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ON SOME FUNCTIONS OF FAST INCREASE

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ABSTRACT. This article looks at some theorems of functions which satisfy the condition $\lim\limits_{\nu\to\infty}\dfrac{\ln\nu}{\ln\phi(\nu)} = 0$. This function is labelled function of fast increase. To show the applicability of such functions, some general results on a sequence s_n of positive integers that satisfy the asymptomatic rule $s_n \sim n^r \ln\phi(n)$ where $\phi(\nu)$ is a function of fast increase are derived.

1. INTRODUCTION

Drawing inspiration from functions of slow increase in [1, 2] the function of fast increase is defined as follows.

Definition 1.1. *Let $\phi(\nu)$ be a function from $[a,\infty)$ (where $a > 0$) to $(0,\infty)$ and $\phi(\nu) > 0$ with continuous derivative $\phi'(\nu) > 0$ and $\lim\limits_{\nu\to\infty}\phi(\nu) = \infty$. The function $\phi(\nu)$ is said to be function of fast increase if $\lim_{\nu\to\infty}\frac{\ln\nu}{\ln\phi(\nu)} = 0$. i.e., to every $\sigma > 0\ \exists\ k_\sigma \ni \nu > k_\sigma$ and $\left|\frac{\ln\nu}{\ln\phi(\nu)}\right| < \sigma$*

$$\Leftrightarrow |\ln\nu| < \sigma\,|\ln\phi(\nu)|\,,\ \forall v > k_\sigma$$

$$\Leftrightarrow e^{\ln\nu} < e^{\sigma\ln\phi(\nu)},\ \ \forall\nu > k_\sigma, \text{where } (\ln\nu,\ \ln\phi(\nu) > 0).$$

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Some functions of fast increase are $\phi(\nu) = e^{\nu}$, $\phi(\nu) = e^{e^{\nu}}$, $\phi(\nu) = a^{\nu}$ $(a \geq 2)$, and $\phi(\nu) = \dfrac{\Gamma(\nu)}{\Gamma'(\nu)}$ where $\Gamma(\nu) = \displaystyle\int_0^{\infty} t^{\nu-1} e^{-t} dt$ etc.

Note. 1) If $\phi(\nu)$ is function of fast increase, then $\lim_{\nu\to\infty} \frac{\phi(\nu)}{\nu\phi'(v)} = 0$.

2) Write $F = \{\phi | \phi \text{ is f.f.i}\}$.

Theorem 1.1. *Let $\phi, \psi \in F$ and let $\alpha, d > 0$ be two constants, then $\phi + d$, $\phi - d$, $d\phi$, $\phi\psi$, ϕ^{α}, $\phi \circ \psi$, e^{ϕ} and $\phi + \psi$ all lies in F.*

The proof is immediate consequence from the definition.

Theorem 1.2. *Let $\phi, \psi \in F$ and $\mu(\nu) = \phi(\nu^{\alpha})$ and $\eta(\nu) = \phi(\nu^{\alpha}\psi(\nu))$ for each v and $\alpha > 1$, then $\mu, \eta \in F$.*

Proof. (i) As μ satisfies the conditions of a function of fast increase, we have

$$\lim_{\nu\to\infty} \frac{\mu(\nu)}{\nu\mu'(\nu)} = \lim_{\nu\to\infty} \frac{\phi(\nu^{\alpha})}{\nu\alpha\nu^{\alpha-1}\phi'(\nu^{\alpha})}$$

$$= \lim_{\nu\to\infty} \frac{\phi(\nu^{\alpha})}{\nu^{\alpha}\phi'(\nu^{\alpha})} = 0 \quad (\text{since } \nu \to \infty \text{ then } \nu^{\alpha} \to \infty)$$

Therefore $\mu = \phi(\nu^{\alpha}) \in F$.

(ii) As μ satisfies the conditions of a function of fast increase, we have

$$0 \leq \lim_{\nu\to\infty} \frac{\eta(\nu)}{\nu\eta'(\nu)} = \lim_{\nu\to\infty} \frac{\phi(\nu^{\alpha}\psi(\nu))}{\nu\phi'(\nu^{\alpha}\psi(\nu))\left[\alpha\nu^{\alpha-1}\psi(\nu) + \nu^{\alpha}\psi'(\nu)\right]}$$

$$\leq \lim_{\nu\to\infty} \frac{\phi(\nu^{\alpha}\psi(\nu))}{\nu^{\alpha}\psi(\nu)\phi'(\nu^{\alpha}\psi(\nu))} = 0 \ (\text{since } \nu \to \infty \text{ and } \psi(\nu) \to \infty \text{ then } \nu^{\alpha}\psi(\nu) \to \infty)$$

Therefore $\eta = \phi(\nu^{\alpha}\psi(\nu)) \in F$. □

Theorem 1.3. *Let $\phi, \psi \in F$ such that $\lim_{\nu\to\infty} \dfrac{\phi(\nu)}{\psi(\nu)} = \infty$ and if $\dfrac{d}{d\nu}\left(\ln \dfrac{\phi(\nu)}{\psi(\nu)}\right) > 0$, then $\dfrac{\phi}{\psi} \in F$.*

Proof. Let $\mu = \dfrac{\phi(\nu)}{\psi(\nu)}$ where $\phi, \psi \in F$ and $\mu' = \dfrac{\phi'\psi - \phi\psi'}{\psi^2}$. Then

$$\lim_{\nu\to\infty} \frac{\mu}{\nu\mu'} = \lim_{\nu\to\infty} \frac{\frac{\phi}{\psi} \times \psi^2}{\nu(\phi'\psi - \phi\psi')} = \lim_{\nu\to\infty} \frac{1}{\nu\left(\frac{\phi'\psi - \phi\psi'}{\phi\psi}\right)} = 0$$

$$\left(\frac{d}{dx}\left(1\text{n}\frac{\phi(\nu)}{\psi(\nu)}\right) = \frac{\phi'\psi - \phi\psi'}{\phi\psi} > 0\right).$$

Therefore $\quad \mu = \dfrac{\phi}{\psi} \in F.$ □

Theorem 1.4. *Let* $\phi(\nu)$ *be a function from* $[a,\infty)\,(a>0)$ *to* $(0,\infty)$ *such that* $\phi(\nu)>0$ *with continuous derivative* $\phi'(\nu)>0$ *and* $\lim_{\nu\to\infty}\phi(\nu)=\infty$.

(i) *Define* $\mu(\nu)=\phi(\ln\nu)$, *then* $\mu\in F \Leftrightarrow \lim_{\nu\to\infty}\frac{\phi(\nu)}{\phi'(\nu)}=0$

(ii) *Define* $\eta(\nu)=e^{\phi(\nu)}$, *then* $\eta\in F \Leftrightarrow \lim_{\nu\to\infty}\frac{1}{\nu\phi'(\nu)}=0\,.$

Proof. (i) Suppose $\mu\in F \Rightarrow \lim_{\nu\to\infty}\frac{\mu(\nu)}{\nu\mu'(\nu)}=0 \Rightarrow \lim_{\nu\to\infty}\frac{\phi(1n\nu)}{\phi'(1n\nu)}=0$. If $y=\ln\nu$ and $\nu\to\infty$ then $y\to\infty$ then $\lim_{y\to\infty}\frac{\phi(y)}{\phi'(y)}=0$ or $\lim_{\nu\to\infty}\frac{\phi(\nu)}{\phi'(\nu)}=0\,.$

Converse follows from the above proof.

(ii) Suppose $\eta\in F \Rightarrow \lim_{\nu\to\infty}\frac{\eta(\nu)}{\nu\eta'(\nu)}=0 \Rightarrow \lim_{\nu\to\infty}\frac{1}{\nu\phi'(\nu)}=0\,.$

Converse follows from the above proof. □

Theorem 1.5. *Let* ϕ *be the function of fast increase if and only if to every* $\alpha>0$ *then there exist* ν_α *such that* $\dfrac{d}{d\nu}\left[\dfrac{\phi(\nu)}{\nu^\alpha}\right]>0,\ \forall\nu>\nu_\alpha\,.$

Proof. We have

$$\frac{d}{d\nu}\left[\frac{\phi(\nu)}{\nu^\alpha}\right]=\frac{\alpha\phi'(\nu)}{\nu^\alpha}\left[\frac{1}{\alpha}-\frac{\phi(\nu)}{\nu\phi'(\nu)}\right]. \tag{1.1}$$

Suppose $\phi\in F$. Then $\lim\limits_{\nu\to\infty}\dfrac{\phi(\nu)}{\nu\phi'(\nu)}=0\,$, *i.e.* for every $\alpha>0\ \exists\nu_\alpha\ \ni\ \forall\,\nu>\nu_\alpha$ and $\left|\dfrac{\phi}{\nu\phi}\right|<\dfrac{1}{\alpha},\quad \forall\,\nu>\nu_\alpha$

$$\Rightarrow \frac{1}{\alpha}-\frac{\phi}{\nu\phi}>0,\quad \forall\,\nu>\nu_\alpha$$

$$\Rightarrow \frac{d}{d\nu}\left[\frac{\phi(\nu)}{\nu^\alpha}\right]>0,\ \forall\nu>\nu_\alpha\,,$$

from (1.1). Converse follows from the above proof. □

Theorem 1.6. *If* $\phi\in F$ *then* $\lim\limits_{\nu\to\infty}\dfrac{\phi(\nu)}{\nu^\beta}=\infty,\ \forall\beta>0\,.$

Proof follows from the definition.

Note. We know that every $\phi\in F$ is an increasing function. Moreover by the above theorem it is clear that $\lim\limits_{\nu\to\infty}\dfrac{\phi(\nu)}{\nu^\beta}=\infty,\ \forall\beta>0\,.$

This shows that the increasing nature of ϕ is fast.

Corollary 1.1. *If* $\phi \in F$ *then* $\lim_{\nu\to\infty} \dfrac{\phi(\nu)}{\nu} = \infty$ *and* $\lim_{\nu\to\infty} \phi'(\nu) = \infty$.

The proof is immediate consequence of Theorem 1.6.

Theorem 1.7. *If* $\phi \in F$, *for any then* $\alpha \geq -1$ *and* $\beta \in R^+$, *the series* $\sum_{j=1}^{\infty} j^\alpha \phi(j)^\beta$ *diverges to* ∞.

Proof. Write $\sum_{j=1}^{\infty} j^\alpha \phi(j)^\beta = \sum_{j=1}^{\infty} \left[j^{\alpha+1}\phi(j)^\beta\right] \frac{1}{j}$. We know that $\sum_{j=1}^{\infty} \frac{1}{j}$ diverges to ∞ .
Given $\alpha \geq -1 \Rightarrow \alpha + 1 \geq 0$ and $\beta \in R^+$, $\Rightarrow \lim_{j\to\infty} j^{\alpha+1}\phi(j)^\beta = \infty$.

Therefore $\sum_{j=1}^{\infty} j^\alpha \phi(j)^\beta$ diverges to ∞ . □

Theorem 1.8. *Let* $\phi \in F$, *for any* $\alpha \geq -1$ *and* $\beta \in R^+$ *Then*

$$\lim_{v\to\infty} \frac{\int_a^v x^\alpha \phi(x)^{\beta-1}\phi'(x)dx}{\frac{1}{\beta} v^\alpha \phi(\nu)^\beta} = 1.$$

Proof. From Theorem 1.7., $\lim_{v\to\infty} \frac{1}{\beta} v^\alpha \phi(v)^\beta = \infty,\ \alpha \geq -1,\ \beta \in R^+$ and

$$\sum_{v=1}^{\infty} v^\alpha \phi(\nu)^\beta = \infty$$

$$\Rightarrow \lim_{\nu\to\infty} \int_a^\nu x^\alpha \phi(x)^{\beta-1}\phi'(x)dx = \infty$$

$$\Rightarrow \lim_{v\to\infty} \frac{\int_a^v x^\alpha \phi(x)^{\beta-1}\phi'(x)dx}{\frac{1}{\beta} v^\alpha \phi(v)^\beta} = \lim_{v\to\infty} \frac{v^\alpha \phi(v)^{\beta-1}\phi'(\nu)}{\nu^\alpha \phi(\nu)^{\beta-1}\phi'(\nu)\left[\frac{\alpha}{\beta}\frac{\phi(\nu)}{\nu\phi'(\nu)} + 1\right]} = 1\,,$$

by L'Hospital's Rule and $\phi \in F$. □

Corollary 1.2. *If* $\phi \in F$, *then the following results holds.*

(i) $\int_a^v x^\alpha \phi'(x)dx \sim \nu^\alpha \phi(\nu)$.

(ii) $\int_a^v \frac{\phi'(x)}{x}dx \sim \frac{\phi(\nu)}{\nu}$.

Proof is a particular case of Theorem 1.8.

Theorem 1.9. *If* $\phi \in F$ *and* $\lim_{\nu\to\infty} \frac{\phi'(\nu)}{\phi(\nu)} = \mathrm{M}$ *then* $\lim_{\nu\to\infty} \dfrac{1n\phi(\nu+k)}{1n\phi(\nu)} = 1$ *for every* $k \in R$.

Proof. Let $\psi(\nu) = \ln\phi(\nu)$, Suppose $k > 0,\ \exists x \in (\nu, \nu + k)$ such that $\psi(\nu + k) - \psi(\nu) = (\nu + k - \nu)\,\psi'(x)$(By LMVT)

$$\Rightarrow \frac{\psi(\nu + k)}{\psi(\nu)} - 1 = \frac{k\psi'(x)}{\psi(\nu)}$$

$$\Rightarrow \lim_{\nu\to\infty} \frac{\psi(\nu + k)}{\psi(\nu)} - 1 = k \lim_{\nu\to\infty} \left(\frac{\phi'(\nu)}{\phi(\nu)} \times \frac{1}{1n\phi(\nu)} \right) = 0\,,$$

$$\Rightarrow \lim_{\nu\to\infty} \frac{1n\phi(\nu + k)}{1n\phi(\nu)} = 1\,.$$

Suppose $k < 0,\ \exists x \in (\nu + k, \nu)$ such that $\psi(\nu) - \psi(\nu + k) = (\nu - \nu - k)\,\psi'(x)$(By LMVT)

$$\Rightarrow \lim_{\nu\to\infty} \frac{\psi(\nu + k)}{\psi(\nu)} - 1 = -k \lim_{\nu\to\infty} \left(\frac{\phi'(\nu)}{\phi(\nu)} \times \frac{1}{1\text{n}\phi(\nu)} \right) = 0$$

$$\Rightarrow \lim_{\nu\to\infty} \frac{1n\phi(\nu + k)}{1n\phi(\nu)} = 1.$$

□

2. Applications Of Functions Of Fast Increasing To Some Sequences Of Integers

Definition 2.1. *Let $\phi \in F,\ (s_n)$ be a sequence of positive integers and is strict1y increasing such that $s_1 \geq 1$ and*

$$\lim_{n\to\infty} \frac{s_n}{n^r \ln\phi(n)} = 1 \text{ for some } r \geq 1 \tag{2.1}$$

$i.e.\quad s_n \sim n^r \ln\phi(n)$*, see reference [3].*

For example, (i) $(s_n) = n^2,\ \phi(\nu) = e^\nu$ and $r = 1$
(ii) $(s_n) = ne^n,\ \phi(\nu) = e^{e^\nu}andr = 1$

Definition 2.2. *Let (s_n) be the sequence defined as above and $\nu > 0$ then $f(\nu) = \sum_{s_k \leq \nu} 1$, i.e. the member of (s_n) and is not exceeding ν .*

Theorem 2.1. *Let (s_n) be the sequence satisfying and $\phi, \psi \in F$ then*

(i) $s_{n+1} \sim s_n$
(ii) $\lim_{n\to\infty} \frac{s_{n+1} - s_n}{s_n} = 0$
(iii) $\ln s_{n+1} \sim \ln s_n$

(iv) $\psi(s_{n+1}) \sim \psi(s_n)$

(v) $\lim\limits_{\nu\to\infty} \dfrac{f(\nu)}{\nu} = 0$.

Proof. The proofs are immediate consequence of (2.1) and Theorem 1.9. □

Theorem 2.2. *Let (s_n) be the sequence satisfying 2.1 and $\psi \in F$ and $p \geq 1$ then*

$$\psi(s_n) \sim p\,\psi(n) \Leftrightarrow \psi(f(\nu)) \sim \frac{1}{p}\,\psi(\nu)\ .$$

Proof. Let $\psi(f(\nu)) \sim \frac{1}{p}\,\psi(\nu)$

$$\Rightarrow \psi(f(s_n)) \sim \frac{1}{p}\,\psi(s_n) \Rightarrow \psi(s_n) \sim p\psi(n) \quad (f(s_n) = n)\ .$$

Conversely, let

$$\psi(s_n) \sim p\,\psi(n) \Rightarrow \lim_{n\to\infty} \frac{\psi(f(s_n))}{\frac{1}{p}\,\psi(s_n)} = 1 \qquad (\text{since} \quad f(s_n) = n)\,. \tag{2.2}$$

If $s_n \leq \nu \leq s_{n+1}$ then $\psi(f(s_n)) \leq \psi(f(\nu)) \leq \psi(f(s_{n+1}))$, and $\frac{1}{p}\psi(s_n) \leq \frac{1}{p}\psi(\nu) \leq \frac{1}{p}\psi(s_{n+1})$.

$$\Rightarrow \lim_{n\to\infty} \frac{\psi(f(s_n))}{\frac{1}{p}\psi(s_{n+1})} \leq \lim_{\nu\to\infty} \frac{\psi(f(\nu))}{\frac{1}{p}\psi(\nu)} \leq \lim_{n\to\infty} \frac{\psi(f(s_{n+1}))}{\frac{1}{p}\psi(s_n)}$$

$$\Rightarrow 1 \leq \frac{\psi(f(\nu))}{\frac{1}{p}\psi(\nu)} \leq 1 \qquad (s_{n+1} \sim s_n)$$

and from (2.2),

$$\Rightarrow \psi(f(\nu)) \sim \frac{1}{p}\,\psi(\nu)\ .$$

□

Lemma 2.1. *Let $\sum_{n=1}^{\infty} a_n$and $\sum_{n=1}^{\infty} b_n$ be series of positive terms and $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1$. If $\sum_{n=1}^{\infty} b_n$ is divergent then $\dfrac{\sum_{k=1}^{n} a_k}{\sum_{k=1}^{n} b_k} = 1$.*

Theorem 2.3. *Let (s_n) be the sequence satisfying (2.1) $\psi \in F$ space, $p \geq 1$, $\psi(s_n) \sim \psi(n)$ and $\psi'(s_n) \sim \psi'(n)$ then*

$$s_n \sim \frac{1}{n}\psi(n) \Leftrightarrow f(\nu) \sim \frac{\psi(\nu)}{\nu} \Leftrightarrow f(\nu) \sim \int_a^{\nu} \frac{\psi'(x)}{x}dx \Leftrightarrow \sum_{s_k\leq\nu} k\psi'(s_k) \sim \frac{\psi(\nu)^2}{\nu}\,.$$

Proof. Suppose $f(\nu) \sim \dfrac{\psi(\nu)}{\nu} \Rightarrow f(s_n) \sim \frac{\psi(s_n)}{s_n}$,

$$\Rightarrow n \sim \frac{\psi(s_n)}{s_n} \Rightarrow s_n \sim \frac{1}{n}\psi(n) \qquad (\text{Since } f(s_n) = n).$$

Conversely, suppose

$$s_n \sim \frac{1}{n}\psi(n) \Rightarrow \lim_{n\to\infty} \frac{f(s_n)}{\left(\frac{\psi(s_n)}{s_n}\right)} = 1.$$

If $s_n \le \nu \le s_{n+1}$ then $f(s_n) \le f(\nu) \le f(s_{n+1})$, and $\dfrac{\psi(s_n)}{s_n} \le \dfrac{\psi(\nu)}{\nu} \le \dfrac{\psi(s_{n+1})}{s_{n+1}}$.

$$\Rightarrow \lim_{n\to\infty} \frac{f(s_n)}{\left(\frac{\psi(s_{n+1})}{s_{n+1}}\right)} \le \lim_{\nu\to\infty} \frac{f(\nu)}{\left(\frac{\psi(\nu)}{\nu}\right)} \le \lim_{n\to\infty} \frac{f(s_{n+1})}{\left(\frac{\psi(s_n)}{s_n}\right)}$$

$$\Rightarrow 1 \le \lim_{\nu\to\infty} \frac{f(\nu)}{\left(\frac{\psi(\nu)}{\nu}\right)} \le 1\,(s_{n+1} \sim s_n) \Rightarrow f(\nu) \sim \frac{\psi(\nu)}{\nu}.$$

We have

$$\int_a^v \frac{\psi'(x)}{x}dx \sim \frac{\psi(\nu)}{\nu} \Rightarrow f(\nu) \sim \int_a^\nu \frac{\psi'(x)}{x}dx.$$

Also we have $\int_a^v x\,\psi'(x)dx \sim \nu\,\psi(\nu)$and $\psi'(x)$ is increasing

$$\Rightarrow \sum_{k=1}^{n} k\psi'(k) = \int_a^\nu x\psi'(x)\,dx + h(n)$$

$$\Rightarrow \int_a^v x\,\psi'(x)\,dx + h(n) \sim \nu\,\psi(\nu).$$

From Lemma 2.1., we can write $\displaystyle\sum_{k=1}^{n} k\psi'(s_k) \sim \sum_{k=1}^{n} k\psi'(k)$

$$\Rightarrow \sum_{k=1}^{n} k\psi'(s_k) \sim n\psi(n)$$

$$\Rightarrow \sum_{s_k \le s_n} k\psi'(s_k) \sim n\psi(s_n)$$

$$\Rightarrow \sum_{s_k \le s_n} k\psi'(s_k) \sim f(s_n)\,\psi(s_n)$$

$$\Rightarrow \lim_{n\to\infty} \frac{f(s_n)}{\left(\frac{\sum_{s_k \le s_n} k\psi'(s_k)}{\phi(s_n)}\right)} = 1. \tag{2.3}$$

If $s_n \le \nu \le s_{n+1}$ then $f(s_n) \le f(\nu) \le f(s_{n+1})$, and

$$\Rightarrow \sum_{s_k \le s_n} k\psi'(s_k) \le \sum_{s_k \le x} k\psi'(s_k) \le \sum_{s_k \le s_{n+1}} k\psi'(s_k) ,$$

$$\Rightarrow \lim_{n\to\infty} \frac{f(s_n)}{\left(\frac{\sum_{s_k \le s_{n+1}} k\psi'(s_k)}{\psi(s_n)}\right)} \le \lim_{\nu\to\infty} \frac{f(\nu)}{\left(\frac{\sum_{s_k \le \nu} k\psi'(s_k)}{\psi(\nu)}\right)} \le \lim_{n\to\infty} \frac{f(s_{n+1})}{\left(\frac{\sum_{s_k \le s_n} k\psi'(s_k)}{\psi(s_{n+1})}\right)}$$

$\Rightarrow 1 \le \lim_{\nu\to\infty} \frac{f(\nu)}{\left(\frac{\sum_{s_k \le \nu} k\psi'(s_k)}{\psi(\nu)}\right)} \le 1$ (since $s_{n+1} \sim s_n$) and from (2.3)

$$\Rightarrow f(\nu) \sim \frac{\sum_{s_k \le \nu} k\psi'(s_k)}{\psi(\nu)}$$

$$\Rightarrow \frac{\psi(\nu)}{\nu} \sim \frac{\sum_{s_k \le \nu} k\psi'(s_k)}{\psi(\nu)}$$

$$\Rightarrow \sum_{s_k \le \nu} k\psi'(s_k) \sim \frac{\psi(\nu)^2}{\nu} .$$

□

Theorem 2.4. *Let* (s_n) *be the sequence satisfying (2.1)* , $\psi \in F$, $p \ge 1$, $l \ge 1$, $\psi(s_n) \sim l\psi(n)$ *and* $\psi'(s_n) \sim l\psi'(n)$ *then*

$$s_n \sim \frac{1}{n^p}\psi(n) \Leftrightarrow f(\nu) \sim \frac{\psi(\nu)^{1/p}}{l^{1/p}\nu^{1/p}} \Leftrightarrow f(\nu) \sim \frac{1}{pl^{1/p}} \int_a^v \frac{\psi(x)^{\frac{1}{p}-1}\psi'(x)}{x^{1/p}} dx$$

$$\Leftrightarrow \sum_{s_k \le \nu} k\psi'(s_k) \sim \frac{\psi(\nu)^{\frac{1}{p}+1}}{l^{1/p}\nu^{1/p}} .$$

Proof. Suppose $f(\nu) \sim \frac{\psi(\nu)^{1/p}}{l^{1/p}\nu^{1/p}} \Rightarrow f(s_n) \sim \frac{\psi(s_n)^{1/p}}{l^{1/p}{s_n}^{1/p}}$

$$\Rightarrow n \sim \frac{\psi(s_n)^{1/p}}{l^{1/p}{s_n}^{1/p}} \qquad (\text{Since } f(s_n) = n)$$

$$\Rightarrow s_n \sim \frac{1}{n^p}\psi(n) \quad (\text{Since } \psi(s_n) \sim l\psi(n)) .$$

Conversely, suppose $s_n \sim \frac{1}{n^p}\psi(n)$

$$\Rightarrow \lim_{n\to\infty} \frac{f(s_n)}{\left(\frac{\psi(s_n)^{1/p}}{l^{1/p}{s_n}^{1/p}}\right)} = 1 . \tag{2.4}$$

If $s_n \leq \nu \leq s_{n+1}$ then $f\left(s_n\right) \leq f\left(\nu\right) \leq f\left(s_{n+1}\right)$, and $\dfrac{\psi\left(s_n\right)}{s_n} \leq \dfrac{\psi\left(\nu\right)}{\nu} \leq \dfrac{\psi\left(s_{n+1}\right)}{s_{n+1}}$.

$$\Rightarrow \lim_{n\to\infty} \frac{f\left(s_n\right)}{\left(\frac{\psi(s_{n+1})^{1/p}}{l^{1/p}{s_{n+1}}^{1/p}}\right)} \leq \lim_{\nu\to\infty} \frac{f\left(\nu\right)}{\left(\frac{\psi(\nu)^{1/p}}{l^{1/p}\nu^{1/p}}\right)} \leq \lim_{n\to\infty} \frac{f\left(s_{n+1}\right)}{\left(\frac{\psi(s_n)^{1/p}}{l^{1/p}{s_n}^{1/p}}\right)}$$

$$\Rightarrow 1 \leq \lim_{\nu\to\infty} \frac{f\left(\nu\right)}{\left(\frac{\psi(\nu)^{1/p}}{l^{1/p}\nu^{1/p}}\right)} \leq 1 \text{ (Since } s_{n+1} \sim s_n)$$

and from (2.4)

$$\Rightarrow f\left(\nu\right) \sim \frac{\psi(\nu)^{1/p}}{l^{1/p}\nu^{1/p}} .$$

We have $\displaystyle\int_a^v x^\alpha \psi(x)^{\beta-1}\psi'(x)dx \sim \tfrac{1}{\beta} v^\alpha \psi(\nu)^\beta$.

By taking $\alpha = \frac{1}{p},\ \beta = -\frac{1}{p}$ then we get $\displaystyle f\left(\nu\right) \sim \frac{1}{pl^{1/p}} \int_a^v \frac{\psi(x)^{\frac{1}{p}-1}\psi'\left(x\right)}{x^{1/p}} dx$.

The proof that $\sum_{s_k\leq\nu} k\psi'\left(s_k\right) \sim \frac{\psi(\nu)^{\frac{1}{p}+1}}{l^{1/p}\nu^{1/p}}$ is the same as in Theorem 2.3. $\square$

3. Conclusions

The results discussed in this article are employed in examples to show their applicability in number theory.

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