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THE UPPER RESTRAINED EDGE GEODETIC DOMINATION NUMBER OF A GRAPH

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ABSTRACT. A set S of vertices of a connected graph G is a restrained edge geodetic dominating set, if either $S = V$ or S is edge geodetic dominating set with the subgraph $G[V-S]$ induced by $V-S$ has no isolated vertices. The minimum cardinality of a restrained edge geodetic dominating set of G is called the restrained edge geodetic domination number and is denoted by $\gamma_{ger}(G)$. A restrained edge geodetic dominating set S in a connected graph G is called a minimal restrained edge geodetic dominating set of G if no proper subset of S is a restrained edge geodetic dominating set of G. The upper restrained edge geodetic domination number $\gamma^+_{ger}(G)$ is the maximum cardinality of a minimal restrained edge geodetic dominating set of G. The upper restrained edge geodetic domination number of certain classes of graphs are determined. It is shown that for every pair of integers a, b with $3 \leq a \leq b$, there exist a connected graph G of order b such that $\gamma^+_{ger}(G) = a$. Also, for any four integers a, b, c and d with $2 \leq a \leq b \leq c \leq d \leq p$, there exists a connected graph G of order p such that $g_e(G) = a$, $\gamma_{ge}(G) = b$, $\gamma_{ger}(G) = c$ and $\gamma^+_{ger}(G) = d$.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology refer [3] and [6]. The neighborhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The closed neighborhood of a vertex v is the set $N[v] = N(v) \cup N\{v\}$. A vertex v is an extreme vertex if the subgraph induced by its neighbors is complete. A vertex v is a semi-extreme vertex of G if the subgraph induced by its neighbors has a full degree vertex in $N(v)$. In particular, every extreme vertex is a semi-extreme vertex and a semi-extreme vertex need not be an extreme vertex refer [2]. For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called $u - v$ geodesic. A geodetic set of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The geodetic number $g(G)$ of G is the minimum order of its geodetic sets (refer [4], [7]).

A dominating set in a graph G is a subset of vertices of G such that every vertex outside the subset has neighbor in it. The size of a minimum dominating set in a graph G is called the domination number of G and is denoted by $\gamma(G)$. An edge geodetic set of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in an edge geodesic joining some pair of vertices in S. The edge geodetic number $g_e(G)$ of G is the minimum order of its edge geodetic set. An edge geodetic dominating set of G is a subset of $V(G)$ which is both edge geodetic set and dominating set of G. The minimum cardinality of an edge geodetic dominating set is an edge geodetic domination number and is denoted by $\gamma_{ge}(G)$ (refer [5], [8]). A set S of vertices of a connected graph G is a restrained edge geodetic set, if either $S = V$ or S is an edge geodetic set with the subgraph $G[V - S]$ induced by $V - S$ has no isolated vertices. A set S of vertices of a connected graph G is a restrained edge geodetic dominating set, if either $S = V$ or S is an edge geodetic dominating set with the subgraph $G[V - S]$ induced by $V - S$ has no isolated vertices. The minimum cardinality of a restrained edge geodetic dominating set of G is called the restrained edge geodetic domination number and is denoted by $\gamma_{ger}(G)$ (refer [1]).

The following theorems, which can be find in [1], are used in sequel.

Theorem 1.1. *Each extreme vertex of a connected graph G belongs to every restrained edge geodetic dominating set of G.*

Theorem 1.2. *Every restrained edge geodetic dominating set of a connected graph G contains its semi-extreme vertex of G.*

2. THE UPPER RESTRAINED EDGE GEODETIC DOMINATION NUMBER OF A GRAPH

Definition 2.1. *A restrained edge geodetic dominating set S in a connected graph G is called a minimal restrained edge geodetic dominating set if no proper subset of S is a restrained edge geodetic dominating set of G. The upper restrained edge geodetic domination number $\gamma^+_{ger}(G)$ is the maximum cardinality of a minimal restrained edge geodetic dominating set of G.*

Example 1. *For the graph G given in Figure 1, $S_1 = \{v_1, v_4, v_6\}$ be the minimum restrained edge geodetic dominating set of G so that $\gamma_{ger}(G) = 3$, $S_2 = \{v_1, v_2, v_5, v_6, v_7\}$ is a minimal restrained edge geodetic dominating set, so that $\gamma^+_{ger}(G) \geq 6$. It is easily verified that no six elements set of G is a restrained edge geodetic dominating set of G. Hence $\gamma^+_{ger}(G) = 5$.*

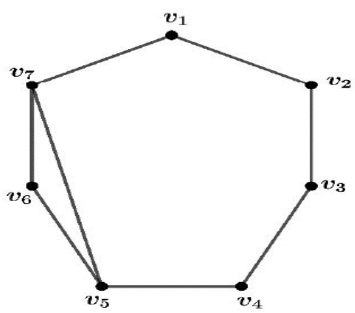

FIGURE 1

Theorem 2.1. *Each extreme vertex of a connected graph G belongs to every minimal restrained edge geodetic dominating set of G.*

Proof. Since every minimal restrained edge geodetic dominating set is a restrained edge geodetic dominating set of G. By Theorem 1.1, it is also belonging to every minimal restrained edge geodetic dominating set of G. □

Theorem 2.2. *Each semi-extreme vertex of a connected graph G belongs to every minimal restrained edge geodetic dominating set of G.*

Proof. By Theorem 1.2 and Theorem 2.1, every semi-extreme vertex belongs to every restrained edge geodetic dominating set. Since minimal restrained edge geodetic dominating set is itself a restrained edge geodetic dominating set. □

Theorem 2.3. *Let G be a connected graph of order p. If G have a semi-extreme vertex of order p then $\gamma^+_{ger}(G) = p$.*

Proof. Let G have a semi-extreme vertex of order p and by Theorem 2.2, it belongs to every minimal restrained edge geodetic dominating set. The result follows. □

Theorem 2.4. *If G is a connected graph with extreme vertices and if the set S of all extreme vertices is a restrained edge geodetic dominating set of G, then $\gamma_{ger}(G) = \gamma^+_{ger}(G)$.*

Proof. Suppose that G is a graph with extreme vertices and the set of all extreme vertices forms a restrained edge geodetic dominating set. Since any minimal restrained edge geodetic dominating set contains all the extreme vertices, it follows that the minimum restrained edge geodetic dominating sets are nothing but the minimal restrained edge geodetic dominating sets. Hence $\gamma_{ger}(G) = \gamma^+_{ger}(G)$. □

Theorem 2.5. *Let G be a connected graph with cut-vertices and let S be a minimal restrained edge geodetic dominating set of G. If v is a cut-vertex of G, then every component of $G - v$ contains some vertices of S.*

Proof. Let v be a cut-vertex of G and S be a minimal restrained edge geodetic dominating set of G. Suppose, there is a component G_1 of $G - v$ such that G_1 contains no vertices of S. By Theorem 2.1, G_1 contains at least one vertex, say u. Since S is a minimal restrained edge geodetic dominating set, there exists vertices $x, y \in S$ such that u lies on the $x - y$ geodetic path $P : x = u_0, u_1, ..., u, ..., u_t = y$ in G. Let P_1 be a $x - u$ sub path of P and P_2 be a $u - y$ sub path of P. Since v is a cut-vertex of G, both P_1 and P_2 contains v so that P is not a path, which is a contradiction. Thus, every component of $G - v$ contains an element of S. □

Theorem 2.6. *For any connected graph G, $3 \leq \gamma_{gr}(G) \leq \gamma^+_{gr}(G) \leq p$.*

Proof. A restrained edge geodetic dominating set needs at least three vertices and therefore $\gamma_{ger}(G) \geq 3$. Also, since every minimal restrained edge geodetic dominating set is a restrained edge geodetic dominating set of G and then $\gamma_{ger}(G) \leq \gamma^+_{ger}(G)$. Also, since $V(G)$ is a restrained edge geodetic dominating set of G, it is clear that $\gamma^+_{ger}(G) \leq p$. Thus $3 \leq \gamma_{ger}(G) \leq \gamma^+_{ger}(G) \leq p$. □

Theorem 2.7. *For a complete graph $K_p (p \geq 2)$, $\gamma^+_{ger}(K_p) = p$.*

Proof. Since every vertex of the complete graph $K_p (p \geq 2)$ is an extreme vertex, the vertex set of K_p is the restrained edge geodetic dominating set which contains all the vertices of K_p. Thus $\gamma^+_{ger}(K_p) = p$. □

Theorem 2.8. *For any connected graph G, $\gamma_{ger}(G) = p$ if and only if $\gamma^+_{ger}(G) = p$.*

Proof. Let $\gamma^+_{ger}(G) = p$. Then $S = V(G)$ is the unique minimal restrained edge geodetic dominating set of G. Since no proper subset of S is a restrained edge geodetic dominating set, it is clear that S is the minimum restrained edge geodetic dominating set of G and so $\gamma_{ger}(G) = p$. The converse follows from Theorem 2.1. □

Theorem 2.9. *Let G be a connected graph of order p and $u \in V(G)$. If $deg(u) = 1$ then $\gamma^+_{ger}(G - u) \leq \gamma^+_{ger}(G)$.*

Proof. Let $u \in V(G)$ and $deg(u) = 1$. Let S be the minimal restrained edge geodetic dominating set of $G - u$ with maximum cardinality. So $\gamma^+_{ger}(G) = |S|$. Since $deg(u) = 1$, u is an end vertex and u is adjacent to exactly one vertex, say v. By Theorem 2.1, every minimal restrained edge geodetic dominating set of G contains u. We consider two cases.

Case (i) Let $v \in S$. Since S is a restrained edge geodetic dominating set of $G - u$, there exist a vertex $w \in V(G - u)$ such that $w \in I[v, x] \subseteq I[S]$, $w \in N(S)$ and $d(v, x) \leq 3$. If $d(v, x) = 3$, then consider the set $S_1 = S - \{v\} \cup \{u, w\}$. If $d(v, x) \leq 2$ then consider the set $S_2 = S - \{v\} \cup \{u\}$. It is straightforward to verify that S_1 is a minimal restrained geodetic dominating set of G so that $\gamma^+_{ger}(G - u) = |S| \leq |S_1| \leq \gamma^+_{ger}(G)$.

Case (ii) Let $v \notin S$. Then consider the set $S_1 = S \cup \{u\}$. It is straight forward to verify that S_1 is a minimal restrained geodetic dominating set of G so that $\gamma^+_{ger}(G-u) = |S| \leq |S_1| \leq \gamma^+_{ger}(G)$. Hence in both cases, $\gamma^+_{ger}(G-u) \leq \gamma^+_{ger}(G)$.

□

Remark 2.1. *The sharpness of the bound, we take $G = P_5$. Let u be an end vertex of G.*

Theorem 2.10. *For the connected graph G of order p, the following are equivalent*

(i) $\gamma^+_{ger}(G) = p$;
(ii) $\gamma_{gr}(G) = p$;
(iii) $G = K_p$.

Proof. (i) ⇒ (ii)

Let $\gamma^+_{ger}(G) = p$. Then $S = V(G)$ is the unique minimal restrained edge geodetic dominating set of G. Since no proper subset of S is a restrained edge geodetic dominating set, it is clear that S is the unique minimum restrained edge geodetic dominating set of G and so $\gamma_{gr}(G) = p$.

(ii) ⇒ (iii)

Let $\gamma_{gr}(G) = p$. If $G \neq K_p$, then by Theorem 2.9, $\gamma_{ger}(G) \leq p-1$, which is a contradiction. There for $G = K_p$.

(iii) ⇒ (i).

Let $G = K_p$. Then by Theorem 2.7, $\gamma^+_{ger}(G) = p$. □

Theorem 2.11. *Let G be a connected graph of order p with $\gamma_{ger}(G) \leq p-2$. Then $\gamma^+_{ger}(G) = p$ or $p-2$.*

Proof. Given $\gamma_{ger}(G) \leq p-2$. Hence by Theorem 2.6, $\gamma^+_{ger}(G) \geq p-2$ therefore $\gamma^+_{ger}(G)$ is either p or $p-2$. If $\gamma^+_{ger}(G) = p$ then by Theorem 2.8, $\gamma_{ger}(G) = p$ which is a contradiction. Therefore $\gamma^+_{ger}(G) = p-2$. □

Theorem 2.12. *For a connected graph G,*

$$2 \leq g_e(G) \leq \gamma_{ge}(G) \leq \gamma_{ger}(G) \leq \gamma^+_{ger}(G) \leq p\,.$$

Proof. A geodetic set needs at least two vertices and therefore $g_e(G) \geq 2$. Also, every edge geodetic set is a edge geodetic dominating set of G and then $g_e(G) \leq \gamma_{ge}(G)$. If $\gamma_{ge}(G) = p$ or $p-1$ then $\gamma_{ger}(G) = p$. Also, every minimal restrained edge geodetic dominating set of G is a restrained edge geodetic dominating set of G but the converse is not true there for $\gamma_{ger}(G) < \gamma^+_{gr}(G)$. If, suppose

$\gamma_{ger}(G) = p - 2$ then clearly $\gamma^{+}_{ger}(G) = p - 2$, there fore $\gamma_{ger}(G) = \gamma^{+}_{ger}(G)$. It follows that $2 \leq g_e(G) \leq \gamma_{ge}(G) \leq \gamma_{ger}(G) \leq \gamma^{+}_{ger}(G) \leq p$. □

2.1. **Realization results.**

Theorem 2.13. *For every pair a, b of integers with $3 \leq a \leq b$, there exist a connected graph G of order b such that $\gamma^{+}_{ger}(G) = a$.*

Proof. Let $X = \{x, y\}$ and $Y = \{u_1, u_2, ..., u_{b-a}\}$ be two set of vertices. Let G be the graph obtained from X and Y by adding new vertices $z_i(1 \leq i \leq a-1)$ and joining each $z_i(1 \leq i \leq a-1)$ to y and also join each $u_i(1 \leq i \leq b-a)$ to both x and y. The resulting graph G is given in Figure 2.

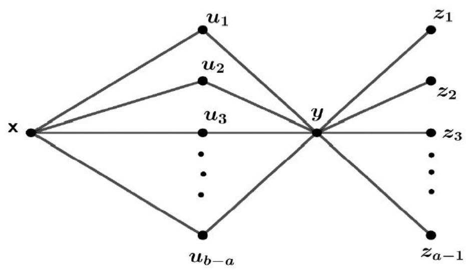

FIGURE 2

Let $S = \{z_1, z_2, z_3, ..., z_{a-1}\}$ be the set of all extreme vertices of G. By Theorem 2.1, S is the subset of every edge geodetic set, edge geodetic dominating set, restrained edge geodetic dominating set and upper restrained edge geodetic dominating set and clearly it is not an edge geodetic set of G. Obviously $S_1 = S \cup \{x\}$ is an edge geodetic set and edge geodetic dominating set and restrained edge geodetic dominating set of G. Also S_1 is the minimal restrained edge geodetic dominating set of G so that

$$\gamma^{+}_{ger}(G) = \{z_1, z_2, z_3, ..., z_{a-1}, x\} = a - 1 + 1 = a\,.$$

□

Theorem 2.14. *For every pair a, b of integers with $3 \leq a \leq b$, there exist a connected graph G such that $\gamma_{ger}(G) = a$ and $\gamma^{+}_{ger}(G) = b$.*

Proof. Let $P_1 : v_1, v_2, v_3, v_4$ be a path of order 4. Let H be the graph obtained from P_1 by adding the new vertices $z_i(1 \leq i \leq a-2)$ and joining each $z_i(1 \leq i \leq a-2)$ to v_4. Let G be the graph obtained from H by adding the new vertices

$u_i(1 \leq i \leq b-a)$ and join each $u_i(1 \leq i \leq b-a)$ with v_1 and v_4. The resulting graph G is given in Figure 3.

Let $S = \{z_1, z_2, z_3, ..., z_{a-2}\}$ be the set of all extreme vertices of G. By Theorem 2.1, S is the subset of every edge geodetic set, edge geodetic dominating set, restrained edge geodetic dominating set and upper restrained edge geodetic dominating set and clearly it is not an edge geodetic set of G. It is clear that $S_1 = S \cup \{v_1, v_2\}$ is an edge geodetic set, edge geodetic dominating set and restrained edge geodetic dominating set of G so that $\gamma_{ger}(G) = a - 2 + 2 = a$. Now, we see that $S_2 = S_1 \cup \{u_1, u_2, ..., u_{b-a}\}$ is the minimal restrained edge geodetic dominating set of G so that

$$\gamma^+_{ger}(G) = a - 2 + 2 + b - a = b\,.$$

□

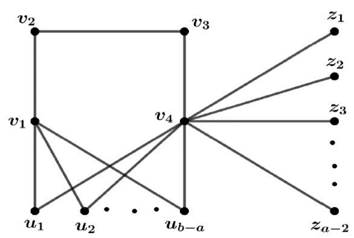

FIGURE 3

Theorem 2.15. *For any four integers a, b, c and d with $2 \leq a \leq b \leq c \leq d \leq p$, there exists a connected graph G such that $g_e(G) = a$, $\gamma_{ge}(G) = b$, $\gamma_{ger}(G) = c$ and $\gamma^+_{ger}(G) = d$.*

Proof. Case 1. $2 < a < b + 1 = c < d$, we take $V(G) - k = d$.

Let $P_1 : v_1, v_{2,3\,,4}\,, v_5$ be a path of order 5. Let H be the graph obtained from P_1 by adding $a - 1$ new vertices $z_1, z_2, z_3, ..., z_{a-1}$ and join each $z_i(1 \leq i \leq a-1)$ with v_1. Let G be the graph obtained from H by adding a cycle C of even order n. Take the vertices of C be $x_1, x_2, x_3, ..., x_n$. Now, we isolate the vertices v_5 from a path P_1 with a vertex x_1 from the cycle C. The resulting graph G given in Figure 4.

Let $S = \{z_1, z_2, z_3, ..., z_{a-1}\}$ be the set of all extreme vertices of G. By Theorem 2.1, S is the subset of every edge geodetic set, edge geodetic

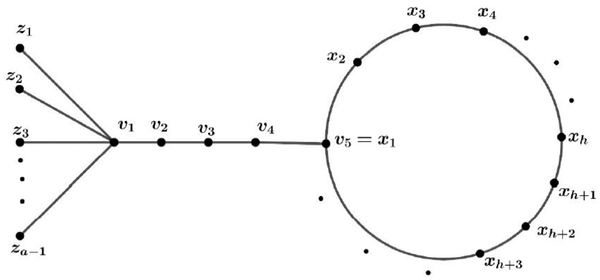

FIGURE 4

dominating set, restrained edge geodetic dominating set and upper edge restrained geodetic dominating set and clearly it is not an edge geodetic set of G. Clearly $S_1 = S \cup \{x_h\}$ is a minimal edge geodetic set of G so $g_e(G) = a-1+1 = a$. Obviously $S_2 = S_1 \cup \{v_2, v_5 = x_1, x_4, x_7, ..., x_{b-a-1}\}$ is a minimal edge geodetic dominating set of G so that $\gamma_{ge}(G) = a + b - a - 1 + 1 = b$. Clearly $S_3 = S_2 \cup \{v_1\}$ is a minimal restrained edge geodetic dominating set of G so that $\gamma_{ger}(G) = b + 1 = c$. Also, clearly $S_4 = V(G) - \{v_3, v_4, x_2, x_3, x_5, x_6, ..., x_k\}$ is a minimal upper restrained edge geodetic dominating set of G so that $\gamma^+_{ger}(G) = V(G) - k = d$.

Case 2 $2 \leq a + 1 = b = c < d$

Consider the cycle $C_5 : v_1, v_2, v_3, v_4, v_5, v_6, v_1$. Let H be the graph obtained from C by adding new vertices $x_1, x_2, x_3, ..., x_{a-1}$ and join each $x_i (1 \leq i \leq a-1)$ with v_4. The resulting graph G is given in Figure 5.

Let $S = \{x_1, x_2, x_3, ..., x_{a-1}\}$ be the set of all extreme vertices of G. By Theorem 2.1, S is the subset of every edge geodetic set, edge geodetic dominating set, restrained edge geodetic dominating set and upper restrained edge geodetic dominating set and clearly it is not an edge geodetic set of G. Clearly $S_1 = S \cup \{v_1\}$ is a minimal edge geodetic set of G so $g_e(G) = a - 1 + 1 = a$. Now $S_2 = S_1 \cup \{v_4\}$ is the edge geodetic dominating set of G so that $\gamma_{ge}(G) = a + 1 = b$ it is also a minimal restrained edge geodetic dominating set of G so that $\gamma_{ger}(G) = b = c$. Now, it is seen that $S_2 = V(G) - \{v_1, v_6\}$ is the minimal upper restrained edge geodetic dominating set of G so that $\gamma^+_{ger}(G) = |V(G) - \{v_1, v_6\}| = d$.

Case 3. $2 = a = b < c = d$

Let $P : v_1, v_2, v_3$ be a path of order 3. Let G be the graph obtained from P by adding a new vertices $w_1, w_2, w_3, ..., w_{c-b-1}$ and joining each

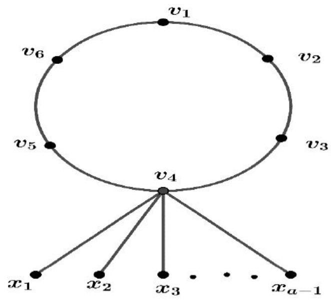

FIGURE 5

$w_i (1 \leq i \leq c - b - 1)$ to the vertices v_1 and v_3. The resulting graph G is given in Figure 6.

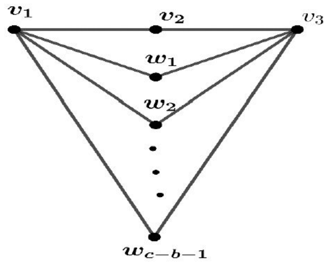

FIGURE 6

Let $S_1 = \{v_1, v_3\}$ is a minimal edge geodetic set and minimal edge geodetic dominating set of G so $g_e(G) = 2 = a = \gamma_{ge}(G) = b$. Now, we see that $S_2 = S_1 \cup \{w_1, w_2, w_3, ..., w_{c-b-1}, v_2\}$ is a minimal restrained edge geodetic dominating set of G so that $\gamma_{ger}(G) = b + c - b - 1 + 1 = c$, which is also a minimal upper restrained edge geodetic dominating set of G so that $\gamma^+_{ger}(G) = c = d$.

Case 4. $2 < a + 1 = b + 1 = c < d$

Let $P : v_1, v_2, v_3, v_4, v_5$ be a path of order 5. Let G be the graph obtained from P by adding new vertices $z_1, z_2, z_3, ..., z_{a-1}$ and joining each $z_i (1 \leq i \leq a - 1)$ to the vertices v_1. The resulting graph G is given in Figure 7.

Let $S = \{z_1, z_2, z_3, ..., z_{a-1}\}$ be the set of all extreme vertices of G. By Theorem 2.1, S is the subset of every edge geodetic set, edge geodetic dominating set, restrained edge geodetic dominating set and upper

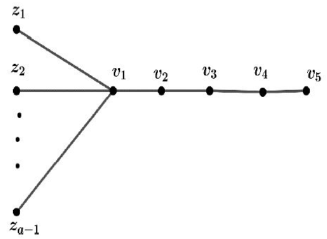

FIGURE 7

restrained edge geodetic dominating set and clearly it is not an edge geodetic set of G. Clearly $S_1 = S \cup \{v_5\}$ is a minimal edge geodetic set of G so that $g_e(G) = a - 1 + 1 = a$. Now, we see that $S_2 = S_1 \cup \{v_2\}$ is a minimal edge geodetic dominating set of G so that $\gamma_{ge}(G) = a + 1 = b$. Obviously $S_3 = S_2 \cup \{v_1\}$ is a minimal restrained edge geodetic dominating set of G so that $\gamma_{ger}(G) = b + 1 = c$. Clearly $S_4 = V(G) - \{v_3, v_4\}$ is a minimal upper restrained edge geodetic dominating set of G so that $\gamma^+_{ger}(G) = |V(G) - \{v_3, v_4\}| = d$.

□

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