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TOTAL GRAPH OF REGULAR GRAPHS

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ABSTRACT. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, the total graph $T(G)$ of G has vertex set $V(G) \cup E(G)$ and two vertices in $T(G)$ are adjacent if and only if they are adjacent or incident in G. In this paper we characterize the total graph of regular graphs and complete graphs. We prove that there doesn't exist a complete graph K_n, whose total graph is also complete for $n > 2$. Also we prove that there doesn't exists a graph whose total graph is a complete bipartite graph and some properties of total graph of cycle C_p.

1. INTRODUCTION

Graph coloring plays an important role in practical applications as well as theoretical challenges. Beside the classical types of problems, different limitations can also be solved using graph coloring. It has even reached popularity with the general public in the form of the popular number puzzle Sudoku. Graph coloring is still a very active area of research in the field of graph theory. Total coloring arises naturally since it is simply a union of vertex and edge colorings. It can be simply done as a vertex colouring of its total graph. The concept of total graph was introduced by Mehdi Behzan [1]. The total parameters of graph can be simply viewed as a graph parameter of its total graph.

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A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$, where an edge is an unordered pair of distinct vertices of G. We will usually use uv or (u, v) to denote an edge. If uv is a edge, then we say that u and v are adjacent or that v is a *neighbor* of u, and denote this by writing $u \sim v$. A vertex v is incident with an edge e if it is one of the two ends of the edge e.

The *degree* of a vertex u is the number of neighbors of u or number of edges incident on u and is denoted by $\deg_G(u)$ or simply $d(u)$.

A graph G is said to be *complete* if every distinct pair of vertices are adjacent, and the complete graph on n vertices is denoted by K_n. A bipartite graph is a graph whose vertices can be divided into two disjoint sets U and W such that every edge connects a vertex in U to one in W. Vertex sets U and W are usually called the partition of V.

The *line graph* of a graph G denoted by $L(G)$ whose vertices are the edges of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges are adjacent in G. If $e = uv$ is an edge of G, then the degree of x in the line graph $L(G)$ is $\deg_{L(G)} u + \deg_{L(G)} v - 2$.

Two graphs G and H are isomorphic if there is a bijection ϕ from $V(G)$ to $V(H)$ such that $x \sim y$ in G if and only if $\phi(x) \sim \phi(y)$ in H. We say that ϕ is an isomorphism from G to H. If G and H are isomorphic then we write $G \cong H$.

Let G and H be two graphs. A mapping f from $V(G)$ to $V(H)$ is a homomorphism if $f(x) \sim f(y)$ in H whenever $x \sim y$ in G. If there exists a homomorphism f then we say that G is homomorphic to$H$. All the preliminary definitions are in [4].

Definition 1.1. *[1] Let G be a graph with vertex set $V = \{v_i : i = 1, 2, \ldots, v_p\}$ and edge set $E = \{e_i : i = 1, 2, \ldots, q\}$. Then the total graph of G denoted by $T(G)$, which has the vertex set $V' = V \cup E$, say $(u_1, u_2, \ldots, u_{pq})$. Two vertices u_i and u_j in $T(G)$ are adjacent if and only if any one of the following holds*

i) *if u_i and u_j are vertices corresponding to the vertices v_i and v_j of G , then v_i and v_j are adjacent in G*
ii) *if u_i and u_j are vertices corresponding to the edges e_i and e_j of G , then e_i and e_j are adjacent in G*
iii) *if u_i is a vertex corresponding to the vertex v_i (or an edge e_j) and u_j is vertices corresponding to the edge e_j (or a vertex v_i) of G , then e_j is incident with v_i in G*

Example 1. *A graph G and its line graph $L(G)$, total graph of $T(G)$ are displayed in Figure 1.*

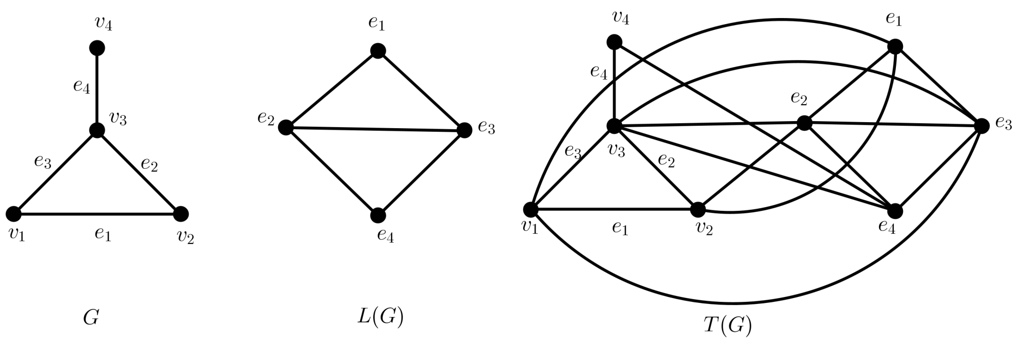


FIGURE 1

Result 1. *[2, 3] Let v be a vertex of a graph G and u be the corresponding vertex in $T(G)$, then $\deg_{T(G)} v = 2\deg_G v$.*

Result 2. *[2, 3] Let $T(G)$ be the total graph of the graph G and let $x = v_1v_2$ be an edge of G, u be the corresponding vertex in $T(G)$ then,*

$$\deg_{T(G)} u = \deg_G v_1 + \deg_G v_2 \,.$$

2. MAIN RESULTS

In this section we try to determine the regularity of certain classes of graphs.

Theorem 2.1. *The total graph of a complete graph is regular. More over, $T(K_n)$ is $2(n-1)-$regular.*

Proof. Let $\{v_i : 1, 2, \ldots, n\}$ be the vertices and $\left\{e_i : i = 1, 2, \ldots, \frac{n(n-1)}{2}\right\}$ be the edges of a complete graph K_n. Then, $T(G)$ has the vertex set

$$\left\{u_1, u_2, \ldots, u_n, u_{n+1}, u_{\frac{n(n-1)}{2}}\right\} .$$

Since $\deg_{K_n} v_i = n - 1$ for $i = 1, 2, \ldots, n$, by Result 1, $\deg_{T(K_n)} v_i = 2(n-1)$ for $i = 1, 2, \ldots, n$. Also, by Result 2, $\deg_{T(K_n)} x_i = 2(n-1)$. Hence $T(K_n)$ is $2(n-1)-$regular. □

Naturally there exists a question, Does there exists any complete graph whose total graph is complete? For we have the following theorem.

Theorem 2.2. *There does not exists a graph whose total graph be a complete graph K_n for $n > 2$.*

Proof. For $n > 2$, suppose there exists a graph G such that $T(G) = K_n$. Since K_n is $n-1$ regular, all vertices in G has of degree $\frac{n-1}{2}$, which gives that G has atleast $\frac{n-1}{2} + 1 = \frac{n+1}{2}$ vertices in G. Hence G has $n - \frac{n+1}{2} = \frac{n-1}{2}$ edges, which gives a contradiction. Hence the result. □

Corollary 2.1. *K_1 and K_2 are the only two graphs whose total graph is complete.*

Proof. Let K_n be the complete graph. Since K_n has $\frac{n(n-1)}{2}$ edges, it follows that $T(K_n)$ has $n + \frac{n(n-1)}{2}$ vertices. If $T(k_n)$ is complete, then it is $n + \frac{n(n-1)}{2} - 1$ regular. By Theorem 2.1, $T(K_n)$ is $2(n-1)$ regular. Using these two we get,

$$\begin{aligned} n + \frac{n(n-1)}{2} - 1 &= 2(n-1) \\ n^2 - 3n + 2 &= 0 \qquad (2.1) \end{aligned}$$

On solving equation (2.1) we get $n = 1$ or $n = 2$. Hence K_1 and K_2 are the only two graphs whose total graph is complete. □

Theorem 2.1, characterize the total graph of complete graphs. Now our aim is to characterize the total graph of regular graphs.

Theorem 2.3. *There does not exists a graph G whose total graph is isomorphic to a $(2n+1)-$regular graph for $n \geq 1$.*

Proof. By Result 2, the degree of all vertices in a total graph of a graph corresponding to the vertex of the graph is even. Here all degree are odd. □

Theorem 2.4. *Let G be a $2n-$regular graph with p vertices, then G is a total graph of a graph H if $p \geq \frac{(n+1)(n+2)}{2}$.*

Proof. Suppose H has k vertices, then H has $p-k$ edges. Since G is $2n$ regular, each vertex of H has degree n. Using fundamental theorem of graph theory,

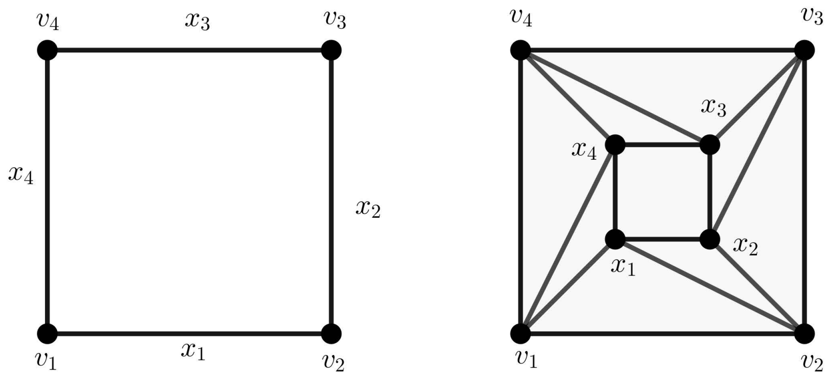
$$\begin{aligned} kn &= 2(p-k) \\ \Leftrightarrow \quad k &= \frac{2p}{n+2} \end{aligned} \tag{2.2}$$

Since every vertex of H is of degree n, so H has at least $n+1$ vertices, that is, $k \geq n+1$, then equation (2.2) becomes,

$$\frac{2p}{n+2} \geq n+1$$
$$p \geq \frac{(n+1)(n+2)}{2}$$

□

Consider C_4 and its total graph $T(C_4)$



C_4 $T(C_4)$

is a 4 regular graph.

In general, let C_p be a cycle, then its total graph has $2p$ vertices and is 4 regular. Conversely if G be a graph with $2p$ vertices and is $4-$regular, then it is a total graph of a cycle C_p. Then the following theorem immediately follows.

Theorem 2.5. *Total graph of a cycle C_p is 4 regular and has $2p$ vertices. More over, if G has $2p$ vertices and $4-$regular then G is isomorphic to $T(C_p)$.* □

Theorem 2.6. *There does not exists a graph G whose total graph, $T(G)$ is isomorphic to a complete bipartite graph.*

Proof. Suppose G be a (p, q) graph with $T(G)$ is isomorphic to $K_{m,n}$. Then $m + n = p + q$, since $T(G)$ has $p + q$ vertices, the sum of all degree of its vertices is $2(p + q)$. The sum of all degrees of vertices of $K_{m,n}$ is mn, so $mn = 2(p + q)$. Now consider,

$$\begin{aligned}(m-n)^2 &= (m+n)^2 - 4mn \\ &= (p+q)^2 - 4 \cdot 2(p+q) \\ &= (p+q)^2 - 8(p+q)\end{aligned}$$

Which is not a perfect square so we cannot find m and n as integers. Which complete the proof. □

Theorem 2.7. *The total graph of C_P,can be decomposed into $2p$ copies of K_3.*

Proof. Let $\{v_1, v_2, \ldots, v_p\}$ be the vertices of C_p and $x_i = v_i v_{i+1},\ i = 1, 2, \ldots, p - 1,$ and $x_p = v_p v_1$ be the edges of C_p. In $T(C_p)$, v_i, v_{i+1} and x_i are adjacent, hence $v_i v_{i+1} x_i$ forms a K_3. We can construct p such K_3. Now in $T(C_p)$, x_i, x_{i+1} and v_{i+1} are adjacent, so, $x_i x_{i+1} v_{i+1}$ forms a K_3. p such possibilities are there. □

Theorem 2.8. *The total graph $T(C_p)$ of C_p is homomorphic to*

$$\begin{cases} K_3 & \textit{if } 3 \equiv 0 \mod P \\ K_4 & \textit{if } p \not\equiv 0 \mod 3 \end{cases}.$$

Proof. Let $\{v_1, v_2, \ldots, v_p\}$ be the vertices of C_p and $x_i = v_i v_{i+1},\ i = 1, 2, \ldots, p - 1,$ and $x_p = v_p v_1$ be the edges of C_p.

Case 1: Suppose $p \equiv 0 \mod 0$. Let u_1, u_2, u_3 be the vertices of K_3. Define the function $\phi : V(T(C_p)) \to V(K_3)$ by,

$$\phi(v_i) = \begin{cases} u_1 & \text{if } i \equiv 1 \mod 3 \\ u_2 & \text{if } i \equiv 2 \mod 3 \\ u_3 & \text{if } i \equiv 0 \mod 3\,, \end{cases}$$

and

$$\phi(x_i) = \begin{cases} u_1 & \text{if } i \equiv 1 \mod 3 \\ u_2 & \text{if } i \equiv 2 \mod 3 \\ u_3 & \text{if } i \equiv 0 \mod 3\,. \end{cases}$$

Suppose u and v are two adjacent vertices in $T(C_p)$ then there are three possibilities:

(1) If u and v are vertices of C_p, then either $u = v_i$ and $v = v_{(i+1) \mod 3}$ or $u = v_i$ and $v = v_{(i-1) \mod 3}$, in either cases $\phi(u)$ and $\phi(v)$ are adjacent.

(2) Similarly for u and v are the edges of C_p.

(3) If u is a vertex and v is an edge of C_p, since u and v are adjacent, so, if $u = v_i$ then $v = x_i or x_{(i-1) \mod 3}$ in either cases $\phi(u)$ and $\phi(v)$ are adjacent.

Case 2: Let u_1, u_2, u_3, u_4 be the vertices of K_4. Suppose $p \not\equiv 0 \mod 3$ and p is even. Define the function $\xi : V(T(C_p)) \to V(K_4)$ by,

$$\xi(v_i) = \begin{cases} u_1 & \text{if } i \text{ is even} \\ u_2 & \text{if } i \text{ is odd} \end{cases} \quad \text{and } \xi(x_i) = \begin{cases} u_3 & \text{if } i \text{ is even} \\ u_4 & \text{if } i \text{ is odd}. \end{cases}$$

Then $T(C_p)$ is homomorphic to K_4.

Suppose $p \not\equiv 0 \mod 3$ and p is odd.

Define the function $\xi : V(T(C_p)) \to V(K_4)$ by, for $i = 1, 2, \ldots, p-1$

$$\xi(v_i) = \begin{cases} u_1 & \text{if } i \text{ is odd} \\ u_2 & \text{if } i \text{ is even} \end{cases} \qquad \xi(x_i) = \begin{cases} u_3 & \text{if } i \text{ is even} \\ u_4 & \text{if } i \text{ is odd}. \end{cases}$$

$\xi(v_p) = u_3$ and $\xi(x_p) = u_1$. Then $T(C_p)$ is homomorphic to K_4.

□

Corollary 2.2. *The chromatic number of $T(C_P)$ is* $\begin{cases} 3 & \text{if } p \equiv 0 \mod 3 \\ 4 & \text{if } p \not\equiv 0 \mod 3 \end{cases}$.

Proof. Follows from Theorem 2.8. □

Corollary 2.3. *The total chromatic number of C_P is* $\begin{cases} 3 & \text{if } p \equiv 0 \mod 3 \\ 4 & \text{if } p \not\equiv 0 \mod 3 \end{cases}$.

Proof. Follows from Definition 1.1 and Theorem 2.8. □

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