(d-c)}{2(e^{-i\rho-1})}\left\{\left(\int_0^1\frac{e^{-i\rho tp}}{(tc+(1-t)d)^{2p}}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{K}'\left(\frac{cd}{tc+(1-t)d)}\right)\right|dt\right)^{\frac{1}{q}}\right\}\\
&+\frac{cd(d-c)}{2(e^{-i\rho-1})}\left\{\left(\int_0^1\frac{e^{-i\rho tp}}{(tc+(1-t)d)^{2p}}dt\right)^{\frac{1}{p}}\left(\left|\mathcal{K}'\left(\frac{cd}{tc+(1-t)d)}\right)\right|dt\right)^{\frac{1}{q}}\right\}\\
&\leq\frac{cd(d-c)}{2(e^{-i\rho-1})}(B_1^{\frac{1}{p}}+B_2^{\frac{1}{p}})\left(\int_0^1 t\left|\mathcal{K}'(c)\right|^q+(1-t)\left|\mathcal{K}'(d)\right|^q dt\right)^{\frac{1}{q}},
\end{aligned}$$

where,

$$\mathcal{B}_1=\int_0^1\frac{e^{-i\rho(1-t)p}}{(tc+(1-t)d)^{2p}}dt\quad and\quad \mathcal{B}_2=\int_0^1\frac{e^{-i\rho tp}}{(tc+(1-t)d)^{2p}}dt\,,$$

which is required result. □

Theorem 2.4. *Let $\mathcal{K}:\mathbb{I}\subseteq(0,\infty)\to\mathbb{R}$ be a continuously differentiable function on $\mathcal{I}$ with $c<d$, $c,d\in\mathbb{I}$, If $|f'|$ is harmonically convex on $[a,b]$ for $q>1$, then the following inequality for Fourier integral transform holds:*

$$\begin{aligned}
&\mathcal{K}\left(\frac{c+d}{2}\right)-\frac{i\xi}{2(1-e^{-i\rho})}\left(\widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi+\frac{1}{d})+\widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi-\frac{1}{c})\right)\\
&\leq\frac{cd(d-c)}{2(e^{-i\rho}-1)}\mathcal{C}_1\frac{e^{-i\rho q}}{2i\rho q}\left(|\mathcal{K}'(d)|^q+|\mathcal{K}'(c)|^q\right)
\end{aligned}\tag{2.5}$$

where,

$$\mathcal{C}_1 = \int_0^1 \frac{1}{(tc+(1-t)d)^{2p}} dt\,.$$

Proof. Using Theorem 2.2 and Lemma 2.1, we have:

$$\begin{aligned}
&\left|\mathcal{K}\left(\frac{c+d}{2}\right) - \frac{i\xi}{2(1-e^{-i\rho})}\left(\widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi+\frac{1}{d}) + \widehat{F}(\mathcal{K}\circ\mathcal{G})(\xi-\frac{1}{c})\right)\right| \\
&\leq \frac{cd(d-c)}{2(e^{-i\rho-1})}\int_0^1 \left|\frac{e^{-i\rho(1-t)} - e^{-i\rho t}}{(tc+(1-t)d)^2}\right| \left|\mathcal{K}'\left(\frac{cd}{tc+(1-t)d)}\right)\right| dt \\
&\leq \frac{cd(d-c)}{2(e^{-i\rho-1})}\left(\int_0^1 \frac{1}{(tc+(1-t)d)^{2p}}\right)^{\frac{1}{p}} \left(\int_0^1 \left|e^{-i\rho(1-t)} - e^{i\rho t}\right|^q \left|\mathcal{K}'\left(\frac{cd}{tc+(1-t)d}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{cd(d-c)}{2(e^{-i\rho-1})}\left(\int_0^1 \frac{1}{(tc+(1-t)d)^{2p}}\right)^{\frac{1}{p}} \left(\int_0^1 \left|e^{-i\rho(1-2t)}\right|^q \left[t\left|\mathcal{K}'(d)\right|^q + (1-t)\left|\mathcal{K}'(c)\right|^q\right] dt\right)^{\frac{1}{q}} \\
&\leq \frac{cd(d-c)}{2(e^{-i\rho-1})}\left(\int_0^1 \frac{1}{(tc+(1-t)d)^{2p}}\right)^{\frac{1}{p}} \left(\left(\int_0^1 \left|e^{-i\rho(1-2t)}\right|^q tdt\right)\left|\mathcal{K}'(d)\right|^q\right. \\
&\quad \left.+\left(\int_0^1 \left|e^{-i\rho(1-2t)}\right|^q (1-t)dt\right)\left|\mathcal{K}'(c)\right|^q\right),
\end{aligned}$$

where,

$$\begin{aligned}
\mathcal{C}_1 &= \int_0^1 \frac{1}{(tc+(1-t)d)^{2p}} dt \\
\mathcal{C}_2 &= \int_0^1 \left|e^{-i\rho(1-2t)}\right|^q tdt = \int_0^{\frac{1}{2}} e^{-i\rho q(1-2t)} tdt + \int_{\frac{1}{2}}^1 e^{-i\rho q(2t-1)} tdt \\
&= \left(\frac{te^{-i\rho(1-2t)}}{2i\rho q} - \frac{e^{-i\rho(1-2t)}}{(2i\rho q)^2}\right)_0^{\frac{1}{2}} + \left(\frac{te^{-i\rho(1-2t)}}{2i\rho q} - \frac{e^{-i\rho(2t-1)}}{(2i\rho q)^2}\right)_{\frac{1}{2}}^1 = \frac{e^{-i\rho q}}{2i\rho q} \\
\mathcal{C}_3 &= \int_0^1 \left|e^{-i\rho(1-2t)}\right|^q (1-t)dt = \frac{-e^{-i\rho q}}{2i\rho q}\,.
\end{aligned}$$

Thus, if we use the last three equations, we obtain the inequality in (2.5). This completes the proof. □

3. APPLICATION

(1) Let $\mathcal{K} : \mathbb{I} \in (-\infty, \infty) \to \mathbb{R}$, defined by $f(x) = x$ is harmonic convex function on $\mathbb{I}$. Therefore, by Theorem 2.1, we have:

$$\frac{2cd}{c+d} \leq \frac{cd}{d-c}(logd - logc) \leq \frac{c+d}{2}$$

which gives relation between hamonic, logorithmic and Arithmetic means.

(2) Let $\mathcal{K} : \mathbb{I} \in (-\infty, \infty) \to \mathbb{R}$, defined by $f(x) = e^{-2x}$ is harmonic convex function on $\mathbb{I}$. We have,

$$e^{\frac{-4cd}{c+d}} \leq \frac{cd}{d-c}\int_c^d \frac{e^{-2x}}{x^2}dx \leq \frac{e^{-2c}+e^{-2d}}{2}.$$

(3) Since $f(x) = cosx$ is harmonic convex function in $(0, \pi)$, therefore, by using Theorem 2.1 for $c, d \in (0, \infty)$, we have:

$$\cos\left(\frac{2cd}{c+d}\right) \leq \frac{cd}{d-c}\int_c^d \frac{\cos x}{x^2}dx \leq \frac{\cos c + \cos d}{2}.$$

Similarly, we can calculate some definite integrals.

4. CONCLUSION

We have established some non-numeric calculations of well known definite integrals related to Hermite-Hadamard type of inequalities of harmonic convex function by using Fourier transform. This new class unifies several classes of harmonically convex functions which may inspire further research in this fields.

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