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QUALITATIVE ANALYSIS OF DELAY DIFFERENTIAL EQUATIONS FROM MEDICINE

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ABSTRACT. The purpose of this article is to present the system of delay differential equations for understanding the normal and abnormal regulatory mechanisms of living systems. The results of a qualitative analysis show that the trivial equilibrium position is stable, non-trivial equilibria can loss own stability and there are oscillatory solutions.

1. INTRODUCTION

In recent years the theory of delay differential equations has rapidly developed [1] and its results have been applied in modeling of regulatory mechanisms of cancer grow [2-4], cardiac activity and liver function [5-6]. We offer the following system of delay differential equations for understanding the normal and abnormal regulatory mechanisms of living systems:

$$\frac{dX_1(t)}{dt} = aX_1(t-\rho)X_4(t-\rho)X_5(t-\rho)e^{-\sum\limits_{j=1}^{6}\delta_j X_j(t-\rho)} + a_2X_2(t-\rho) - a_1X_1(t);$$

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$$\begin{aligned} \frac{dX_2(t)}{dt} &= a_1X_1(t) - (a_2+a_3)X_2(t); \\ \frac{dX_3(t)}{dt} &= a_3X_2(t) - (a_4+a_5)X_3(t); \\ \frac{dX_4(t)}{dt} &= a_4X_3(t) - a_6X_4(t); \\ \frac{dX_5(t)}{dt} &= a_5X_3(t) - a_6X_5(t); \\ \frac{dX_6(t)}{dt} &= a_6\left(X_4(t)+X_5(t)\right) - a_7X_6(t). \end{aligned} \tag{1.1}$$

The equations of this class make it possible to take into account the spatial separation of processes, have oscillatory solutions and are used in modeling populations, cellular processes, immune systems and the functioning of the organism as a whole [7].

2. The equilibrium positions

Let us consider the equilibrium positions $(\xi_1, \xi_2, ..., \xi_6)$ of system of equations (1.1). We have

$$\begin{aligned} a\xi_1\xi_4\xi_5 e^{-\sum\limits_{j=1}^{6}\delta_j\xi_j} + a_2\xi_2 - a_1\xi_1 &= 0; \\ a_1\xi_1 - (a_2+a_3)\xi_2 &= 0; \\ a_3\xi_2 - (a_4+a_5)\xi_3 &= 0; \\ a_4\xi_3 - a_6\xi_4 &= 0; \\ a_5\xi_3 - a_6\xi_5 &= 0; \\ a_6(\xi_4+\xi_5) - a_7\xi_6 &= 0, \end{aligned}$$

which show what ξ_1 can be determined from

$$P\xi_1^3 e^{-\delta\xi_1} = \alpha\xi_1,$$

where

$$P = \frac{a\,a_1^2\,a_3^2\,a_4\,a_5}{a_6^2\,(a_2+a_3)^2(a_4+a_5)^2};$$
$$\delta = \delta_1 + \frac{a_1}{a_2+a_3}\left(\delta_2 + \frac{a_3}{a_4+a_5}\left(\delta_3 + \frac{a_4}{a_6}\delta_4 + \frac{a_5}{a_6}\delta_5\right) + \frac{a_3}{a_7}\delta_6\right); \quad \alpha = a_1\left(1 - \frac{a_2}{a_2+a_3}\right)$$

and coordinates of the equilibrium position can be calculated based on the formulas:

$$\xi_2 = \frac{a_1}{a_2 + a_3}\xi_1;$$
$$\xi_3 = \frac{a_1 a_3}{(a_2 + a_3)(a_4 + a_5)}\xi_1;$$
$$\xi_4 = \frac{a_1 a_3 a_4}{a_6(a_2 + a_3)(a_4 + a_5)}\xi_1;$$
$$\xi_5 = \frac{a_1 a_3 a_5}{a_6(a_2 + a_3)(a_4 + a_5)}\xi_1;$$
$$\xi_6 = \frac{a_1 a_3}{a_7(a_2 + a_3)}\xi_1 .$$

From here we can see the existence of a trivial equilibrium position (for positive values of the coefficients) and $\alpha > 0,\ \rho > 0$,which shows the possibility of the existence of a nontrivial equilibrium position, the first coordinate of which is determined from the equation

$$P\xi_1^2 e^{-\delta\xi_1} = \alpha. \tag{2.1}$$

Introducing the notation $Z = \delta\xi_1$, we write (2.1) in the form

$$Z^2 e^{-Z} = \mu, \tag{2.2}$$

where

$$\mu = \frac{\alpha\delta^2}{P}.$$

It can be seen from Figure 1 that for the existence of a nontrivial equilibrium position it is necessary that M > μ. Define the maximum value of M function

$$Y(Z) = Z^2 e^{-Z}.$$

We obtain

$$Y_Z' = (2Z - Z^2)e^{-Z} = Z(2 - Z)e^{-Z},$$

which shows that $M = Y(2) = 4e^{-2}$.

Therefore, for $P \geq \frac{\alpha(\delta e)^2}{4}$ there is a nontrivial equilibrium position of system (1.1) and

$$P^* = \frac{\alpha(\delta e)^2}{4}\,,$$

is a bifurcation point. We can build a bifurcation diagram (see Figure 2).

Thus, for system (1.1) if $P < P^*$, there is only a trivial equilibrium position, which split (undergoes bifurcation) into trivial and non-trivial equilibrium positions $\xi = \xi^*$ at $P = P^*$. Further growth of P leads to a splitting of the nontrivial equilibrium position into two, one of which evolves to the origin, and the other

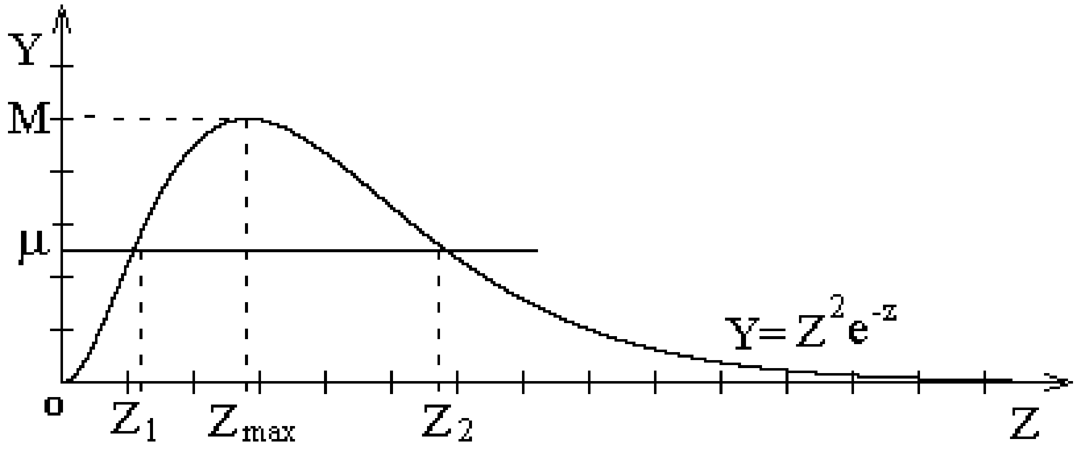

FIGURE 1. The existence of a nontrivial equilibrium positions (2.2).

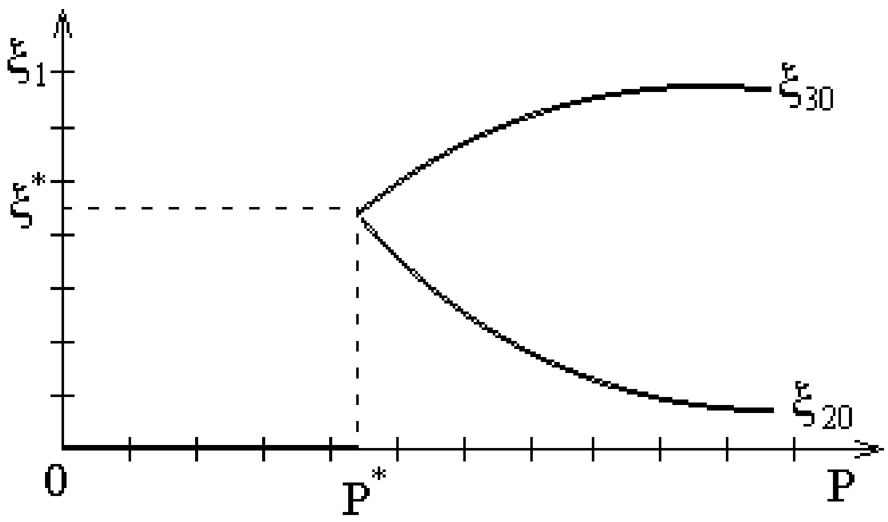

FIGURE 2. Bifurcation diagram for equilibrium positions (1.1).

to infinity. Therefore, if $P > P^{*}$the considered biosystem has three stationary modes of functioning: rest mode, weak mode and normal mode. These regularities of a general nature, determined using numerical integration of the system's functioning on computer, showed the stability of the resting position and the existence of one stable functioning mode for a wide range of coefficient values.

Let us consider the behavior of solutions near equilibrium positions. Let be $\xi(\xi_1, \xi_2, ..., \xi_6)$ the equilibrium position. We have

$X_i(t) = \xi_i + Y_i(t)$, $i = 1, 2, ..., 6$,

where $Y_i(t)$ are very small, we can linearize (1.1) around the considered equilibrium position. We obtain

$$\begin{aligned}
\frac{dY_1(t)}{dt} &= -\sum_{j=1}^{6} \gamma_j Y_j(t-\rho) + a_2 Y_2(t) - a_1 Y_1(t);\\
\frac{dY_2}{dt} &= a_1 Y_1(t) - (a_2 + a_3) Y_2(t);\\
\frac{dY_3(t)}{dt} &= a_3 Y_2(t) - (a_4 + a_5) Y_3(t);\\
\frac{dY_4(t)}{dt} &= a_4 Y_3(t) - a_6 Y_4(t);\\
\frac{dY_5(t)}{dt} &= a_5 Y_3(t) - a_6 Y_5(t);\\
\frac{dY_6(t)}{dt} &= a_6 (Y_4(t) + Y_5(t)) - a_7 Y_6(t),
\end{aligned}$$

where

$$\begin{aligned}
\gamma_1 &= \delta \xi_4 \xi_5 (\delta_1 \xi_1 - 1);\\
\gamma_2 &= \delta \delta_2 \xi_1 \xi_4 \xi_5;\\
\gamma_3 &= \delta \delta_3 \xi_1 \xi_4 \xi_5;\\
\gamma_4 &= \delta \xi_1 \xi_5 (\delta_4 \xi_4 - 1);\\
\gamma_5 &= \delta \xi_1 \xi_4 (\delta_5 \xi_5 - 1);\\
\gamma_6 &= \delta \delta_6 \xi_1 \xi_4 \xi_5;
\end{aligned}$$

$$\delta = a e^{-\sum_{i=1}^{6} \delta_i \xi_i}.$$

Calculation of the determinant shows that there is a root $\lambda_1 = -a_6$.

For other roots, we have

$$(\lambda + \gamma_1 e^{-\rho\lambda} + a_1)(\lambda + a_1 + a_2)(\lambda + a_4 + a_5)(\lambda + a_6)(\lambda + a_7) - a_4(a_2 - \gamma_2 e^{-\rho\lambda})\cdot$$
$$\cdot(\lambda + a_4 + a_5)(\lambda + a_6)(\lambda + a_7) + a_3\gamma_3 e^{-\rho\lambda}(\lambda + a_6)(\lambda + a_7) +$$
$$+ a_4 a_3 \gamma_4 e^{-\rho\lambda}(\lambda + a_7) + a_6 a_4 a_3 \gamma_6 e^{-\rho\lambda} = 0.$$

To analyze the roots of this equation, we introduce the notation $Z = \rho\lambda$ and we obtain:

$$((Z + a_1\rho)e^Z + \gamma_1)(Z + (a_1 + a_2)\rho)(Z + (a_4 + a_5)\rho)(Z + a_6\rho)(Z + a_7\rho) -$$
$$- a_1\rho^2(a_2 e^Z - \gamma_2)(Z + (a_4 + a_5)\rho)(Z + a_6\rho)(Z + a_7\rho) + a_3\gamma_3\rho^3(Z + a_6\rho)\cdot$$
$$\cdot(Z + a_7\rho) + a_4 a_3 \gamma_4 \rho^4 (Z + a_7\rho) + a_6 a_4 a_3 \gamma_6 \rho^5 = 0.$$

As an example, we consider the trivial equilibrium position. We have

$$e^Z(Z+\rho+a_1)(Z+\rho(a_1+a_2))(Z+\rho(a_4+a_5))(Z+\rho a_6)(Z+\rho a_7))-$$
$$-a_1a_2\rho^2e^Z(Z+\rho(a_4+a_5))(Z+\rho a_6)(Z+\rho a_7)=0.$$

There are the following roots:

$$Z_1=-\rho(a_4+a_5); \qquad Z_2=-\rho a_{6;} \qquad Z_3=-\rho a_7\,,$$

and for other roots we have

$$(Z+\rho a_1)(Z+\rho(a_1+a_2))-a_1a_2\rho^2=0.$$

An analysis of the solutions of this equation shows that it has two negative roots, since it reduces to

$$Z^2+\rho(2a_1+a_2)-\rho^2a_1^2=0$$

and

$$Z_{1,2}=\rho\frac{-2a_1-a_2\pm\sqrt{(2a_1+a_2)^2-4a_1^2}}{2}=\rho\frac{-2a_1-a_2\pm\sqrt{4a_1a_2+a_2^2}}{2}.$$

Therefore, there are only negative roots, which implies the stability of the trivial equilibrium position.

The results of a qualitative analysis of the basic equations of the dynamics of living systems show the presence of trivial and two nontrivial equilibrium positions. The trivial equilibrium position is always stable, the equilibrium position (X_{20}) close to the origin (0) is always unstable, the third equilibrium position is stable when the delay time is neglected. Taking into account the time delay leads to the loss of stability of the third equilibrium position for a certain range of parameters values of the equation (1.1). Violation of the stability of the equilibrium position leads to Hopf bifurcation, to the existence of oscillations around X_{30}.

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