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DIFFUSION ANALYSIS OF A PREY-PREDATOR MODEL WITH HOLLING TYPE II FUNCTIONAL RESPONSE

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ABSTRACT. In this article, a prey predator model with Holling type II functional response with diffusive parameters is proposed to analyse the dynamics and effects of diffusion and delay further. Local and global stability is discussed both analytically and graphically. An appropriate time delay is introduced in the model to catch the effects of the delay and considering delay as bifurcation parameter to examine the Hopf-bifurcation, which allows us to draw some interesting findings to discuss in both analytical and graphical view. Finally numerical simulations are performed to validate the results by using MATLAB.

1. INTRODUCTION

Mathematical modelling is a significant tool in numerous fields like natural science, applied mathematics, economics and engineering science. Modelling of collaborating populations can give profitable bits of knowledge into varieties of populations over a period of time. In this article, we consider prey-predator framework with a Holling type II association. In present examination, populace models showing up in various fields of numerical science have been inferred and contemplated comprehensively because of their all-inclusive presence and significance [1]. The most significant urgent component in prey-predator framework is the kinds of practical reaction. It portrays the quantity of prey devoured

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per predator per unit time for given amounts of prey and predator. The helpful and significant practical reaction is known as Holling type-II is as in the form of $R(p_1) = kp_1/(n + p_1)$, where p_1 and p_2 are the population densities of the prey and predator, respectively. k is the maximal predator per capita consumption rate, the maximum number of prey that can be eaten by a predator in each time unit; n is the half capturing saturation constant. i.e., the number of prey necessary to achieve one half of the maximum rate k. Marine saves likewise help in ensuring bio assorted variety and eco framework structure. In the writing [2], C. W .Clark examines broadly the ideal administration inexhaustible assets as fishery. Leung [3], B. S. Ghosh [4], Mesterton-Gibbons.M [5, 6] contemplated the affordable and organic of sustainable assets by displaying dynamic model for fishery assets. The stage organized prey predator model with reaping is a significant model and it is considered by numerous creators [7–12]. The impact of dispersal and spatial heterogeneity is progressively significant in the populace elements as it assumes real job in the dependability of the environment. The subjective hypothesis on the diffusive frameworks was built and created by numerous analysts [11–18]. The organization of this article is as follows that the mathematical model is available in section 2; analysis of stability of non-diffusive system is in section 3 whereas section 4 consists of stability of diffusive system. We add some numerical simulations in section 5. The conclusions are provided in section 6.

2. MATHEMATICAL MODEL

We consider an ecological system, where prey and predator species living together with the spatiotemporal effect along with Holling type II interaction between the species. Assume that the prey species are growing logistically and the predator purely depends on prey for food with Holling type-II interaction. Also, it is ignoring the mortality rate of prey and considering the mortality rate of predator species. Let p_1 and p_2 represents the density of prey and predators respectively at any time t, c_1 represents carrying capacity of prey species, i_1 represents intrinsic growth rate of prey species, m_1 is the mortality rate of predator species, the term $(r_1 p_1)/(1 + r_2 p_1)$ represents the functional response of the predator, (r_1/r_2) is the maximum number of prey that eaten by each predator in unit time, $(1/r_2)$ is the density of prey necessary to achieve one half that rate,

m_2 is the conversion factor denoting the number of newly born predators for each captured prey. D_{p_1}, D_{p_2} represent the constant diffusion coefficient of the prey, predator species. Keeping these in view, the mathematical model of the system is governed by the following equations:

$$\frac{\partial p_1}{\partial t} = i_1 p_1 \left(1 - \frac{p_1}{c_1}\right) - \frac{r_1 p_1 p_2}{1 + r_2 p_1} + D_1 \frac{\partial^2 p_1}{\partial x^2}\,, \tag{2.1}$$

$$\frac{\partial p_2}{\partial t} = p_2 \left(\frac{r_1 m_2 p_1}{1 + r_2 p_1} - m_1\right) + D_2 \frac{\partial^2 p_2}{\partial x^2}\,. \tag{2.2}$$

We consider the following conditions of the population

$$p_1(u,t) \text{ and } p_2(u,t) \text{ in } 0 \leq u \leq LL > 0, \tag{2.3}$$

$$\frac{\partial p_1(0,t)}{\partial t} = \frac{\partial p_1(L,t)}{\partial t} = \frac{\partial p_2(0,t)}{\partial t} = \frac{\partial p_2(L,t)}{\partial t} = 0. \tag{2.4}$$

3. Dynamics of the system in the absence of diffusion

Equilibria: The probable steady situations of the scheme (2.1)-(2.2) are

$$E_1(0,0), E_2({p_1}^{\theta}, 0), E_3(0, {p_2}^{\varphi}), E_4({p_1}^*, {p_2}^*).$$

Since we are studying the stability of the given system around interior equilibrium point $E_4({p_1}^*, {p_2}^*)$, the concentration is on $E_4({p_1}^*, {p_2}^*)$ only.

$${p_1}^* = m_1/(r_1 m_2 - r_2 m_1); {p_2}^* = (1 + r_2 p_1)\left(i_1 - (i_1 p_1/c_1)\right)/r_1$$

For ${p_1}^*$ and ${p_2}^*$ are to be positive, provided

$$m_2 > (m_1 r_2/r_1) and c_1 > {p_1}^*$$

Local Stability: To check the local stability at $E_4({p_1}^*, {p_2}^*)$, it is required to form Jacobean matrix of the system (2.1)-(2.2). Let this matrix be in the form of $J = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$, where

$$P = -\frac{i_1 p_1}{c_1} + \frac{r_1 r_2 p_1 p_2}{(1 + r_2 p_1)^2}, \quad Q = \frac{-r_1 p_1}{1 + r_2 p_1}, \quad R = \frac{p_2 r_1 m_2}{(1 + r_2 p_1)^2}, \quad S = 0.$$

The characteristic equation of J is

$$\lambda^2 - P\lambda - (QR) = 0.$$

Clearly, the system is locally asymptotically stable provided $(i_1/c_1) < M$.

Global Stability:

Theorem 3.1. *: The positive equilibrium $E_4(p_1{}^*, p_2{}^*)$ of system (2.1)-(2.2) is globally asymptotically stable if $(i_1 > (Ac_1)/(1 + r_2p_1))$ and $(1 + r_2p_1) > 0$, where $A = (r_1r_2p_2{}^*)/(1 + r_2p_1{}^*)$.*

4. Dynamics of the system in the presence of diffusion

We consider the effect of diffusion of ecological population on the above model (2.1)-(2.2). Here D_{p_1} and D_{p_2} represents the constant diffusion coefficients of the prey and predator species. The model system (2.1)-(2.2) are in homogeneous as the reaction diffusion system is added at the top. For such introduction of the diffusion terms of the populations, it has become a spatio-temporal one dimensional system. We consider the conditions (2.3)-(2.4). The zero isoclines of model equations (2.1)-(2.2) also give the steady state which are same as we have obtained for homogeneous system. Now we consider linearized the system of (2.1)-(2.2) about the interior steady state as

$$\frac{\partial p_1}{\partial t} = i_1p_1p_1{}^* + c_1rp_2p_1{}^* + D_1\frac{\partial^2 p_1}{\partial x^2}, \tag{4.1}$$

$$\frac{\partial p_2}{\partial t} = r_1m_2p_1p_2{}^* - m_1r_2p_2p_2{}^* + D_2\frac{\partial^2 p_2}{\partial x^2}, \tag{4.2}$$

by putting $p_1 = p_1{}^* + P_1$ and $p_2 = p_2{}^* + P_2$. Let us assume the solutions in the form

$$P_1(u,t) = \alpha_1 e^{\lambda t}e^{iku}, P_2(u,t) = \alpha_2 e^{\lambda t}e^{iku},$$

where λ and k are frequency and wave numbers respectively. Then the characteristic equation of the model (4.1)-(4.2) is given by

$$\lambda^2 + A_1\lambda + A_2 = 0, \tag{4.3}$$

$$A_1 = i_1p_1{}^* - k^2(D_1 + D_2) + m_1r_2p_2{}^*,$$

$$A_2 = D_1D_2k^4 + k^2(D_1i_1c_1r_2p_1{}^* - D_2m_1r_2p_2{}^*) + p_1{}^*p_2{}^*r_2(m_1 - c_1r_1m_2).$$

Let us rewrite A_2 as a function of k^2 say $G(k^2)$ and is as follows. $G(k^2) = D_1D_2k^4 + k^2(D_1i_1c_1r_2p_1{}^* - D_2m_1r_2p_2{}^*) + p_1{}^*p_2{}^*r_2(m_1 - c_1r_1m_2)$, our main goal is to derive the criteria for diffusive instability of model system (2.1)-(2.2).The system (2.1)-(2.2) is unstable if one of the above roots of the equation (4.3) is positive. A necessary condition for a root to be positive is that $i_1 {p_1}^* - k^2(D_1 + D_2) + m_1 r_2 {p_2}^* > 0$ and $(i_1 {p_1}^* + m_1 r_2 {p_2}^*) - k^2(D_1 + D_2) > 0$. This implies that

$$k^2 < \left(i_1 {p_1}^* + m_1 r_2 {p_2}^*\right) / (D_1 D_2). \tag{4.4}$$

Thus, the necessary condition for diffusive instability of the system is $i_1 {p_1}^* + m_1 r_2 {p_2}^* > 0$. The sufficient condition for positivity of one of the roots of the equation (4.3) is $G(k^2) < 0$. Since $G(k^2)$ is an expression in k^2 where k the wave number, non-zero positive quantity, the minimum of $G(k^2)$ occurs. Let $(k^2)_{min}$ be the corresponding value of k^2 for minimum value of $G(k^2)$. The corresponding minimum value of $G(k^2)$ is

$$G(k^2) \frac{(i_1 {p_1}^*)^2 D_2}{4 D_1},$$

provided

$$\Gamma < \frac{4 m_1 r_2}{i_1 {p_1}^*}; \quad \Gamma = \frac{D_1}{D_2}. \tag{4.5}$$

Thus the diffusion of the prey predator populations derives the ecological system into unstable oscillation when (4.4) and (4.5) are satisfied.

Theorem 4.1.

(i) *if the interior equilibrium of the non-diffusive system is globally stable, and then the respective uniform steady state of the diffusive model (2.1)-(2.2) under (2.3) and (2.4) is also globally asymptotically stable.*

(ii) *If the interior equilibrium of the non-diffusive system is unstable, then the respective uniform steady state of the diffusive model (2.1)-(2.2) under (2.3) and (2.4) can be made stable by increasing diffusion coefficients appropriately.*

Proof. Let us define the function $V_1(t) = \int_0^R V\left(P_1, P_2\right) ds$. Differentiating V_1 with respect to t along the solutions of the diffusive model (2.1)-(2.2), we get,

$$\frac{dV_1}{dt} = \int_0^R \left(\frac{\partial V}{\partial P_1} \frac{\partial P_1}{\partial t} + \frac{\partial V}{\partial P_2} \frac{\partial P_2}{\partial t} \right) ds = I_1 + I_2, \tag{4.6}$$

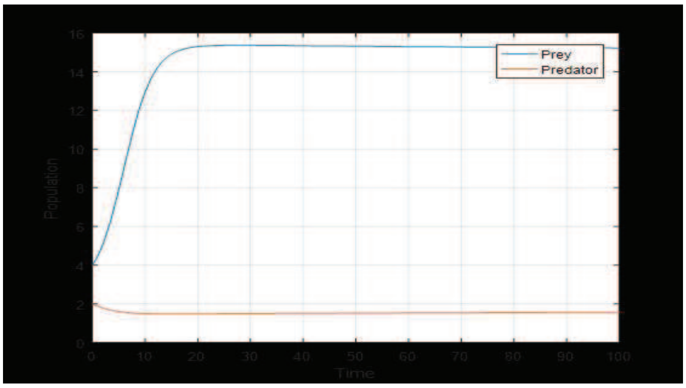

FIGURE 1

where

$$I_1 = \int_0^R \frac{dV}{dt} dx; I_2 = \int_0^R \left(D_1 \frac{\partial V}{\partial P_1} \frac{\partial^2 P_1}{\partial s^2} + D_2 \frac{\partial V}{\partial P_2} \frac{\partial^2 P_2}{\partial s^2} \right) ds \tag{4.7}$$

and

$$\begin{aligned} I_2 &= -D_1 \int_0^R \frac{\partial^2 V}{\partial P_1^2} \left(\frac{\partial P_1}{\partial s} \right)^2 ds - D_2 \int_0^R \frac{\partial^2 V}{\partial P_2^2} \left(\frac{\partial P_2}{\partial s} \right)^2 ds \\ &= -D_1 \int_0^R \frac{P_1^*}{P_1^2} \left(\frac{\partial P_1}{\partial s} \right)^2 ds - D_2 \int_0^R \frac{P_2^*}{P_2^2} \left(\frac{\partial P_2}{\partial s} \right)^2 ds. \end{aligned}$$

From (4.6) and (4.7), it can clearly be observed that if $I_1 < 0$ then $V_1'(t)$ is negative. If $I_1 > 0$, then it can be noted that by increasing the diffusion coefficients D_1 and D_2 sufficiently large, $V_1'(t)$ can be made negative. □

5. NUMERICAL SIMULATIONS

In this section, we established the analytical findings through numerical simulations using MATLAB. Figure 1 and Figure 2 represent the variation of the population against time and phase portrait between prey and predator species respectively with the parameters $c_1 = 18, m_2 = 0.4, i_1 = 0.5$.

Figure 3 and Figure 4 denote the steady fluctuations of the prey and predator populace against space and time with the parameters $m_2 = 0.4, i_1 = 0.5, D_1 = 0.1, D_2 = 0.2$.

Figure 5 and Figure 6 denote the steady fluctuations of the prey and predator

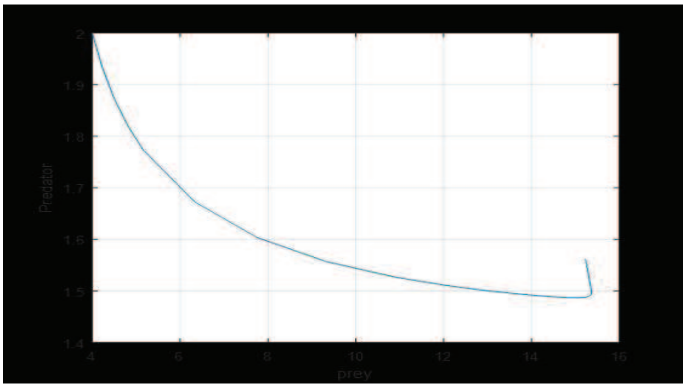

FIGURE 2

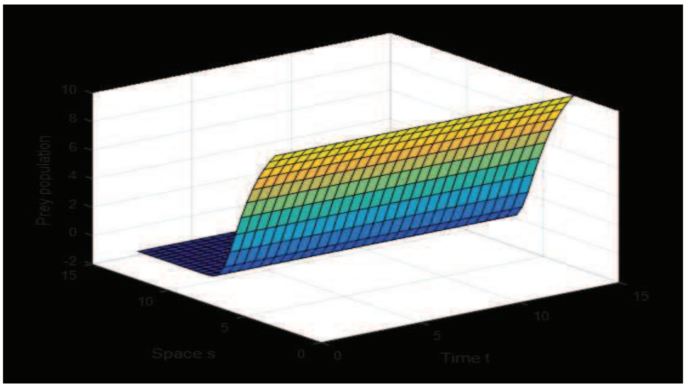

FIGURE 3

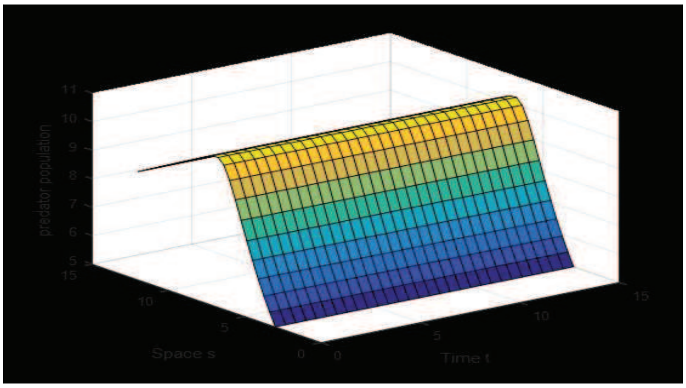

FIGURE 4

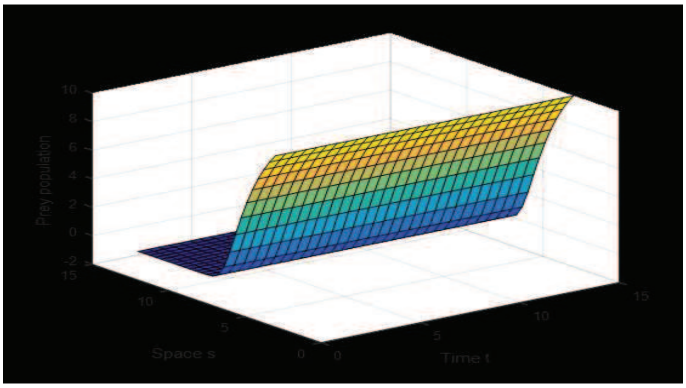

FIGURE 5

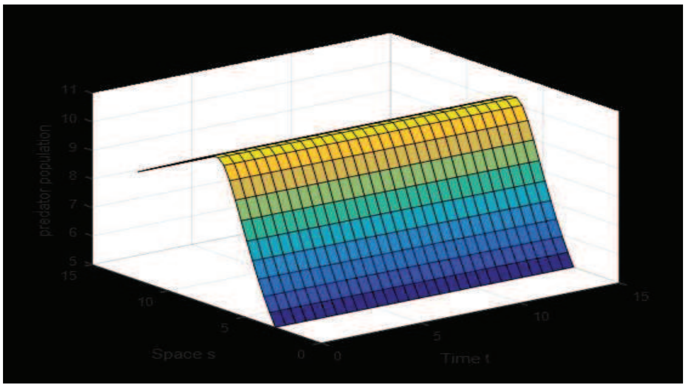

FIGURE 6

populace against space and time with the parameters $m_2 = 0.4, i_1 = 0.5, D_1 = 0.0001, D_2 = 0.0002$.

6. CONCLUSION

In this article, it is discussed about a prey-predator model with diffusion for both prey and predator. We obtained all possible equilibrium points and analysed for stability using various mathematical tools. It is shown that the dynamics of deterministic system in the figures 1, 2 and also local and global stabilities are analysed using Routh-Hurwitz criteria and Lyapunov function respectively. It is also verified the stable oscillations of the prey and predator populations against time and space in figures 3-6. Finally, numerical simulations have substantiated the analytical findings.

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