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OSCILLATORY BEHAVIOR OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS OF PARTIAL FRACTIONAL DIFFERENCE EQUATIONS

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ABSTRACT. The aim in this work is to investigate the oscillation of solutions for a class of fractional partial difference equations subject to the Robin boundary condition. We establish some new sufficient conditions for the oscillation of fractional partial difference equations based on discrete Gaussian formula, generalized Riccati technique and some inequalities. Examples are presented to show the validity of the theoretical results.

1. INTRODUCTION

Fractional calculus has garnered phenomenal interest because of its various applications in multiple areas of science and engineering ranging from electric circuits, signal and image processing to viscoelasticity, industrial robotics and numerous other branches of both physical and biological sciences. Fractional differential equations have been established as an apt tool to depict the hereditary properties of various materials and real processes. Presently, fractional calculus and in particular the theory of fractional differential equations has become a prevalent gambit, see the monographs and papers [8, 9].

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The study of oscillation theory as part of the qualitative theory of the solutions for various equations like ordinary and partial differential equations, dynamic equations on time scales, difference equations and fractional differential equations is an exciting field of investigation with wide scope for research. Recently, the fundamental theory of fractional partial differential equations with different arguments has undergone intensive development [5, 10, 11, 15].

The theory of discrete fractional equations has been explored by very few researchers [3, 6, 12]. In recent years, the research on the oscillation theory of solutions of nonlinear fractional difference equations has gained momentum and some important results have been established, see [2, 4, 13]. Nevertheless, very little is known to the best of the authors' knowledge about the oscillatory behavior of fractional partial difference equations with boundary conditions which involve the Riemann-Liouville fractional partial difference operator [7]. In this paper, we aim to study some new oscillation criteria for a class of nonlinear fractional partial difference equations of the form

(1.1)

$$\Delta\left[b(\theta)g\left(p(\theta)+r(\theta)\Delta_\theta^\alpha v(y,\theta)\right)\right]+q(y,\theta)f\left[\sum_{\tau=\theta_0}^{\theta-1+\alpha}(\theta-\tau-1)^{(-\alpha)}v(y,\tau)\right]=k(\theta)Lv(y,\theta),$$

for $(y,\theta)\in\Omega\times\mathbb{N}_a$, where $\alpha\in(0,1]$ is the fractional order, Ω is a convex connected solid net, L is the discrete Laplacian on Ω (for the details on Ω and L, we refer to [7, 14]), $\Delta_\theta^\alpha v(y,\theta)$ denotes the Riemann-Liouville fractional difference operator of order α of v with respect to θ, $\mathbb{N}_a=\{a,a+1,\cdots\}$ and $a\geq 0$ is a real number.

The following assumptions hold throughout this paper:

(A_1) $b(\theta)$, $k(\theta)$ and $r(\theta)$ are positive sequences on $\theta\in[\theta_0,\infty)$ for a certain $\theta_0>0$ and $p(\theta)$ is a nonpositive sequence on $\theta\in[\theta_0,\infty)$ for a certain $\theta_0>0$. There exists a constant $M>0$ such that $r(\theta)\leq M$ for $\theta_0>0$;

(A_2) $\Delta\left[\frac{-p(\theta)}{r(\theta)}\right]\neq 0$ for $\theta\in[\theta_0,\infty)$ and $\lim_{\theta\to\infty}\sum_{\tau=\theta_0}^{\theta-1}\left[\frac{-p(\tau)}{r(\tau)}\right]<\infty$;

(A_3) The functions $f,g:\mathbb{R}\to\mathbb{R}$ are continuous functions with $vf(v)>0$, $vg(v)>0$ for $v\neq 0$, there exist positive constants μ, β such that $\frac{f(v)}{v}\geq\mu$, $\frac{v}{g(v)}\geq\beta$ for all $v\neq 0$;

(A_4) $g^{-1} \in C(\mathbb{R}, \mathbb{R})$ is a continuous function with $vg^{-1}(v) > 0$ for $v \neq 0$, there exists a positive constant d such that $g^{-1}(uv) \leq dg^{-1}(u)g^{-1}(v)$ for $uv < 0$ and $g^{-1}(uv) \geq dg^{-1}(u)g^{-1}(v)$ for $uv > 0$;

(A_5) $q(y, \theta) \geq 0$ and $\mathcal{Q}(\theta) = \min_{y \in \Omega} q(y, \theta)$, for $(y, \theta) \in \Omega \times \mathbb{N}_a$;

Consider the following Robin boundary condition

$$\Delta_N v(y-1, \theta) + \gamma(y, \theta) v(y, \theta) = 0, \quad (y, \theta) \in \partial\Omega \times \mathbb{N}_a, \tag{1.2}$$

where N is the exterior unit normal vector to $\partial\Omega$ and $\gamma(y, \theta) \geq 0$, $(y, \theta) \in \partial\Omega \times \mathbb{N}_a$. For the details on $\partial\Omega$ and $\Delta_N v(y-1, \theta)$, we refer to the monograph [1] and paper [14], respectively.

A solution $v(y, \theta)$ of (1.1) - (1.2) is said to be *oscillatory* in $\Omega \times \mathbb{N}_a$ if it is neither eventually positive nor eventually negative; otherwise, it is *nonoscillatory*.

2. Materials and Methods

In this section, we give some background materials from discrete fractional calculus, which are used throughout this paper.

Definition 2.1. *[3, 7] Let $\alpha > 0$. The α-th fractional sum f is defined by*

$$\Delta^{-\alpha} f(\theta) = \frac{1}{\Gamma(\alpha)} \sum_{\tau=a}^{\theta-\alpha} (\theta - \tau - 1)^{(\alpha-1)} f(\tau),$$

where f is defined for $\tau = a$ mod(1), $\Delta^{-\alpha} f$ is defined for $\theta = (a + \alpha)$ mod(1) and $\theta^{(\alpha)} = \dfrac{\Gamma(\theta+1)}{\Gamma(\theta - \alpha + 1)}$. The fractional sum $\Delta^{-\alpha} f$ maps functions defined on $\mathbb{N}_a = \{a, a+1, \cdots\}$ to functions defined on $\mathbb{N}_{a+\alpha} = \{a+\alpha, a+\alpha+1, \cdots\}$, where Γ is the gamma function.

Definition 2.2. *[7] Let $\alpha > 0$. The α-th fractional sum with respect to θ of $v(y, \theta)$ is defined by*

$$\Delta_\theta^{-\alpha} v(y, \theta) = \frac{1}{\Gamma(\alpha)} \sum_{\tau=a}^{\theta-\alpha} (\theta - \tau - 1)^{(\alpha-1)} v(y, \tau).$$

Lemma 2.1. *[7] For $\theta_0 \in \mathbb{N}_a$, let*

$$\Psi(\theta) = \sum_{\tau=\theta_0}^{\theta-1+\alpha} (\theta - \tau - 1)^{(-\alpha)} v(\tau), \text{ for } \theta \in \mathbb{N}_a. \tag{2.1}$$

Then

$$\Delta\Psi(\theta) = \Gamma(1-\alpha)\Delta^{\alpha}v(\theta).$$

Lemma 2.2. *(Discrete Gaussian formula) [14] Let Ω be a convex connected solid net. Then*

$$\sum_{y\in\Omega} Lv(y,\theta) = \sum_{y\in\partial\Omega} \Delta_N v(y-1,\theta).$$

3. Oscillation of Boundary Value Problem

In this section, we obtain some new oscillation criteria by using discrete Gaussian formula, generalized Riccati technique and some inequalities. For the sake of convenience, we introduce the following notations:

$$V(\theta) = \sum_{y\in\Omega} v(y,\theta), \quad z(\theta) = p(\theta) + r(\theta)\Delta^{\alpha}V(\theta).$$

Theorem 3.1. *Suppose that (A_1) - (A_5) hold. If the fractional difference inequality*

$$\Delta\left[b(\theta)g(z(\theta))\right] + \mathcal{Q}(\theta)f[\Psi(\theta)] \leq 0, \tag{3.1}$$

has no eventually positive solution, then every solution of (1.1) - (1.2) is oscillatory in $\Omega\times\mathbb{N}_a$.

Proof. On the contrary, suppose that (1.1) - (1.2) has a nonoscillatory solution v. Then, without loss of generality we may assume that $v(y,\theta) > 0$ in $\Omega\times\mathbb{N}_a$ for some $\theta_0 \geq a$. Summing up (1.1) with respect to y over Ω, we have

$$\Delta\left[b(\theta)g\left(p(\theta) + r(\theta)\Delta_{\theta}^{\alpha}\sum_{y\in\Omega} v(y,\theta)\right)\right] + \sum_{y\in\Omega} q(y,\theta)f\left[\sum_{\tau=\theta_0}^{\theta-1+\alpha}(\theta-\tau-1)^{(-\alpha)}v(y,\tau)\right] = k(\theta)\sum_{y\in\Omega} Lv(y,\theta),\ \theta\in\mathbb{N}_a. \tag{3.2}$$

The discrete Gaussian formula and (1.2) yield

$$\sum_{y\in\Omega} Lv(y,\theta) = \sum_{y\in\partial\Omega} \Delta_N v(y-1,\theta) = \sum_{y\in\partial\Omega} -\gamma(y,\theta)v(y,\theta) \leq 0,\ \theta\in\mathbb{N}_a. \tag{3.3}$$

By applying Jensen's inequality and (A_5), we have

(3.4)

$$\sum_{y\in\Omega} q(y,\theta) f\left[\sum_{\tau=\theta_0}^{\theta-1+\alpha}(\theta-\tau-1)^{(-\alpha)}v(y,\tau)\right] \geq \mathcal{Q}(\theta) f\left[\sum_{\tau=\theta_0}^{\theta-1+\alpha}(\theta-\tau-1)^{(-\alpha)}V(\tau)\right].$$

From (2.1), (3.2), (3.3) and (3.4), we have

$$\Delta\left[b(\theta)g(z(\theta))\right] + \mathcal{Q}(\theta) f\left[\Psi(\theta)\right] \leq 0,\ \theta \in \mathbb{N}_a.$$

Therefore, $V(\theta)$ is an eventually positive solution of (3.1), which contradicts our hypothesis. The proof is complete.

Theorem 3.2. *Assume that* $(A_1)-(A_5)$ *hold, and*

$$\sum_{\tau=\theta_0}^{\infty} g^{-1}\left(\frac{1}{b(\tau)}\right) = \infty. \tag{3.5}$$

Furthermore, suppose that there exists a positive function $c(\theta)$ *such that*

$$\limsup_{\theta\to\infty}\sum_{\tau=\theta_0}^{\theta-1}\left[\mu c(\tau)\mathcal{Q}(\tau) - \frac{Mb(\tau)\left[\Delta c(\tau)\right]^2}{4\beta c(\tau)\Gamma(1-\alpha)}\right] = \infty, \tag{3.6}$$

where μ *and* β *are defined in* (A_3). *Then every solution of (3.1) is oscillatory in* $\Omega\times\mathbb{N}_a$.

Proof. On the contrary, suppose that (3.1) has a nonoscillatory solution $V(\theta)$. Then, without loss of generality we may assume that $V(\theta)$ is an eventually positive solution of (3.1). Then there exists $\Psi(\theta) > 0$, $\theta \in [\theta_1,\infty)$, where $\Psi(\theta)$ is defined as in Lemma 2.1. Consequently, it is obvious that

$$\Delta\left[b(\theta)g(z(\theta))\right] \leq -\mathcal{Q}(\theta) f\left[\Psi(\theta)\right] \leq 0,\ \theta\in[\theta_1,\infty). \tag{3.7}$$

Thus, $b(\theta)g(z(\theta))$ is strictly decreasing on $[\theta_1,\infty)$ and (A_3), we see that $z(\theta)$ is eventually of one sign. We claim that $z(\theta) > 0$ for $\theta\in[\theta_1,\infty)$. Otherwise, assume that $z(\theta) < 0$ and there exists $\theta_2 \geq \theta_1$ such that $z(\theta_2) < 0$. Since $b(\theta)g(z(\theta))$ is strictly decreasing on $[\theta_1,\infty)$ and it is obvious that $b(\theta)g(z(\theta)) \leq b(\theta_2)\,g(z(\theta_2)) = \delta < 0$, where δ is a constant for $\theta\in[\theta_2,\infty)$. Therefore, we have

$$z(\theta) \leq g^{-1}\left(\frac{\delta}{b(\theta)}\right).$$

Due to $r(\theta) > 0$ and $g^{-1}(\delta) < 0$, we get

$$\frac{p(\theta)}{r(\theta)} + \Delta^{\alpha} V(\theta) \leq \frac{dg^{-1}(\delta)\, g^{-1}\left(\frac{1}{b(\theta)}\right)}{M}.$$

From Lemma 2.1, the above inequality becomes

$$\frac{p(\theta)}{r(\theta)} + \frac{\Delta\Psi(\theta)}{\Gamma(1-\alpha)} \leq \frac{dg^{-1}(\delta)\, g^{-1}\left(\frac{1}{b(\theta)}\right)}{M}. \tag{3.8}$$

Now summing up (3.8) from θ_2 to $\theta - 1$, we obtain

$$\Psi(\theta) \leq \Psi(\theta_2) + \Gamma(1-\alpha)\left[\frac{dg^{-1}(\delta)}{M}\sum_{\tau=\theta_2}^{\theta-1} g^{-1}\left(\frac{1}{b(\tau)}\right) + \sum_{\tau=\theta_2}^{\theta-1}\left(\frac{-p(\tau)}{r(\tau)}\right)\right].$$

By (A_1) and (3.5), letting $\theta \to \infty$, we obtain $\lim_{\theta\to\infty} \Psi(\theta) = -\infty$, which contradicts $\Psi(\theta) > 0$. Therefore, $z(\theta) > 0$ for $\theta \in [\theta_1, \infty)$ holds. From Lemma 2.1,

$$z(\theta) = p(\theta) + r(\theta)\Delta^{\alpha} V(\theta) = p(\theta) + r(\theta)\frac{\Delta\Psi(\theta)}{\Gamma(1-\alpha)}.$$

Therefore

$$\Delta\Psi(\theta) = \Gamma(1-\alpha)\frac{z(\theta) - p(\theta)}{r(\theta)} \leq \Gamma(1-\alpha)\frac{z(\theta)}{M}. \tag{3.9}$$

Let us define the following generalized Riccati transformation:

$$\omega(\theta) = c(\theta)\frac{b(\theta) g(z(\theta))}{\Psi(\theta)} \quad \text{for } \theta \in [\theta_1, \infty). \tag{3.10}$$

Then we have $\omega(\theta) > 0$ for $\theta \in [\theta_0, \infty)$.

$$\begin{aligned}\Delta\omega(\theta) &= \Delta\left[c(\theta)\frac{b(\theta)g(z(\theta))}{\Psi(\theta)}\right] \\ &= \Delta c(\theta)\frac{b(\theta)g(z(\theta))}{\Psi(\theta)} + \frac{c(\theta+1)\Delta[b(\theta)g(z(\theta))]}{\Psi(\theta+1)} - \frac{b(\theta)c(\theta+1)g(z(\theta))}{\Psi(\theta)\Psi(\theta+1)}\Delta\Psi(\theta).\end{aligned}$$

Now applying (A_1), (3.7), (3.9) and (3.10), we obtain

$$\Delta\omega(\theta) \leq \Delta c(\theta)\frac{\omega(\theta)}{c(\theta)} - \mu c(\theta)\mathcal{Q}(\theta) - \frac{\beta\Gamma(1-\alpha)}{Mc(\theta)b(\theta)}\omega^2(\theta) \tag{3.11}$$

i.e.,
$$\Delta\omega(\theta) \le -\mu c(\theta)\mathcal{Q}(\theta) + \frac{Mb(\theta)[\Delta c(\theta)]^2}{4\beta c(\theta)\Gamma(1-\alpha)}.$$
Summing up the last inequality from θ_0 to $\theta - 1$, we get

$$\sum_{\tau=\theta_0}^{\theta-1}\left[\mu c(\tau)\mathcal{Q}(\tau) - \frac{Mb(\tau)[\Delta c(\tau)]^2}{4\beta c(\tau)\Gamma(1-\alpha)}\right] \le \omega(\theta_0) - \omega(\theta) \le \omega(\theta_0).$$

Now taking limit as $\theta \to \infty$, we have

$$\limsup_{\theta\to\infty}\sum_{\tau=\theta_0}^{\theta-1}\left[\mu c(\tau)\mathcal{Q}(\tau) - \frac{Mb(\tau)[\Delta c(\tau)]^2}{4\beta c(\tau)\Gamma(1-\alpha)}\right] \le \omega(\theta_0) < \infty,$$

which contradicts (3.6). The proof is complete.

For the following theorem, we introduce the double sequence $\mathcal{H}(\theta,\tau)$ satisfying the conditions [11]

$$\mathcal{H}(\theta,\theta) = 0 \quad \text{for } \theta \ge \theta_0;\ \mathcal{H}(\theta,\tau) > 0 \quad \text{for } \theta > \tau \ge \theta_0;$$
$$\Delta_\tau\mathcal{H}(\theta,\tau) = \mathcal{H}(\theta,\tau+1) - \mathcal{H}(\theta,\tau) \le 0 \quad \text{for } \theta > \tau \ge \theta_0.$$

Theorem 3.3. *Assume that (A_1) - (A_5) and (3.5) hold. Furthermore, suppose that there exists a positive function $c(\theta)$ such that,*

$$\limsup_{\theta\to\infty}\frac{1}{\mathcal{H}(\theta,\theta_0)}\sum_{\tau=\theta_0}^{\theta-1}\left[\mu c(\tau)\mathcal{Q}(\tau)\mathcal{H}(\theta,\tau) - \frac{Mb(\tau)c(\tau)h_+^2(\theta,\tau)}{4\beta\Gamma(1-\alpha)\mathcal{H}(\theta,\tau)}\right] = \infty, \tag{3.12}$$

where μ, β and $c(\theta)$ are defined as in Theorem 3.2. Then every solution of (3.1) is oscillatory in $\Omega \times \mathbb{N}_a$.

Proof. On the contrary, suppose that (3.1) has a nonoscillatory solution $V(\theta)$. Then, without loss of generality, we can assume that $V(\theta)$ is an eventually positive solution of (3.1). Proceeding as in the proof of Theorem 3.2 and from assumption (A_3), we arrive at the inequality (3.11). Now

$$\mu c(\theta)\mathcal{Q}(\theta) \le -\Delta\omega(\theta) + \Delta c(\theta)\frac{\omega(\theta)}{c(\theta)} - \frac{\beta\Gamma(1-\alpha)}{Mb(\theta)c(\theta)}\omega^2(\theta).$$

Let us multiply the above inequality by $\mathcal{H}(\theta,\tau)$ on both sides and take summation from θ_2 to $\theta-1$, to get

$$\sum_{\tau=\theta_2}^{\theta-1}[\mu c(\tau)\mathcal{Q}(\tau)\mathcal{H}(\theta,\tau)] \le -\sum_{\tau=\theta_2}^{\theta-1}\Delta\omega(\tau)\mathcal{H}(\theta,\tau)+\sum_{\tau=\theta_2}^{\theta-1}\Delta c(\tau)\frac{\omega(\tau)}{c(\tau)}\mathcal{H}(\theta,\tau) - \sum_{\tau=\theta_2}^{\theta-1}\frac{\beta\Gamma(1-\alpha)}{Mb(\tau)c(\tau)}\omega^2(\tau)\mathcal{H}(\theta,\tau). \tag{3.13}$$

Using summation by parts formula, we get

$$-\sum_{\tau=\theta_2}^{\theta-1}\Delta\omega(\tau)\mathcal{H}(\theta,\tau) < \omega(\theta_2)\mathcal{H}(\theta,\theta_2)+\sum_{\tau=\theta_2}^{\theta-1}\omega(\tau+1)\Delta_\tau\mathcal{H}(\theta,\tau). \tag{3.14}$$

Substituting (3.14) in (3.13), we obtain

$$\sum_{\tau=\theta_2}^{\theta-1}\left[\mu c(\tau)\mathcal{Q}(\tau)\mathcal{H}(\theta,\tau)-\frac{h_+^2(\theta,\tau)}{4}\cdot\frac{Mb(\tau)c(\tau)}{\beta\Gamma(1-\alpha)\mathcal{H}(\theta,\tau)}\right]\le \mathcal{H}(\theta,\theta_0)\omega(\theta_2),$$

where $h_+(\theta,\tau)=\Delta_\tau\mathcal{H}(\theta,\tau)+\dfrac{\Delta c(\tau)}{c(\tau)}\mathcal{H}(\theta,\tau)$. Since, $0<\mathcal{H}(\theta,\tau)\le\mathcal{H}(\theta,\theta_0)$ for $\theta>\tau\ge\theta_0$ and $0<\dfrac{\mathcal{H}(\theta,\tau)}{\mathcal{H}(\theta,\theta_0)}\le 1$ for $\theta>\tau\ge\theta_0$, we have

$$\frac{1}{\mathcal{H}(\theta,\theta_0)}\sum_{\tau=\theta_0}^{\theta-1}\left[\mu c(\tau)\mathcal{Q}(\tau)\mathcal{H}(\theta,\tau)-\frac{h_+^2(\theta,\tau)}{4}\cdot\frac{Mb(\tau)c(\tau)}{\beta\Gamma(1-\alpha)\mathcal{H}(\theta,\tau)}\right] \le \sum_{\tau=\theta_0}^{\theta_2-1}\mu c(\tau)\mathcal{Q}(\tau)+\omega(\theta_2).$$

For $\theta\to\infty$, we obtain

$$\limsup_{\theta\to\infty}\frac{1}{\mathcal{H}(\theta,\theta_0)}\sum_{\tau=\theta_0}^{\theta-1}\left[\mu c(\tau)\mathcal{Q}(\tau)\mathcal{H}(\theta,\tau)-\frac{Mb(\tau)c(\tau)h_+^2(\theta,\tau)}{4\beta\Gamma(1-\alpha)\mathcal{H}(\theta,\tau)}\right] \le \sum_{\tau=\theta_0}^{\theta_2-1}\mu c(\tau)\mathcal{Q}(\tau)+\omega(\theta_2)<\infty,$$

which contradicts (3.12) and the proof is complete.

4. Applications

As applications of our main results, we consider the following examples.

Example 1. *Consider the nonlinear discrete partial fractional equation*

$$\Delta\left[\theta^{\frac{1}{3}}\left(-\frac{1}{\theta}+\theta^{-3}\Delta_\theta^\alpha v(y,\theta)\right)\right]+\frac{2\theta}{y}\cdot\sum_{\tau=1}^{\theta-1+\alpha}(\theta-\tau-1)^{(-\alpha)}v(y,\tau)=\theta^{\frac{1}{3}}Lv(y,\theta), \tag{4.1}$$

for $(y,\theta)\in\mathbb{N}(1,2)\times\mathbb{N}_0$, *with boundary condition*

$$\Delta_N v(0,\theta)=\Delta_N v(3,\theta)=0,\quad \theta\in\mathbb{N}_0, \tag{4.2}$$

where $\alpha\in(0,1]$. *In equation (1.1), we take* $b(\theta)=\theta^{\frac{1}{3}}$, $p(\theta)=-\frac{1}{\theta}$, $r(\theta)=\theta^{-3}$, $q(y,\theta)=\frac{2\theta}{y}$, $k(\theta)=\theta^{\frac{1}{3}}$ *and* $f(v)=g(v)=v$.
Also we set $\frac{f(v)}{v}\geq\mu=1$, $\frac{v}{g(v)}\geq\beta=3$, $M=4$, $c(\theta)=\theta$.
The assumptions (A_1)-(A_5) *hold and moreover* $\mathcal{Q}(\theta)=\min_{y\in\mathbb{N}(1,2)}q(y,\theta)=\theta$, $\Delta\left[\frac{-p(\theta)}{r(\theta)}\right]=\Delta\left[\theta^2\right]\neq 0$, $\lim_{\theta\to\infty}\sum_{\tau=\theta_0}^{\theta-1}\left[\frac{-p(\tau)}{r(\tau)}\right]=\lim_{\theta\to\infty}\sum_{\tau=1}^{\theta-1}\tau^2<\infty$ *and* $\Delta c(\theta)=1$.
Furthermore, $\sum_{\tau=\theta_0}^{\infty}g^{-1}\left[\frac{1}{b(\tau)}\right]=\sum_{\tau=1}^{\infty}\frac{1}{b(\tau)}=\sum_{\tau=1}^{\infty}\frac{1}{\tau^{\frac{1}{3}}}=\infty$
and $\sum_{\tau=\theta_0}^{\infty}\left[\mu c(\tau)\mathcal{Q}(\tau)-\frac{Mb(\tau)\left[\Delta c(\tau)\right]^2}{4\beta c(\tau)\Gamma(1-\alpha)}\right]=\sum_{\tau=1}^{\infty}\left[\tau^2-\frac{1}{3\tau^{\frac{2}{3}}\Gamma(1-\alpha)}\right]=\infty.$
Using Theorem 3.2, every solution of (4.1)-(4.2) is oscillatory in $\mathbb{N}(1,2)\times\mathbb{N}_0$.

Example 2. *Consider the fractional partial difference equation*

$$\Delta\left[\theta\left(-\frac{1}{\theta}+\theta^{\frac{1}{3}}\Delta_\theta^\alpha v(y,\theta)\right)\right]+\frac{\theta}{y}\cdot\sum_{\tau=1}^{\theta-1+\alpha}(\theta-\tau-1)^{(-\alpha)}v(y,\tau)=\theta^{\frac{2}{3}}Lv(y,\theta), \tag{4.3}$$

for $(y,\theta)\in\mathbb{N}(1,3)\times\mathbb{N}_0$, *with boundary condition*

$$\Delta_N v(0,\theta)=\Delta_N v(4,\theta)=0,\quad \theta\in\mathbb{N}_0. \tag{4.4}$$

where $\alpha\in(0,1]$. *By taking* $b(\theta)=\theta$, $p(\theta)=-\frac{1}{\theta}$, $r(\theta)=\theta^{\frac{1}{3}}$, $q(y,\theta)=\frac{\theta}{y}$, $k(\theta)=\theta^{\frac{2}{3}}$ *and* $f(v)=g(v)=v$ *in equation (1.1), we get the above equation.*
Also we take $\frac{f(v)}{v}\geq\mu=1$, $\frac{v}{g(v)}\geq\beta=3$, $M=2$, $c(\theta)=1$. *Obviously assumptions* (A_1)-(A_5) *hold. Moreover* $\mathcal{Q}(\theta)=\min_{y\in\mathbb{N}(1,3)}q(y,\theta)=\frac{\theta}{3}$, $\Delta\left[\frac{-p(\theta)}{r(\theta)}\right]=$

$\Delta\left[\theta^{-\frac{4}{3}}\right] \neq 0$, $\lim_{\theta\to\infty}\sum_{\tau=\theta_0}^{\theta-1}\left[\frac{-p(\tau)}{r(\tau)}\right] = \lim_{\theta\to\infty}\sum_{\tau=1}^{\theta-1}\left[\tau^{-\frac{4}{3}}\right] < \infty$ *and* $\Delta c(\theta) = 0$.
Furthermore, $\sum_{\tau=\theta_0}^{\infty} g^{-1}\left[\frac{1}{b(\tau)}\right] = \sum_{\tau=1}^{\infty}\frac{1}{b(\tau)} = \sum_{\tau=1}^{\infty}\frac{1}{\tau} = \infty$.
The double sequence $\mathcal{H}(\theta,\tau)$ *is defined as follows:*
$\mathcal{H}(\theta,\tau) = (\theta-\tau)^2 > 0$ *for* $\theta > \tau > 1$; $\mathcal{H}(\theta,1) = (\theta-1)^2 > 0$ *for* $\theta > \tau = 1$;
$\Delta_\tau\mathcal{H}(\theta,\tau) = [\theta-(\tau+1)]^2 - (\theta-\tau)^2 = 2\tau - 2\theta + 1 < 0$ *for* $\theta > \tau \geq 1$.
Also

$$h_+(\theta,\tau) = \Delta_\tau\mathcal{H}(\theta,\tau) + \frac{\Delta c(\tau)\mathcal{H}(\theta,\tau)}{c(\tau)} = (2\tau - 2\theta + 1).$$

Then

$$\begin{aligned}&\frac{1}{\mathcal{H}(\theta,\theta_0)}\sum_{\tau=\theta_0}^{\theta-1}\left[\mu c(\tau)\mathcal{Q}(\tau)\mathcal{H}(\theta,\tau) - \frac{h_+^2(\theta,\tau)Mb(\tau)c(\tau)}{4\beta\Gamma(1-\alpha)\mathcal{H}(\theta,\tau)}\right]\\ &\qquad = \frac{1}{(\theta-1)^2}\sum_{\tau=1}^{\theta-1}\left[\frac{\tau(\theta-\tau)^2}{3} - \frac{\tau(2\tau-2\theta+1)^2}{6(\theta-\tau)^2\Gamma(1-\alpha)}\right].\end{aligned} \tag{4.5}$$

It follows from (4.5) that

$$\limsup_{\theta\to\infty}\frac{1}{\mathcal{H}(\theta,\theta_0)}\sum_{\tau=\theta_0}^{\theta-1}\left[\mu c(\tau)\mathcal{Q}(\tau)\mathcal{H}(\theta,\tau) - \frac{h_+^2(\theta,\tau)Mb(\tau)c(\tau)}{4\beta\Gamma(1-\alpha)\mathcal{H}(\theta,\tau)}\right] = \infty.$$

From Theorem 3.3, every solution of (4.3)-(4.4) is oscillatory in $\mathbb{N}(1,3)\times\mathbb{N}_0$.

5. Concluding remarks

In this paper, by using a suitable generalized Riccati transformation and Riemann-Liouville difference operator, we obtained new sufficient conditions for the oscillation of nonlinear fractional partial difference equations in the presence of Robin boundary condition. With the support of basic theory of discrete fractional calculus, discrete Gaussian formula and fractional difference operator, the proofs are presented in a descriptive manner. The findings include some new methods for examining the oscillation of solutions of fractional partial difference equations with boundary conditions.

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