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FGP APPROACH TO BI-LEVEL MULTI-OBJECTIVE QUADRATIC FRACTIONAL PROGRAMMING WITH PARAMETRIC FUNCTIONS

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ABSTRACT. This paper propounded a methodology for obtaining compromised solution to a Quadratically constrained Bi-level Multi-objective Quadratic Fractional Programming Model(BLMOQFPM) by using the approach of parametric functions and Fuzzy Goal Programming(FGP). The Parametric approach converts the fractional model into a non-fractional one and then Fuzzy Goal Programming(FGP) is used to convert that multi-objective model into a new single objective model. An algorithm and numerical example are presented in the last to explain and validate the proposed methodology.

1. INTRODUCTION

BLMOQFPP is a two stage decision structure which contains two decision makers at two levels known as Upper level and Lower level. In BLQFPP, objectives which are required to be optimized are taken as a fraction where both numerator and denominator are quadratic functions. Objectives at the upper level are optimized by Upper level decision maker (ULDM), called as leader. Objectives at the Lower level are optimized by Lower level decision maker (LLDM), called as follower. Decision maker has its control variable at both the levels and

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is interested in optimizing its objectives. Due to opposing nature of objective functions, it is not possible to optimize every objective function simultaneously. In this case, Pareto-optimal solutions are found and Decision makers can choose the solution according to their preferences. BLQFPP have various applications such as in health care, banking sector,financial planning, IT sector, science and technology etc.

Quadratic Fractional programming problem with parametric approach was first developed by [7]. Then, [6] worked with parametric approach to find solution for interval coefficient linear fractional programming(LFP)and also [5] worked with the same approach to obtain solution to quadratic fractional problems. Fuzzy Programming also gained importance as there are many situations which are uncertain. [1] propounded FGP approach for decentralised bi-level optimization. [2] proposed a method for aiding in treatment decision based on fuzzy set theory by optimizing membership function parameters. [3] proposed a framework for FBP with risk tolerances and management requirement by forming two hybrid bilevel programming models. [4] proposed FGP to solve Multiobjective linear fractional programming problem (MOLFPP). This paper proposed a new technique to obtain a compromised solution to Bi-Level Multiobjective Quadratic Fractional Programming (BLMOQFP) problem efficiently and fastly. Fuzzy goals and Parametric approach are both applied for this purpose. Fractional objectives are converted into non-fractional ones with the help of parametric approach and then the concept of membership functions with fuzzy goals are implemented to get single and linear objectives subjected to a set of conditions. Weights corresponding to each objective at both the levels are found which appear in our final model with under deviational variables.

2. Preliminaries

In this paper, the space of n_i-dimensional real vector is denoted by R^{n_i}. Also, x^T represents transpose of any vector x. The superscripts 'I' and 'II' are used for denoting Upper Level and Lower Level respectively. 'S' denotes the collection of constraints.

3. PROBLEM FORMULATION OF BLMOQFPP PROBLEM

In BLMOQFPP, we have two decision levels which are Upper Level and Lower Level and decision makers at the two levels have their individual objective vectors and control any one of the variables say $x_1 \, and \, x_2$ respectively. We consider that the decision makers(DMs) co-operate each other in achieving their targets completely.

The Bi-level multi-objective quadratic fractional programming (BLMOQFP) model is defined as follows:

<u>Upper Level:</u> $\underset{x_1}{Min}\ F^I(x) = \underset{x_1}{Min}\ \left\{F_1^I(x),\ F_2^I(x),\ \ldots F_{p_1}^I(x)\right\}$, where x_1 is the decision variable and $F_i^I\ (x) = \frac{F_{i1}^I(x)}{F_{i2}^I(x)} = \frac{\frac{1}{2}x^T\ D_{i1}^I\ x + C_{i1}^I x + d_{i1}^I}{\frac{1}{2}\ x^T\ D_{i2}^I x + C_{i2}^I x + d_{i2}^I};\ i = 1, 2, \ldots,\ p_1$.

<u>Lower Level:</u> $\underset{x_2}{Min}\ F^{II}(x) = \underset{x_2}{Min}\ \left\{F_1^{II}(x),\ F_2^{II}(x) \ldots,\ F_{p_2}^{II}(x)\right\}$, where x_2 is the decision variable $F_j^{II}(x) = \frac{F_{j1}^{II}(x)}{F_{j2}^{II}(x)} = \frac{\frac{1}{2}x^T D_{j1}^{II} x + C_{j1}^{II} x + d_{j1}^{II}}{\frac{1}{2}x^T D_{j2}^{II} x + C_{j2}^{II} x + d_{j2}^{II}}; j = 1, 2, \ldots, p_2$ such that

$x \in S$, where $S = \left\{ x \in R^n |\ \frac{1}{2}x^T A_m x + B_m x + d_m \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} 0\ ;\ x \geq 0 \right\}$.

Here $x = (x_1,\ x_2) \in (R^{n_1},\ R^{n_2})$ such that $n_1 + n_2 = n$, i.e.

$$x_i = (x_{i1},\ x_{i2},\ \ldots x_{in_i}) \in R^{n_i}.$$

Each $D_{i1}^I,\ D_{i2}^I,\ D_{j1}^{II},\ D_{j2}^{II}$ are 2×2 real matrices and $C_{i1}^I,\ C_{i2}^I,\ C_{j1}^{II},\ C_{j2}^{II}$ are all of order 1×2; $\forall\ 1 \leq i \leq p_1$ and $\forall\ 1 \leq j \leq p_2$ and also

$$d_{i1}^I,\ d_{i2}^I,\ d_{j1}^{II},\ d_{j2}^{II} \in R\ , A_m \in R^{m \times 2},\ B_m \in R^m,\ d_m \in R\ \ \forall\ m = 1, 2,\ \ldots, s.$$

The decision makers at the two defined levels can refine their objectives by controlling their decision variables $x_1,\ x_2$.

4. PROPOSED METHODOLOGY

The parametric approach was initially proposed by [7] to obtain a solution to fractional programming problems. This approach is helpful in converting a fractional programming problem into a non- fractional problem. In this, we assign a parametric vectors to every fractional objective function and convert it into a non-fractional parametric function.

BLMOQFPM is also solved here by using parametric approach as described.

For this take each Min $F_i^I(x) = \alpha_i \; \forall \; i = 1, 2, \; \ldots, p_1$ and each Min$F_i^{II}(x) = \beta_i \;\; \forall \, i = 1, 2, \; \ldots, p_2$

Let $\alpha = \; (\alpha_1, \; \alpha_2, \; \ldots, \alpha_{p_1})$ be the parametric vector to upper level objective $F^I(x)$ and $\beta = \; (\beta_1, \; \beta_2, \; \ldots, \beta_{p_2})$ be the parametric vector to lower level objective $F^{II}(x)$.

Consider, $P_i^I(x) \; = \; F_{i1}^I(x) - \alpha_i \; F_{i2}^I(x); \;\; \forall \; i \; = \; 1, \; 2, \; \ldots, \; p_1 \; P_i^{II} \; (x) \; = F_{i1}^{II} \; (x) - \; \beta_i \; F_{i2}^{II}(x) \; ; \; \forall \, i \; = \; 1, \; 2, \; \ldots\ldots p_2$

So, by using parametric approach, the above fractional model gets converted into a non-fractional optimization model.

After this, with the help of FGP, the above multiobjective model gets transformed into the following single objective model.FGP is used to obtain a compromised solution of BLMOQFPP which is called a Pareto optimal solution by forming Fuzzy Goals corresponding to each objective function which is associated with its membership function. Further, for defining membership functions, maximum and minimum value of each and every objective function is calculated in isolation subjected to the set of initial constraints. For this, we define

$$L_i^I = \; Min\left\{P_i^I \; (x)\,\right\} \; = \; Min\left\{F_{i1}^I \; (x) - \alpha_i F_{i2}^I \; (x)\,\right\} \; ; 1 \leq i \leq n_1,$$

$$L_i^{II} = \; Min\left\{P_i^{II} \; (x)\right\} \; = \; Min\left\{F_{i1}^{II} \; (x) - \beta_i F_{i2}^{II} \; (x)\right\} ; 1 \leq i \leq n_2,$$

$$U_i^I = \; Max\left\{P_i^I \; (x)\,\right\} \; = \; Max\left\{F_{i1}^I \; (x) - \alpha_i F_{i2}^I \; (x)\,\right\} \; ; 1 \leq i \leq n_1,$$

$$U_i^{II} = \; Max\left\{P_i^{II} \; (x)\right\} \; = \; Max\left\{F_{i1}^{II} \; (x) - \beta_i F_{i2}^{II} \; (x)\right\} ; 1 \leq i \leq n_2.$$

Let $\left(x_1^{L_i^I}, \; x_2^{L_i^I}, L_i^I\right)$, $\left(x_1^{U_i^I}, \; x_2^{U_i^I}, U_i^I\right)$ be the best and $\left(x_1^{L_i^{II}}, \; x_2^{L_i^{II}}, L_i^{II}\right)$, $\left(x_1^{U_i^{II}}, \; x_2^{U_i^{II}}, U_i^{II}\right)$ be the worst solutions for the upper level and lower level decision makers. Membership functions are used to form Fuzzy Goals with aspiration level equal to unity. These goals satisfy decision vectors by allowing objective functions to reach up to their maximum aspiration levels. Fuzzy goals are then considered as the additional constraints with new single, easy and linear objective. Then, membership functions for ULDM and LLDM are given by:

$$\mu_{P_i^I}\left(P_i^I(x)\right) = \begin{cases} 1 & P_i^I(x) \leq L_i^I \\ \frac{U_i^I - P_i^I(x)}{U_i^I - L_i^I} & L_i^I \leq P_i^I(x) \leq U_i^I \\ 0 & P_i^I(x) \geq U_i^I \end{cases} \; ; \, x \in S; \;\; 1 \leq i \leq p_1,$$

$$\mu_{P_i^{II}}\left(P_i^{II}(x)\right) = \begin{cases} 1 & P_i^{II}(x) \le L_i^{II} \\ \frac{U_i^{II}-P_i^{II}(x)}{U_i^{II}-L_i^{II}} & L_i^{II} \le P_i^{II}(x) \le U_i^{II} \\ 0 & P_i^{II}(x) \ge U_i^{II} \end{cases} ;\, x \in S;\, 1 \le i \le p_2.$$

Membership functions for both the levels are converted into fuzzy goals to obtain efficient and compromised solution of BLMOQFPM. Fuzzy goals corresponding to these objectives are given as

$$\mu_{P_i^{I}}\left(P_i^{I}(x)\right) + d_i^{I-} - d_i^{I+} = 1\,; i = 1, 2, ..., p_1,$$

$$\mu_{P_i^{II}}\left(P_i^{II}(x)\right) + d_i^{II-} - d_i^{II+} = 1\,; i = 1, 2, ..., p_2,$$

where $d_i^{I-}, d_i^{II-}\, and\, d_i^{I+}, d_i^{II+}$ are under and over deviational variables such that all variables are not less than 0.

In BLMOQFPP, upper level DM has first priority, so controlled decision variable of ULDM will also appear as fuzzy goal associated with membership function defined as:

$$\mu_{x1}\left(x_1\right) = \begin{cases} 1 & x_1 \le x_1^w \\ \frac{x_1 - x_1^w}{x_1^m - x_1^w} & x_1^w \le x_1 \le x_1^m \\ 0 & x_1 \ge x_1^m \end{cases},$$

with $\mu_{x_1}\left(x_1\right) + d_1^- - d_1^+ = 1\,;$

This will become part of the constraint set for the final model. Final model consist of a single objective which is sum of the weighted under deviational variables. Here, our target is to minimize under deviational variables to reach up to maximum possible aspiration variable. So final mathematical model for obtaining optimal solution is

$$\min(z) = \sum_{\mathrm{i=1}}^{\mathrm{p_1}} \mathrm{w_i^I d_i^{I-}} + \sum_{\mathrm{i=1}}^{\mathrm{p_2}} \mathrm{w_i^{II} d_i^{II-}},$$

where $\mathrm{w_i^I} = \frac{1}{\mathrm{U_i^I - L_i^I}}$ and $\mathrm{w_i^{II}} = \frac{1}{\mathrm{U_i^{II} - L_i^{II}}}$.

Subject to

$$\mu_{P_i^{I}}\left(P_i^{I}(x)\right) + d_i^{I-} - d_i^{I+} = 1\,; i = 1, 2, ..., p_1$$

$$\mu_{P_i^{II}}\left(P_i^{II}(x)\right) + d_i^{II-} - d_i^{II+} = 1\,; i = 1, 2, ..., p_2$$

$$\mu_{x_1}\left(x_1\right) + d_1^- - d_1^+ = 1$$

$$x \in S\,; x \ge 0,\ d_i^{I-}, d_i^{II-}, d_{1j}^- \ge 0,\ d_i^{I+}, d_i^{II+}, d_{1j}^+ \ge 0$$

5. ALGORITHM

Step 1: Find $\alpha = \ (\alpha_1, \ \alpha_{\ 2}, \ \ldots, \ \alpha_{p_1})$ and $\beta = \ (\beta_1, \ \beta_{\ 2}, \ \ldots, \ \beta_{p_2})$.

Step 2: Substitute α and β to obtain $P_i^I(x) \forall \ \ 1 \leq i \leq \ p_1andP_i^{II}(x) \forall \ \ 1 \leq i \leq p_2$.

Step 3: Obtain maximum and minimum values of $P_i^I(x)$ and $P_i^{II}(x)$ and define membership functions $\mu_{P_i^I}\left(P_i^I(x)\right)$ and $\mu_{P_i^{II}}\left(P_i^{II}(x)\right)$ for both upper and lower levels respectively.

Step 4: Determine x_1^w, x_1^m and define membership function corresponding to lower level.

Step 5: Define equations for fuzzy goals corresponding to both the DMs and upper level decision variable.

Step 6: Determine all the weights as defined above.

Step 7: Put all the values in final model and execute it on Lingo software to obtain optimal solution.

Step 8: If both the DMs agree with the obtained solution, then proceed to step 10. Otherwise, proceed to step 9.

Step 9: Modify upper and lower tolerance limits defined by ULDM and LLDM.

Step 10: Stop.

6. NUMERICAL ILLUSTRATION

Ist level:

$$\min_{x_1}\left(\frac{2x_1^2+3x_2^2+x_1+5}{x_1^2-x_1x_2+2x_1+2}, \frac{5x_1^2-3x_1x_2+2x_1+x_2+3}{2x_1^2+x_2^2+x_1+1}\right)$$

IInd level:

$$\min_{x_2}\left(\frac{3x_1^2-4x_2^2+2x_2+2}{2x_2^2+x_1x_2+2}, \frac{x_1^2+x_2^2+x_1+2x_2+3}{x_1^2+2x_2^2+2x_1-1}\right)$$

Subject to

$$\begin{aligned}
&x_1^2+2x_2 \leq 4\\
&2x_1-2x_1x_2 \geq 1\\
&4x_1+7x_2^2 \geq 4\\
&x_1 \geq 0,\ x_2 \geq 0
\end{aligned}$$

Solution:

According to the procedure explained in the algorithm, Solution of single objective programming problem is obtained using LINGO 15 as $p_1^I = 1.521, p_2^I = 2.14,\ p_1^{II} = 4.179,\ p_2^{II} = 1.631$ at $x_1 = 1.456$ and $x_2 = 0$ which is our best preferred solution.

7. Conclusion

Simple and efficient solution of Bi-level Multi-objective Quadratic Fractional Programming Problem is obtained using Parametric and FGP approach. Both approaches are explored to convert multi-objective problem into single objective and solution obtained of single objective problem is the equivalent solution of Multi-objective problem.

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