Advances in Mathematics: Scientific Journal **9** (2020), no.6, 3631–3642
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.6.41

THE HYPER-ZAGREB INDEX OF SOME COMPLEMENT GRAPHS

MOHAMMED SAAD ALSHARAFI[1], MAHIOUB MOHAMMED SHUBATAH,
AND ABDU QAID ALAMERI

ABSTRACT. In this study, the Hyper-Zagreb index for some complement graphs operations has been computed, that have been applied to compute the Hyper-Zagreb index for complement molecular graph of a nanotorus and titania nanotubes.

1. INTRODUCTION

Mathematical chemistry is a branch of theoretical chemistry in which we use mathematical methods to analyze and predict the chemical structure. Chemical graph theory is a branch of mathematical chemistry where we use tools from graph theory to mathematically model the chemical phenomenon. This theory plays an important role in Function in the Chemical Sciences [9]. Throughout this paper, we consider a finite connected graph G that has no loops or multiple edges with vertex and edge sets $V(G)$, and $E(G)$, respectively. For a graph G, the degree of a vertex u is the number of edges incident to u, denoted by $\delta_G(u)$. The complement of G, denoted by $\overline{G}$, is a simple graph on the same set of vertices $V(G)$ in which two vertices u and v are adjacent, i.e., connected by an edge uv, if and only if they are not adjacent in G. Hence, $uv \in E(\overline{G})$, if and only if $uv \notin E(G)$. Obviously $E(G) \cup E(\overline{G}) = E(K_n)$, and $\overline{m} = |E(\overline{G})| =$

[1]*corresponding author*

2010 *Mathematics Subject Classification.* 05C09.
Key words and phrases. Zagreb index, Hyper-Zagreb index, complement graph, graph operation.

$\binom{n}{2} - m$, the degree of a vertex u in $\overline{G}$, is the number of edges incident to u, denoted by $\delta_{\overline{G}}(u) = n-1-\delta_G(u)$ [11]. The first and second Zagreb indices have been introduced by Gutman and Trinajestic in 1972 [10]. They are respectively defined as:

$$M_1(G) = \sum_{v\in V(G)} {\delta_G}^2(v) = \sum_{uv\in E(G)} [\delta_G(u)+\delta_G(v)], \qquad M_2(G) = \sum_{uv\in E(G)} \delta_G(u)\,\delta_G(v),$$

In 2013, G.H. Shirdel, H. Rezapour and A.M. Sayadi [5] iintroduced distance-based of Zagreb indices named Hyper-Zagreb index which is defined as:

$$HM(G) = \sum_{uv\in E(G)} (\delta_G(u) + \delta_G(v))^2 .$$

Furtula and Gutman in 2015 introduced forgotten index (F-index) [4] which defined as:

$$F(G) = \sum_{uv\in E(G)} \left({\delta_G}^2(u) + {\delta_G}^2(v)\right).$$

In 2020, computed exact formulas for the Y-index of some graph operations by A. Alameri et al [1]. They defined a new distance-based of forgotten indices named Yemen-index (Y-index) defined as:

$$Y(G) = \sum_{u\in V(G)} \delta_G^4(u) = \sum_{uv\in E(G)} [\delta_G^3(u) + \delta_G^3(v)].$$

2. Preliminaries

In this section we give some basic and preliminary concepts which we shall use later.

Lemma 2.1. *[2] Let G_1 and G_2 be two connected graphs with $|V(G_1)| = n_1$, $|V(G_2)| = n_2$, $|E(G_1)| = m_1$, and $|E(G_2)| = m_2$. Then*

(1) $|V(G_1 \times G_2)| = |V(G_1 \vee G_2)| = |V(G_1 \circ G_2)| = |V(G_1 \otimes G_2)| = |V(G_1 * G_2)| = |V(G_1 \oplus G_2)| = |V(G_1 + G_2)| = n_1n_2$,

(2) $|E(G_1 \times G_2)| = m_1n_2 + n_1m_2$, $|E(G_1 * G_2)| = m_1n_2 + n_1m_2 + 2m_1m_2$,
$|E(G_1 + G_2)| = m_1 + m_2 + n_1n_2$, $|E(G_1 \circ G_2)| = m_1{n_2}^2 + m_2n_1$,
$|E(G_1 \vee G_2)| = m_1{n_2}^2 + m_2{n_1}^2 - 2m_1m_2$, $|E(G_1 \otimes G_2)| = 2m_1m_2$,
$|E(G_1 \oplus G_2)| = m_1{n_2}^2 + m_2{n_1}^2 - 4m_1m_2$.

Corollary 2.1. *[10] The first Zagreb index of some well-known graphs: For path graph P_n and cycle graph C_n, with $n : n \geq 3$ vertices :*

$$M_1(C_n) = 4n, \quad M_1(P_n) = 4n - 6.$$

Corollary 2.2. *[3, 5] The Hyper-Zagreb index of some well-known graphs: For path P_n and cycle graphs C_n, with $n, m \geq 3$ vertices:*

$$HM(C_n) = 16n, \quad HM(P_n) = 16n - 30, \quad M(P_n \times C_m) = 128nm - 150m.$$

Theorem 2.1. *[6] Let G be a simple graph on n vertices and m edges. Then:*

$$\begin{aligned} M_1(\overline{G}) &= n(n-1)^2 - 4m(n-1) + M_1(G)\,, \\ HM(\overline{G}) &= 2n(n-1)^3 - 12m(n-1)^2 + 4m^2 + (5n-6)M_1(G) - HM(G)\,. \end{aligned}$$

3. Main results

In this section, we study the Hyper-Zagreb index of various complement graph binary operations such as Cartesian product $\overline{G_1 \times G_2}$, composition $\overline{G_1 \circ G_2}$, disjunction $\overline{G_1 \vee G_2}$, symmetric difference $\overline{G_1 \oplus G_2}$, join $\overline{G_1 + G_2}$, tensor product $\overline{G_1 \otimes G_2}$, and strong product $\overline{G_1 * G_2}$, of graphs. We use the notation $V(G_i)$ for the vertex set, $E(\overline{G}_i)$ for the edge set, n_i for the number of vertices and m_i, $\overline{m}_i$ for the number of edges of the graph G_i, $\overline{G}_i$ respectively. All graphs here offer are simple graphs.

Proposition 3.1. *The Hyper-Zagreb index of the complement of $(G_1 \otimes G_2)$ is given by:*

$$\begin{aligned} &HM(\overline{G_1 \otimes G_2}) \\ &\quad = 2n_1n_2(n_1n_2-1)^3 - 24m_1m_2(n_1n_2-1)^2 + 16m_1^2m_2^2 \\ &\quad + (5n_1n_2-6)M_1(G_1)M_1(G_2) - F(G_1)F(G_2) + 2M_2(G_1)M_2(G_2). \end{aligned}$$

Proof. By using Theorem 2.1 we have:

$$\begin{aligned} &HM(\overline{G_1 \otimes G_2}) \\ &= 2|V(G_1 \otimes G_2)|(|V(G_1 \otimes G_2)| - 1)^3 - 12|E(G_1 \otimes G_2)|(|V(G_1 \otimes G_2)| - 1)^2 \\ &+4|E(G_1 \otimes G_2)|^2 + (5|V(G_1 \otimes G_2)| - 6)M_1(G_1 \otimes G_2) - HM(G_1 \otimes G_2)\,. \end{aligned}$$

By Lemma 2.1 $|E(G_1 \otimes G_2)| = 2m_1m_2, \qquad |V(G_1 \otimes G_2)| = n_1n_2,$ and by [7] and [8], respectively, we have

$$M_1(G_1 \otimes G_2) = M_1(G_1)M_1(G_2), \quad HM(G_1 \otimes G_2) = F(G_1)F(G_2) + 2M_2(G_1)M_2(G_2),$$

which is complete the proof. □

Proposition 3.2. *The Hyper-Zagreb index of the complement of* $(G_1 + G_2)$ *is given by:*

$$\begin{aligned} &HM(\overline{G_1 + G_2}) \\ &= 2n_1n_2(n_1n_2 - 1)^3 - 12(m_1 + m_2 + n_1n_2)(n_1n_2 - 1)^2 + 4(m_1 + m_2 + n_1n_2)^2 \\ &+(5n_1n_2 - 6)[M_1(G_1) + M_1(G_2) + n_1n_2^2 + n_2n_1^2 + 4m_1n_2 + 4m_2n_1] \\ &-[HM(G_1) + HM(G_2) + 5(n_1M_1(G_2) + n_2M_1(G_1)) \\ &+8[n_1^2m_2 + n_2^2m_1 + m_1m_2] + n_1n_2[(n_2 + n_1)^2 + 4(m_1 + m_2)]]. \end{aligned}$$

Proof. By using Theorem 2.1 we have

$$\begin{aligned} &HM(\overline{G_1 + G_2}) \\ &= 2|V(G_1 + G_2)|(|V(G_1 + G_2)| - 1)^3 - 12|E(G_1 + G_2)|(|V(G_1 + G_2)| - 1)^2 \\ &+4|E(G_1 + G_2)|^2 + (5|V(G_1 + G_2)| - 6)M_1(G_1 + G_2) - HM(G_1 + G_2), \end{aligned}$$

By Lemma 2.1 $|E(G_1 + G_2)| = m_1 + m_2 + n_1n_2, \qquad |V(G_1 + G_2)| = n_1n_2,$ and by [7] and [5], respectively, we have

$$M_1(G_1 + G_2) = M_1(G_1) + M_1(G_2) + n_1n_2^2 + n_2n_1^2 + 4m_1n_2 + 4m_2n_1,$$

$$\begin{aligned} &HM(G_1 + G_2) = \\ &HM(G_1) + HM(G_2) + 5(n_1M_1(G_2) + n_2M_1(G_1)) \\ +\ &8[n_1^2m_2 + n_2^2m_1 + m_1m_2] + n_1n_2[(n_2 + n_1)^2 + 4(m_1 + m_2)], \end{aligned}$$

which is complete the proof. □

Proposition 3.3. *Let* G_1, G_2 *be two simple connected graphs with* n_1, n_2 *vertices and* m_1, m_2 *edges, respectively, Then,*

$$M_1(G_1 * G_2) = (n_2 + 6m_2)M_1(G_1) + 8m_2m_1 + (6m_1 + n_1)M_1(G_2) + 2M_1(G_1)M_1(G_2).$$

Proposition 3.4. *The Hyper-Zagreb index of the complement of* $(G_1 * G_2)$ *is given by:*

$$\begin{aligned} HM(\overline{G_1 * G_2}) & \\ &= 2n_1n_2(n_1n_2-1)^3 - 12(m_1n_2+n_1m_2+2m_1m_2)(n_1n_2-1)^2 \\ &+ 4(m_1n_2+n_1m_2+2m_1m_2)^2 + (5n_1n_2-6)[(n_2+6m_2)M_1(G_1) \\ &+ 8m_2m_1 + (6m_1+n_1)M_1(G_2) + 2M_1(G_1)M_1(G_2)] \\ &- [HM(G_1) + n_1HM(G_2) + 5n_2M_1(G_1) + 5n_1M_1(G_2) \\ &+ 4n_2m_1[2n_2+1] + 8m_2[n_1+m_1] + n_1n_2(n_2^3+2n_2+4m_2)]. \end{aligned}$$

Proof. By using Theorem 2.1 we have

$$\begin{aligned} & HM(\overline{G_1 * G_2}) \\ &= 2|V(G_1 * G_2)|(|V(G_1 * G_2)|-1)^3 - 12|E(G_1 * G_2)|(|V(G_1 * G_2)|-1)^2 \\ &+4|E(G_1 * G_2)|^2 + (5|V(G_1 * G_2)|-6)M_1(G_1 * G_2) - HM(G_1 * G_2)\,. \end{aligned}$$

By Lemma 2.1 $|E(G_1 * G_2)| = m_1n_2 + n_1m_2 + 2m_1m_2$, $|V(G_1 * G_2)| = n_1n_2$, and by Proposition 3.3 and [5], respectively, we have

$$M_1(G_1*G_2) = (n_2+6m_2)M_1(G_1)+8m_2m_1+(6m_1+n_1)M_1(G_2)+2M_1(G_1)M_1(G_2),$$

$$\begin{aligned} HM(G_1 * G_2) = & \\ & HM(G_1) + n_1HM(G_2) + 5n_2M_1(G_1) + 5n_1M_1(G_2) \\ + & \; 4n_2m_1[2n_2+1] + 8m_2[n_1+m_1] + n_1n_2[n_2^3+2n_2+4m_2]\,, \end{aligned}$$

which is complete the proof. □

Proposition 3.5. *The Hyper-Zagreb index of the complement of* $(G_1 \times G_2)$ *is given by:*

$$\begin{aligned} & HM(\overline{G_1 \times G_2}) \\ &= 2n_1n_2(n_1n_2-1)^3 - 12(m_1n_2+m_2n_1)(n_1n_2-1)^2 + 4(m_1n_2+m_2n_1)^2 \\ &+(5n_1n_2-6)[n_2M_1(G_1)+n_1M_1(G_2)+8m_1m_2] - [n_2HM(G_1) \\ &+n_1HM(G_2) + 12m_1M_1(G_2) + 12m_2M_1(G_1)]. \end{aligned}$$

Proof. By using Theorem 2.1 we have

$$\begin{aligned}
&HM(\overline{G_1 \times G_2}) \\
&= 2|V(G_1 \times G_2)|(|V(G_1 \times G_2)| - 1)^3 - 12|E(G_1 \times G_2)|(|V(G_1 \times G_2)| - 1)^2 \\
&+4|E(G_1 \times G_2)|^2 + (5|V(G_1 \times G_2)| - 6)M_1(G_1 \times G_2) - HM(G_1 \times G_2)\,.
\end{aligned}$$

By Lemma 2.1 $|E(G_1\times G_2)| = m_1n_2+n_1m_2, \qquad |V(G_1\times G_2)| = n_1n_2,$ and by [7] and [3] respectively, we have

$$M_1(G_1 \times G_2) = n_2M_1(G_1) + n_1M_1(G_2) + 8m_1m_2,$$

$$HM(G_1 \times G_2) = n_2HM(G_1) + n_1HM(G_2) + 12m_1M_1(G_2) + 12m_2M_1(G_1),$$

which is complete the proof. □

Proposition 3.6. *The Hyper-Zagreb index of the complement of* $(G_1 \circ G_2)$ *is given by:*

$$\begin{aligned}
&HM(\overline{G_1 \circ G_2}) \\
&= 2n_1n_2(n_1n_2 - 1)^3 - 12[m_1n_2^2 + m_2n_1](n_1n_2 - 1)^2 + 4[m_1n_2^2 + m_2n_1]^2 \\
&+(5n_1n_2 - 6)[n_2^3M_1(G_1) + n_1M_1(G_2) + 8n_2m_2m_1] - [n_2^4HM(G_1) \\
&+n_1HM(G_2) + 12n_2^2m_2M_1(G_1) + 10n_2m_1M_1(G_2) + 8m_2m_1].
\end{aligned}$$

Proof. By using Theorem 2.1 we have

$$\begin{aligned}
&HM(\overline{G_1 \circ G_2}) \\
&= 2|V(G_1 \circ G_2)|(|V(G_1 \circ G_2)| - 1)^3 - 12|E(G_1 \circ G_2)|(|V(G_1 \circ G_2)| - 1)^2 \\
&+4|E(G_1 \circ G_2)|^2 + (5|V(G_1 \circ G_2)| - 6)M_1(G_1 \circ G_2) - HM(G_1 \circ G_2),
\end{aligned}$$

By Lemma 2.1 $|E(G_1 \circ G_2)| = m_1{n_2}^2 + m_2n_1, \qquad |V(G_1 \circ G_2)| = n_1n_2,$ and by [10] and [3] respectively, we have

$$M_1(G_1 \circ G_2) = n_2^3M_1(G_1) + n_1M_1(G_2) + 8n_2m_2m_1,$$

$$HM(G_1\circ G_2) = n_2^4HM(G_1)+n_1HM(G_2)+12n_2^2m_2M_1(G_1)+10n_2m_1M_1(G_2)+8m_2m_1,$$

which is complete the proof. □

Proposition 3.7. *The Hyper-Zagreb index of the complement of* $(G_1 \vee G_2)$ *is given by:*

$$\begin{aligned}
&HM(\overline{G_1 \vee G_2})\\
&= 2n_1n_2(n_1n_2-1)^3 - 12[m_1n_2^2+m_2n_1^2-2m_1m_2](n_1n_2-1)^2\\
&+4[m_1n_2^2+m_2n_1^2-2m_1m_2]^2 + (5n_1n_2-6)[(n_1n_2^2-4m_2n_2)M_1(G_1)\\
&+M_1(G_2)M_1(G_1)+(n_2n_1^2-4m_1n_1)M_1(G_2)+8m_1m_2n_1n_2]\\
&-[[n_1^4-2n_2^2m_2]HM(G_2)+[n_2^4-2n_2^2m_2]HM(G_1)+5n_1M_1(G_1)F(G_2)\\
&+5n_2M_1(G_2)F(G_1)+10n_2^2m_2n_1M_1(G_1)+10n_2n_1^2m_1M_1(G_2)\\
&+8n_2^2m_2m_1+8n_1^2m_1m_2-8n_2m_1^2M_1(G_2)-8n_1m_2^2M_1(G_1)\\
&-4n_1^2m_1F(G_2)-4n_2^2m_2F(G_1)-8n_1^2m_1M_2(G_2)-8n_2^2m_2M_2(G_1)\\
&+8m_1M_2(G_2)+8m_2M_2(G_1)-8n_2n_1M_1(G_1)M_1(G_2)+4n_2M_2(G_1)M_1(G_2)\\
&+4n_1M_2(G_2)M_1(G_1)-2F(G_1)F(G_2)-4M_2(G_1)M_2(G_2)].
\end{aligned}$$

Proof. By using Theorem 2.1 we have

$$\begin{aligned}
&HM(\overline{G_1 \vee G_2})\\
&= 2|V(G_1\vee G_2)|(|V(G_1\vee G_2)|-1)^3 - 12|E(G_1\vee G_2)|(|V(G_1\vee G_2)|-1)^2\\
&+4|E(G_1\vee G_2)|^2 + (5|V(G_1\vee G_2)|-6)M_1(G_1\vee G_2) - HM(G_1\vee G_2),
\end{aligned}$$

By Lemma 2.1 $|E(G_1\vee G_2)| = m_1{n_2}^2+m_2{n_1}^2-2m_1m_2, \qquad |V(G_1\vee G_2)| = n_1n_2,$ and by [10] and [8] respectively, we have

$$\begin{aligned}
M_1(G_1\vee G_2) &\\
&= (n_1n_2^2-4m_2n_2)M_1(G_1)+M_1(G_2)M_1(G_1)\\
&+ (n_2n_1^2-4m_1n_1)M_1(G_2)+8m_1m_2n_1n_2,
\end{aligned}$$

$$\begin{aligned}
&HM(G_1\vee G_2) =\\
&[n_1^4-2n_2^2m_2]HM(G_2)+[n_2^4-2n_2^2m_2]HM(G_1)+5n_1M_1(G_1)F(G_2)\\
&+5n_2M_1(G_2)F(G_1)+10n_2^2m_2n_1M_1(G_1)+10n_2n_1^2m_1M_1(G_2)\\
&+8n_2^2m_2m_1+8n_1^2m_1m_2-8n_2m_1^2M_1(G_2)-8n_1m_2^2M_1(G_1)-4n_1^2m_1F(G_2)\\
&-4n_2^2m_2F(G_1)-8n_1^2m_1M_2(G_2)-8n_2^2m_2M_2(G_1)+8m_1M_2(G_2)
\end{aligned}$$

$$+8m_2M_2(G_1) - 8n_2n_1M_1(G_1)M_1(G_2) + 4n_2M_2(G_1)M_1(G_2)$$
$$+4n_1M_2(G_2)M_1(G_1) - 2F(G_1)F(G_2) - 4M_2(G_1)M_2(G_2),$$

which is complete the proof. □

Proposition 3.8. *The Hyper-Zagreb index of the complement of* $(G_1 \oplus G_2)$ *is given by:*

$$HM(\overline{G_1 \oplus G_2})$$
$$= 2n_1n_2(n_1n_2 - 1)^3 - 12[m_1n_2^2 + m_2n_1^2 - 4m_1m_2](n_1n_2 - 1)^2$$
$$+4[m_1n_2^2 + m_2n_1^2 - 4m_1m_2]^2 + (5n_1n_2 - 6)[(n_1n_2^2 - 8m_2n_2)M_1(G_1)$$
$$+4M_1(G_1)M_1(G_2) + (n_2n_1^2 - 8m_1n_1)M_1(G_2) + 8m_1m_2n_1n_2]$$
$$-[[n_1^4 - 4n_2^2m_2]HM(G_2) + [n_2^4 - 4n_2^2m_2]HM(G_1) + 20n_1M_1(G_1)F(G_2)$$
$$+20n_2M_1(G_2)F(G_1) + 10n_2^2m_2n_1M_1(G_1) + 10n_2n_1^2m_1M_1(G_2)$$
$$+8n_2^2m_2m_1 - 16n_2m_1^2M_1(G_2) - 16n_1m_2^2M_1(G_1) - 8n_1^2m_1F(G_2)$$
$$-8n_2^2m_2F(G_1) - 16n_1^2m_1M_2(G_2) - 16n_2^2m_2M_2(G_1) + 32m_1M_2(G_2)$$
$$+32m_2M_2(G_1) - 16n_2n_1M_1(G_1)M_1(G_2) + 16n_2M_2(G_1)M_1(G_2)$$
$$+16n_1M_2(G_2)M_1(G_1) - 16F(G_1)F(G_2) - 32M_2(G_1)M_2(G_2)].$$

Proof. By the similar method in Proposition 3.7 □

4. Application

Corollary 4.1. *The hyper-Zagreb index of complement nanotube Figure 1 is given by*

$$HM(\overline{TiO_2[n,m]}) = 12n(6mn + 6n - 1)^2[6m^2n + 12mn + 6n - 11m - 9]$$
$$+2680m^2n^2 + 4360mn^2 + 1696n^2 - 1036mn - 572n.$$

Proof. By using Theorem 2.1 we have

$$HM(\overline{TiO_2})$$
$$= 2|V(TiO_2)|(|V(TiO_2)| - 1)^3 - 12|E(TiO_2)|(|V(TiO_2)| - 1)^2 + 4|E(TiO_2)|^2$$
$$+(5|V(TiO_2)| - 6)M_1(TiO_2[n,m]) - HM(TiO_2[n,m]),$$

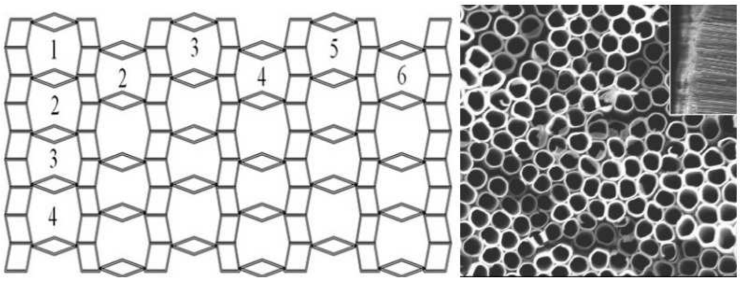

FIGURE 1. The molecular graph of $TiO_2[n,m]$ nanotube.

TABLE 1. The vertex partition of $TiO_2[n,m]$ nanotubes.

Vertex partition	v_2	v_3	v_4	v_5
$Cardinality$	$2mn+4n$	$2mn$	$2n$	$2mn$

TABLE 2. The edge partition of $TiO_2[n,m]$ nanotubes.

Edge partition	$E_6=E_8^*$	$E_7=E_{10}^*\cup E_{12}^*$	$E_8=E_{15}^*$	E_{12}^*	E_{10}^*
$Cardinality$	$6n$	$4mn+4n$	$6mn-2n$	$4mn+2n$	$2n$

and since $M_1(TiO_2[n,m]) = 76mn + 48n$, given in [9]. $HM(TiO_2) = 580mn + 284n$, given in [12]. The partitions of the vertex set and edge set $V(TiO_2), E(TiO_2)$, of $TiO_2[n,m]$ nanotubes are given in Table 1. and Table 2., respectively. We have

$$
\begin{aligned}
&HM(\overline{TiO_2[n,m]})\\
&= 2\sum|V(TiO_2[n,m])|(\sum|V(TiO_2[n,m])|-1)^3\\
&-12\bigcup E(TiO_2[n,m](\sum|V(TiO_2[n,m])|-1)^2+4\bigcup E(TiO_2[n,m]^2\\
&+(5\sum|V(TiO_2[n,m])|-6)M_1(TiO_2[n,m])-HM(TiO_2[n,m])\\
&= 2(6mn+6n)(6mn+6n-1)^3-12[|E_8^*|+|E_{10}^*\cup E_{12}^*|\\
&+|E_{15}^*|](6mn+6n-1)^2+4[|E_8^*|+|E_{10}^*\cup E_{12}^*|+|E_{15}^*|]^2\\
&+(5(6mn+6n)|-6)(76mn+48n)-580mn-284n\\
&= 12n(6mn+6n-1)^2[6m^2n+12mn+6n-11m-9]\\
&+2680m^2n^2+4360mn^2+1696n^2-1036mn-572n.
\end{aligned}
$$

□

Corollary 4.2. *Let $T = T[p,q]$ be the molecular graph of a nanotorus such that $|V(T)| = pq$, $|E(T)| = \frac{3}{2}pq$, Fig. 2. Then:*

a. $HM(\overline{T[p,q]}) = pq[(pq-1)^2[2pq-20] + 54pq - 108]$.

b.

$$HM(\overline{P_n \times T}) = pq[(npq-1)^2[2n^2pq - 32n + 12] + 150pqn^2 - 110pqn + 4pq - 400n + 294.$$

Proof. To proof (a), by using Theorem 2.1 we have

$$HM(\overline{T[p,q]}) = 2|V(T[p,q])|(|V(T[p,q]) - 1)^3 - 12|E(T[p,q])(|V(T[p,q]) - 1)^2 + 4|E(T[p,q])^2 + (5|V(T[p,q]) - 6)M_1(T[p,q]) - HM(T[p,q]).$$

And since $HM(T[p,q]) = 54pq$ by [8]. $M_1(T) = 9pq$ by [10]. Then

$$HM(\overline{T[p,q]}) = pq[(pq-1)^2[2pq-20] + 54pq - 108].$$

To proof (b), by [8]. $HM(P_n \times T) = 250npq - 186pq$, $M_1(T) = 9pq$. $M_1(P_n \times T) = pq(25n - 18)$, and by using Lemma 2.1 $|E(P_n \times T)| = (n-1)pq + \frac{3}{2}npq = pq(\frac{5}{2}n - 1)$, $|V(P_n \times T)| = npq$ and by using Theorem 2.1 we get

$$HM(\overline{P_n \times T}) = 2|V(P_n \times T)|(|V(P_n \times T) - 1)^3 - 12|E(P_n \times T)(|V(P_n \times T) - 1)^2 + 4|E(P_n \times T)^2 + (5|V(P_n \times T) - 6)M_1(P_n \times T) - HM(P_n \times T)$$
$$= pq[(npq-1)^2[2n^2pq - 32n + 12] + 150pqn^2 - 110pqn + 4pq - 400n + 294.$$

□

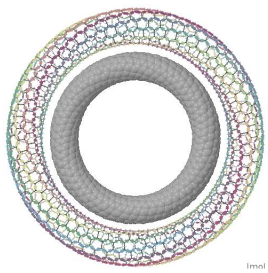

FIGURE 2. Molecular graph of a nanotorus

5. Conclusion

The present study has investigated some of the basic mathematical properties of the Hyper-Zagreb index of complement graphs and obtained explicit formula for their values under several graph operations. and we have studied the Hyper-Zagreb index of molecular complement graph of nanotorus and titania nanotubes $TiO_2[n, m]$.

References

[1] A. Alameria, N. Al-Naggara, M. Alsharafi: *Y-index of some graph operations*, International Journal of Applied Engineering Research (IJAER), **15**(2) (2020), 173-179.

[2] A. Behmaram, H. Yousefi-Azari, A. Ashrafi: *Some New Results on Distance-Based Polynomials*, Commun. Math. Comput. Chem., **65** (2011), 39-50.

[3] B. Basavanagoud, S. Patil: *A Note on Hyper-Zagreb Index of Graph Operations*, Iranian Journal of Mathematical Chemistry, **7**(1) (2016), 89 - 92.

[4] B. Furtula, I. Gutman: *A forgotten topological index*, J. Math. Chem., **53**(4) (2015), 1184-1190.

[5] G. H. Shirdel, H. Rezapour, A. M. Sayadi: *The hyper-Zagreb index of graph operations*, Iranian J. Math. Chem., **4**(2) (2013), 213-220.

[6] I. Gutman: *On Hyper-Zagreb index and coindex,* Bulletin T. CL de Academie serbe des sciences et des arts, **42** (2017), 1-8.

[7] K. Kiruthika: *Zagreb indices and Zagreb coindices of some graph operations*, International Journal of Advanced Research in Engineering and Technology, **7**(3) (2016), 25-41.

[8] M. Al-Sharafi, M. Shubatah: *On the Hyper-Zagreb index of some Graph Binary Operations*, Asian Research Journal of Mathematics, **16**(4) (2020), 12-24.

[9] M. Ali Malik, M. Imran: *On Multiple Zagreb Indices of TiO_2 Nanotubes*, Acta Chim. Slov., **62** (2015), 973-976.

[10] M. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi: *The first and second Zagreb indices of some graph operations*, Discrete applied mathematics, **157**(4) (2009), 804-811.

[11] M. Veylaki, M. J. Nikmehr, H. A.Tavallaee: *The third and hyper-Zagreb coindices of some graph operations*, J. Appl. Math. Comput., **50** (2016), 315-325.

[12] N. De: *On Molecular Topological Properties of TiO_2 Nanotubes*, Hindawi Publishing Corporation Journal of Nanoscience, (2016), Article ID 1028031, 5 pages.

DEPARTMENT OF MATHEMATICS,UNIVERSITY OF SHEBA REGION-YEMEN
E-mail address: alsharafi205010@gmail.com

DEPARTMENT OF STUDIES IN MATHEMATICS,UNIVERSITY OF AL-BAIDA-YEMEN
E-mail address: mahioub70@yahoo.com

DEPARTMENT OF BME,UNIVERSITY OF SCIENCE AND TECHNOLOGY-YEMEN
E-mail address: a.alameri2222@gmail.com