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# F-INDEX OF GENERALIZED HIERARCHICAL PRODUCT OF $N$-GRAPHS

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ABSTRACT. The generalized hierarchical product of graphs was introduced by L. Barriére et al., which is a generalization of both hierarchical product and the Cartesian product of graphs. In this study, “forgotten topological index" or F-index of generalized hierarchical product of $N$-graphs is obtained and hence from the derived results, some results are deduced as special cases.

## 1. INTRODUCTION

All graphs considered in this paper are simple and undirected. Let $G$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $v \in V(G)$ denoted by $d_G(v)$ is the number of vertices adjacent with $v$. A graph can be considered as a mathematical model and a topological index is a numeric value obtained from a graph mathematically which characterize its topology. The study of topological indices was started in 1972 [1] by the Zagreb mathematical chemistry group and successfully used in QSPR and QSAR studies. The first and second Zagreb indices were were introduced in [1] and are respectively defined as

$$M_1(G) = \sum_{v \in V(G)} (d_v)^2 .$$

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The second Zagreb index is defined as,

$$M_2(G) = \sum_{uv \in E(G)} d_u d_v.$$

In the same paper, another topological index was defined and named as forgotten topological index or F-index in [2] and is defined as

$$F(G) = \sum_{v \in V(G)} d_v^3 = \sum_{uv \in E(G)} [d_u^2 + d_v^2].$$

There are various study of this index in recent time. The present author computed some exact expressions of F-index of different graph operations in [3]. Also F-index of different transformation graphs [4], derived graphs [5], lexicographic product graphs [6] were derived. In 2009, Barriére et al. [7] introduced a new product graph, known as hierarchical product of graphs and also generalized this product latter in [8] and named as generalized hierarchical product. Different results on several topological indices under generalized hierarchical product are investigated in [9–13]. In this paper the generalized hierarchical product is extended for more than two graphs, say N-graph with respect to the F-index and hence its particular cases are considered.

## 2. Preliminaries

Let $G_i = (V_i, E_i)$ be $N$ simple connected graphs with vertex set $V_i$ and edge set $E_i$, for $i = 1, 2, ..., N$. Let $U_i,\ i = 1, 2, ..., N-1$ be the non empty vertex subsets, so that $U_i \subseteq V_i$. The generalized hierarchical product of $N$ graph is the graph $H_N$ denoted by

$$H_N = G_N(U_N) \sqcap G_{N-1}(U_{N-1}) \sqcap ..... \sqcap G_2(U_2) \sqcap G_1(U_1)\,,$$

with vertex set $V_N \times V_{N-1} \times ...... \times V_2 \times V_1$ and the edge adjacencies are defined as follows:

$$x_N.....x_3x_2x_1 \sim \begin{cases} x_N....x_3x_2x_1\ if\ y_1 \sim x_1 \in G_1, \\ x_N....x_3y_2x_1\ if\ y_2 \sim x_2 \in G_2\ and\ x_1 \in U_1, \\ \vdots \\ y_N....x_3x_2x_1\ if\ y_N \sim x_N \in G_N\ and\ x_i \in U_i,\ i = 1, 2, ...., N-1. \end{cases}$$

Now from the above definition, we get the following two extreme cases:

(1) If all the subsets $U_i$, $(i = 1, 2, .....N-1)$ are singletons then the resulting graph is the (standard) hierarchical product, which was introduced in [1].
(2) If the subsets $U_i = V_i$ for all $i = 1, 2, ....N-1$, then the graph obtained is the Cartesian product of the $N$-graphs $G_i$. The degrees of the vertices of $H_N$ are illustrated in the following Lemma.

**Lemma 2.1.** *The degree of a vertex $x = (x_N, x_{N-1}, ......., x_2, x_1) \in V(H_N)$ is given by*

$$d(x) = d(x_1) + \chi_{U_1}(x_1)d(x_2) + ....... + [\chi_{U_1}(x_1)....\chi_{U_{N-1}}(x_{N-1})]d(x_N)\,,$$

*where $\chi_{U_i}$ denote the characteristic function on the set $U_i$ which is 1 on $U_i$ and 0 out side $U_i$.*

## 3. Main Results

In this section, we study F-index of the generalized hierarchical product of the N-graphs from definition. It is obvious that the Cartesian product and hierarchical product or cluster product graphs are special cases of generalized hierarchical product of graphs. In the following theorem we obtain the F-index of the generalized hierarchical product of N-graphs.

**Theorem 3.1.** *Let $H_N = G_N \sqcap G_{N-1}(U_{N-1}) \sqcap .... \sqcap G(U_1)$, then*

$$\begin{aligned} F(H_N) &= \sum_{i=1}^{N} F(G_i)\prod_{j=1}^{i-1}|U_j|\prod_{k=i+1}^{N}|V_k| + 6\sum_{i=1}^{N-1}[\sum_{u\in U_i} d(u)^2\{\sum_{j=i+1}^{N}(|E_j|\prod_{k=1}^{j-1}|U_k| \\ &\quad \prod_{r=j+1}^{N}|V_r|)\}] + 3\sum_{i=1}^{N-1}[\sum_{u\in U_i} d(u)\{\sum_{j=i+1}^{N} M_1(G_j)\prod_{k=1}^{j-1}|U_k|\prod_{r=j+1}^{N}|V_r|)\}]\,. \end{aligned}$$

*Proof.* We have prove the above results for $N = 2$ as follows:

$$\begin{aligned} F(G_1 \sqcap G_2(U)) &= |U|F(G_1) + |V_1|F(G_2) + 3M_1(G_1)\sum_{v\in U} d_{G_2}(v) \\ &\quad +6|E_1|\sum_{v\in U} d_{G_2}(v)^2. \end{aligned} \tag{3.1}$$

Again, since the generalized hierarchical product is associative, that is

$$G_N \sqcap G_{N-1}(U_{N-1}) \sqcap .... \sqcap G_1(U_1) = (G_N \sqcap ........ \sqcap G_2(U_2)) \sqcap G(U_1)$$

we have, using Lemma 2. 1 and equation (3.1), and by inductive argument

$$
\begin{aligned}
F(H_{N+1}) &= F(G_{N+1} \sqcap H_N(U_N \times U_{N-1} \times \ldots\ldots \times U_1)) \\
&= |V_{N+1}|F(H_N) + F(G_{N+1}) \prod_{i=1}^{N} |U_i| + 6|E(G_{N+1})| \sum_{u \in (u_N,\ldots\ldots,u_1)} \delta(u)^2 \\
&\quad +3M_1(G_{N+1}) \sum_{u=(u_N,\ldots\ldots,u_1) \in U_N \times U_{N-1} \times \ldots\ldots \times U_1} \delta(u) \\
&= |V_{N+1}|[\sum_{i=1}^{N} F(G_i) \prod_{j=1}^{i-1} |U_j| \prod_{K=i+1}^{N} |V_K| \\
&\quad +6 \sum_{i=1}^{N-1} \{ \sum_{u \in U_i} \delta(u)^2 ( \sum_{j=i+1}^{N} [|E_j| \prod_{K=1}^{j-1} |U_K| \prod_{r=j+1}^{N} |V_r|] ) \} \\
&\quad +3 \sum_{i=1}^{N-1} \{ \sum_{u \in U_i} \delta(u) ( \sum_{j=i+1}^{N} M_1(G_j) \prod_{K=1}^{j-1} |U_K| \prod_{r=j+1}^{N} |V_r| ) \} ] \\
&\quad +F(G_{N+1} \prod_{i=1}^{N} |U_i| + 3M_1(G_{N+1}) [\sum_{i=1}^{N} \{ \prod_{j=1, j \neq i}^{N} |U_j| \sum_{u \in U_i} \delta(u) \} ] \\
&\quad +6|E(G_{N+1})|[\sum_{i=1}^{N} \{ \prod_{j=1, j \neq i}^{N} |U_j| \sum_{u \in U_i} \delta(u)^2 \} ] \\
&= |V_{N+1}|[\sum_{i=1}^{N} F(G_i) \prod_{j=1}^{i-1} |U_j| \prod_{K=i+1}^{N} |V_K| + F(G_{N+1} \prod_{i=1}^{N} |U_i| \\
&\quad +3|V_{N+1}| \sum_{i=1}^{N-1} \{ \sum_{u \in U_i} \delta(u) ( \sum_{j=i+1}^{N} M_1(G_j) \prod_{K=1}^{j-1} |U_K| \prod_{r=j+1}^{N} |V_r| ) \} \\
&\quad +3M_1(G_{N+1}) [\sum_{i=1}^{N} \{ \prod_{j=1, j \neq i}^{N} |U_j| \sum_{u \in U_i} \delta(u) \} \\
&\quad +6|V_{N+1}| \sum_{i=1}^{N-1} \{ \sum_{u \in U_i} \delta(u)^2 ( \sum_{j=i+1}^{N} [|E_j| \prod_{K=1}^{j-1} |U_K| \prod_{r=j+1}^{N} |V_r|] ) \} \\
&\quad +6|E(G_{N+1})|[\sum_{i=1}^{N} \{ \prod_{j=1, j \neq i}^{N} |U_j| \sum_{u \in U_i} \delta(u)^2 \} \\
&= \sum_{i=1}^{N+1} [F(G_i) \prod_{j=1}^{i-1} |U_j| \prod_{K=i+1}^{N} |V_K|]
\end{aligned}
$$

$$+3\sum_{i=1}^{N}[\sum_{u\in U_i}\delta(u)\{\sum_{j=i+1}^{N+1}(M_1(G_j)\prod_{K=1}^{j-1}|U_K|\prod_{r=j+1}^{N+1}|V_r|)\}]$$
$$+6\sum_{i=1}^{N}\{\sum_{u\in U_i}\delta(u)^2(\sum_{j=i+1}^{N}|E_j|\prod_{K=1}^{j-1}|U_K|\prod_{r=j+1}^{N}|V_r|)\}$$

which is the desired result. □

3.1. **Cartesian Product.** The Cartesian product of $G_1$ and $G_2$, denoted by $G_1 \diamond G_2$, is the graph with vertex set $V_1 \times V_2$ and any two vertices $(u_p, v_r)$ and $(u_q, v_s)$ are adjacent if and only if $[u_p = u_q$ and $v_r v_s \in E(G_2)]$ or $[v_r = v_s$ and $u_p u_q \in E(G_1)]$. We know that the the Cartesian product is a special case of generalized hierarchical product of graphs, $U_i = V_i$. Note that, the Cartesian product is both commutative and associative. Now from Theorem 3.1, considering $U_i = V_i$ for $i = 1, 2, ..., N$, the Cartesian product of N-graphs can be obtained as follows:

**Corollary 3.1.** *Let $G_i = (V_i, E_i)$ be graphs for $1 \le i \le N$, then*

$$F(G_N \diamond ... \diamond G_1) = |V|\sum_{i=1}^{N}\frac{F(G_i)}{|V_i|} + 6|V|\sum_{i=j=1}^{N}\frac{M_1(G_i)}{|V_i|}\frac{|E_j|}{|V_j|}$$
$$+8|V|\sum_{p,q,r=1}^{N}\frac{|E_p|}{|V_q|}\frac{|E_q|}{|V_q|}\frac{|E_r|}{|V_r|},$$

$|V| = \prod_{i=1}^{N}|V_i|$. *The above result and its special cases are already derived by the present author in [3]. In particular, if $G_1, G_2, ..., G_N$ be $r_1, r_2, ..., r_N$-regular graphs, respectively then*

$$F(G_1 \diamond G_2 \diamond ... \diamond G_1) = |V_1||V_i|...|V_n|(r_1 + r_2 + ... + r_N)^3,$$

*where $|V_i|$ is the number of vertices of $G_i$ $(1 \le i \le N)$. Again in particular, if $G_1 = G_2 = ... = G_N = G$ and $G^n = G \diamond G \diamond ... \diamond G$ (n-times), then from above result, we get*

$$F(G^n) = |V|^n.N^3.r^3,$$

*where $G$ is a $r$-regular graph.*

3.2. **The hierarchical product of graphs.** Let $G_i = (V_i, E_i)$ be $N$ graphs with each vertex set $V_i,\ 1 \leq i \leq N$ having a distinguished or root vertex, labelled zero. The hierarchical product

$$H = G_N \sqcap ........ \sqcap G_2 \sqcap G_1$$

is the graph with vertices $N$-tuples $x_N......x_2x_1,\ x_i \in V_i$ and edges defined by the adjacencies:

$$x_N.....x_3x_2x_1 \sim \begin{cases} x_N....x_3x_2y_1 \ if \ y_1 \sim x_1 \in G_1, \\ x_N....x_3y_2x_1 \ if \ y_2 \sim x_2 \in G_2 \ and \ x_1 = 0, \\ x_N....y_3x_2x_1 \ if \ y_3 \sim x_3 \in G_3 \ and \ x_1 = x_2 = 0, \\ : \\ y_N....x_3x_2x_1 \ if \ y_N \sim x_N \in G_N \ and \ x_1 = x_2 = ... = x_N = 0. \end{cases}$$

Let $G_1,\ G_2,\ .........,G_N$ be connected rooted graphs with root vertices $r_1, r_2,......,r_n$ respectively. Let $|V_{i,j}| = \prod_{K=i}^{j} |V_K|$ and also if $G = G_n \sqcap G_{n-1} \sqcap ...... \sqcap G_2 \sqcap G_1$ then, $d_G(r) = d_{G_1}(r_1) + d_{G_2}(r_2) + ......... + d_{G_n}(r_n)$. Let $U_i = \{Z_i\}$ are singletons for $1 \leq i \leq N$, then from Theorem 3.1, the hierarchical product of N-graphs can be obtained as follows.

**Corollary 3.2.** *Let $G_i = (V_i, E_i)$ be graphs for $1 \leq i \leq N$, then*

$$\begin{aligned} F(G_N \sqcap ... \sqcap G_1) &= \sum_{i=1}^{N} F(G_i) \prod_{k=i+1}^{N} |V_k| + 6 \sum_{i=1}^{N-1} [d(z_i)^2 \{ \sum_{j=1+1}^{N} (|E_j| \prod_{r=j+1}^{N} |V_r|) \\ &+3 \sum_{j=1+1}^{N} (M(G_j) \prod_{r=j+1}^{N} |V_r|) \}] . \end{aligned}$$

In particular, if $G_1, G_2, ..., G_N$ be $r_1, r_2, ..., r_N$-regular graphs, respectively, then from above result we get

$$\begin{aligned} F(G_N \sqcap ... \sqcap G_1) &= \sum_{i=1}^{N} \prod_{j=i}^{N} r_i^3 |V_j| + 3 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \prod_{k=j}^{N} r_i^2 r_j |V_k| \\ &+3 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \prod_{k=j}^{N} r_i r_j^2 |V_k|. \end{aligned}$$

## 4. CONCLUSION

In this paper, F-index of generalized hierarchical product of $N$-graphs is computed. From the derived results, F-index of the Cartesian product and hierarchical product of graphs is derived. Using that expressions, F-index of Cartesian product and hierarchical product of regular graphs is obtained. For further work, this study can be extended for another degree based indices.

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