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THE UPPER RESTRAINED GEODETIC DOMINATION NUMBER OF A GRAPH

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ABSTRACT. A set S of vertices of a connected graph G is a restrained geodetic dominating set, if either $S = V$ or S is a geodetic dominating set with the subgraph $G[V - S]$ induced by $V - S$ has no isolated vertices. The minimum cardinality of a restrained geodetic dominating set of G is called the restrained geodetic domination number and is denoted by $\gamma_{gr}(G)$. A restrained geodetic dominating set S in a connected graph G is called a minimal restrained geodetic dominating set of G if no proper subset of S is a restrained geodetic dominating set of G. The upper restrained geodetic domination number $\gamma_{gr}^{+}(G)$ is the maximum cardinality of a minimal restrained geodetic dominating set of G. The upper restrained geodetic domination number of certain classes of graphs are determined. It is shown that for every pair of integers a, b with $3 \leq a \leq b$, there exists a connected graph G of order b such that $\gamma_{gr}^{+}(G) = a$. Also, for any four integers a, b, c and d with $2 \leq a \leq b \leq c \leq d \leq p$, there exists a connected graph G of order p such that $g(G) = a$, $\gamma_g(G) = b$, $\gamma_{gr}(G) = c$ and $\gamma_{gr}^{+}(G) = d$.

1. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology, we refer [3, 5]. The neighborhood

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of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The closed neighborhood of a vertex v is the set $N[v] = N(v) \cup \{v\}$. A vertex v is an extreme vertex if the subgraph induced by its neighbors is complete. A vertex v is a semi-extreme vertex of G if the subgraph induced by its neighbors has a full degree vertex in $N(v)$. In Particular, every extreme vertex is a semi-extreme vertex and a semi-extreme vertex need not be an extreme vertex refer [7]. For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic. A geodetic set of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices of S. The geodetic number $g(G)$ of G is the minimum cardinality of its geodetic sets. The geodetic number of a graph was introduced in [2, 6].

A dominating set in a graph G is a subset of vertices of G such that every vertex outside the subset has neighbor in it. The size of a minimum dominating set in a graph G is called the domination number of G and is denoted by $\gamma(G)$. A geodetic dominating set of G is a subset of $V(G)$ which is both geodetic and dominating set of G. The minimum cardinality of a geodetic dominating set is a geodetic domination number and is denoted by $\gamma_g(G)$. The geodetic domination number of a graph was introduced in [4]. A set S of vertices of a connected graph G is a restrained geodetic dominating set, if either $S = V$ or S is a geodetic dominating set with the subgraph $G[V - S]$ induced by $V - S$ has no isolated vertices. The minimum cardinality of a restrained geodetic dominating set of G is called the restrained geodetic domination number and is denoted by $\gamma_{gr}(G)$, we refer [1].

The following Theorems will be used in sequel.

Theorem 1.1. *[1] Each extreme vertex of a connected graph G belongs to every restrained geodetic dominating set of G.*

Theorem 1.2. *[1] Every restrained geodetic dominating set of a connected graph G contains its semi-extreme vertex of G.*

Theorem 1.3. *[1] For the complete graph $G = K_p$ ($p \geq 2$), $\gamma_{gr}(K_p) = p$.*

Throughout the following G denotes a connected graph with at least two vertices.

2. The Upper Restrained Geodetic Domination Number of a Graph

Definition 2.1. *A restrained geodetic dominating set S in a connected graph G is called a minimal restrained geodetic dominating set if no proper subset of S is a restrained geodetic dominating set of G. The upper restrained geodetic domination number $\gamma_{gr}^{+}(G)$ is the maximum cardinality of a minimal restrained geodetic dominating set of G.*

Example 1. *For the graph G given in Figure 1, $S_1 = \{v_1, v_3, v_5\}$ is a minimal restrained geodetic dominating set of G so that $\gamma_{gr}(G) = 3$. The set $S_2 = \{v_1, v_2, v_4, v_5\}$ is a minimal restrained geodetic dominating set, so that $\gamma_{gr}^{+}(G) \geq 4$. It is easily verified that no five elements set of G is a restrained geodetic dominating set of G. Hence $\gamma_{gr}^{+}(G) = 4$.*

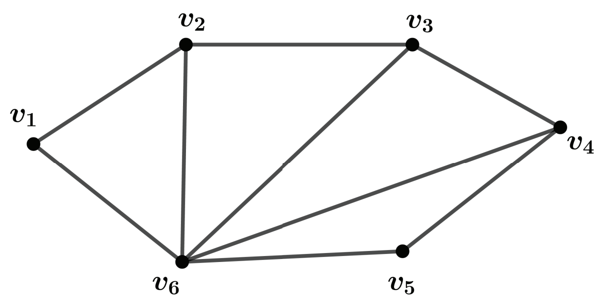

FIGURE 1

Theorem 2.1. *Each extreme vertex of a connected graph G belongs to every minimal restrained geodetic dominating set of G.*

Proof. This follows from Theorem 1.1. □

Theorem 2.2. *Each semi-extreme vertex of a connected graph G belongs to every minimal restrained geodetic dominating set of G.*

Proof. This follows from Theorem 1.2. □

Theorem 2.3. *Let G be a connected graph of order p. If G has a semi-extreme vertex of order p, then $\gamma_{gr}^{+}(G) = p$.*

Proof. Suppose G has semi-extreme vertex of order p. By Theorem 2.2, it belongs to every minimal restrained geodetic dominating set. The result follows. □

Theorem 2.4. *If G is a connected graph with extreme vertices and if the set S of all extreme vertices is a restrained geodetic dominating set of G, then $\gamma_{gr}(G) = \gamma^+_{gr}(G)$.*

Proof. Suppose that G is a graph with extreme vertices and the set of all extreme vertices forms a restrained geodetic dominating set. Since any minimal restrained geodetic dominating set contains all the extreme vertices, it follows that the minimum restrained geodetic dominating sets are nothing but the minimal restrained geodetic dominating sets. Hence $\gamma_{gr}(G) = \gamma^+_{gr}(G)$. □

Theorem 2.5. *Let G be a connected graph with cut-vertices and let S be a minimal restrained geodetic dominating set of G. If v is a cut-vertex of G, then every component of $G - v$ contains an element of S.*

Proof. Let v be a cut-vertex of G and S be a minimal restrained geodetic dominating set of G. Suppose there is a component G_1 of $G - v$ such that G_1 contains no vertices of S. By Theorem 2.1, G_1 contains at least one vertex, say u. Since S is a minimal restrained geodetic dominating set, there exist vertices $x, y \in S$ such that u lies on the $x - y$ geodesic path $P : x = u_0, u_1, \ldots, u, \ldots, u_t = y$ in G. Let P_1 be a $x - u$ subpath of P and P_2 be a $u - y$ subpath of P. Since v is a cut-vertex of G, both P_1 and P_2 contains v so that P is not a path, which is a contradiction. Thus, every component of $G - v$ contains an element of S. □

Theorem 2.6. *For any connected graph G, $3 \le \gamma_{gr}(G) \le \gamma^+_{gr}(G) \le p$.*

Proof. A restrained geodetic dominating set needs at least three vertices and so $\gamma_{gr}(G) \ge 3$. Since every minimal restrained geodetic dominating set is a restrained geodetic dominating set of G, $\gamma_{gr}(G) \le \gamma^+_{gr}(G)$. Also, since $V(G)$ is a restrained geodetic dominating set of G, it is clear that $\gamma^+_{gr}(G) \le p$. Thus $3 \le \gamma_{gr}(G) \le \gamma^+_{gr}(G) \le p$. □

Theorem 2.7. *For any connected graph G, $\gamma_{gr}(G) = p$ if and only if $\gamma^+_{gr}(G) = p$.*

Proof. Let $\gamma^+_{gr}(G) = p$. Then $S = V(G)$ is the unique minimal restrained geodetic dominating set of G. Since no proper subset of S is a restrained geodetic dominating set, it is clear that S is the minimum restrained geodetic dominating set of G and so $\gamma_{gr}(G) = p$. The converse follows from Theorem 2.1. □

Theorem 2.8. *For a complete graph K_p ($p \ge 2$), $\gamma^+_{gr}(K_p) = p$.*

Proof. This follows from Theorem 1.3 and Theorem 2.7. □

Theorem 2.9. *Let G be a connected graph of order p and $u \in V(G)$. If $deg(u) = 1$, then $\gamma_{gr}^+(G-u) \leq \gamma_{gr}^+(G)$.*

Proof. Let $u \in V(G)$ and $deg(u) = 1$. Let S be a minimal restrained geodetic dominating set of $G - u$ with maximum cardinality. So $\gamma_{gr}^+(G) = |S|$. Since $deg(u) = 1$, u is an end vertex and u is adjacent to exactly one vertex, say v. By Theorem 2.1, every minimal restrained geodetic dominating set of G contains u. We consider two cases.

Case (i). Let $v \in S$. Since S is a restrained geodetic dominating set of $G - u$, there exists a vertex $w \in V(G-u)$ such that $w \in I[v,x] \subseteq I[S]$, $w \in N(S)$ and $d(v,x) \leq 3$. If $d(v,x) = 3$, then consider the set $S_1 = S - \{v\} \cup \{u,w\}$. If $d(v,x) \leq 2$ then consider the set $S_2 = S - \{v\} \cup \{u\}$. It is straightforward to verify that S_1 is a minimal restrained geodetic dominating set of G so that $\gamma_{gr}^+(G-u) = |S| \leq |S_1| \leq \gamma_{gr}^+(G)$.

Case (ii). Let $v \notin S$. Then consider the set $S_1 = S \cup \{u\}$. It is straight forward to verify that S_1 is a minimal restrained geodetic dominating set of G so that $\gamma_{gr}^+(G-u) = |S| \leq |S_1| \leq \gamma_{gr}^+(G)$. Hence in both cases, $\gamma_{gr}^+(G-u) \leq \gamma_{gr}^+(G)$. ☐

Theorem 2.10. *For a connected graph G of order p, the following are equivalent.*

(i) $\gamma_{gr}^+(G) = p$;
(ii) $\gamma_{gr}(G) = p$;
(iii) $G = K_p$.

Proof. (i) $\Rightarrow$ (ii). Let $\gamma_{gr}^+(G) = p$. Then $S = V(G)$ is the unique minimal restrained geodetic dominating set of G. Since no proper subset of S is a restrained geodetic dominating set, it is clear that S is the unique minimum restrained geodetic dominating set of G and so $\gamma_{gr}(G) = p$.

(ii) $\Rightarrow$ (iii). Let $\gamma_{gr}(G) = p$. If $G \neq K_p$, then by Theorem 2.11, $\gamma_{gr}(G) \leq p-1$, which is a contradiction. Therefore $G = K_p$.

(iii) $\Rightarrow$ (i). Let $G = K_p$. Then by Theorem 2.7, $\gamma_{gr}^+(G) = p$. ☐

Theorem 2.11. *Let G be a connected graph of order p with $\gamma_{gr}(G) = p-2$. Then $\gamma_{gr}^+(G) = p-2$.*

Proof. Let $\gamma_{gr}(G) = p-2$. By Theorem 2.6, $\gamma_{gr}^+(G) \geq p-2$. Therefore $\gamma_{gr}^+(G)$ is either p or $p-2$. If $\gamma_{gr}^+(G) = p$ then by Theorem 2.8, $\gamma_{gr}(G) = p$ which is a contradiction. Therefore $\gamma_{gr}^+(G) = p-2$. ☐

Theorem 2.12. *For a connected graph G,*

$$2 \leq g(G) \leq \gamma_g(G) \leq \gamma_{gr}(G) \leq \gamma_{gr}^{+}(G) \leq p\,.$$

Proof. A geodetic set needs at least two vertices and therefore $g(G) \geq 2$. Also, every geodetic set is a geodetic dominating set of G and so $g(G) \leq \gamma_g(G)$. If $\gamma_g(G) = p$ or $p-1$ then $\gamma_{gr}(G) = p$. Also, every minimal restrained geodetic dominating set of G is a restrained geodetic dominating set of G, but the converse is not true and therefore $\gamma_{gr}(G) < \gamma_{gr}^{+}(G)$. If $\gamma_{gr}(G) = p-2$ then clearly $\gamma_{gr}^{+}(G) = p-2$, therefore $\gamma_{gr}(G) = \gamma_{gr}^{+}(G)$. It follows that

$$2 \leq g(G) \leq \gamma_g(G) \leq \gamma_{gr}(G) \leq \gamma_{gr}^{+}(G) \leq p\,.$$

□

Realization Results

Theorem 2.13. *For every pair a, b of positive integers with $3 \leq a \leq b$, there exists a connected graph G of order b such that $\gamma_{gr}^{+}(G) = a$.*

Proof. Let $X = \{x, y\}$ and $Y = \{u_1, u_2, \ldots, u_{b-a}\}$ be two sets of vertices. Let G be the graph obtained from X and Y by adding new vertices z_i $(1 \leq i \leq a-1)$ and joining each z_i $(1 \leq i \leq a-1)$ to y and also join each u_i $(1 \leq i \leq b-a)$ to both x and y. The resulting graph G is given in Figure 2.

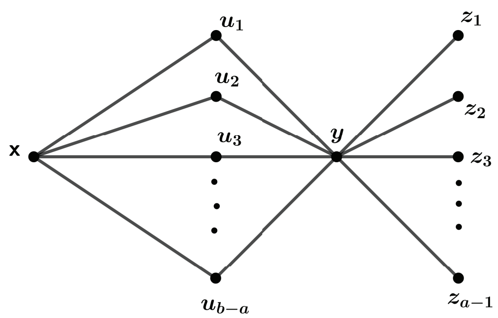

FIGURE 2

Let $S = \{z_1, z_2, z_3, \ldots, z_{a-1}\}$ be the set of all extreme vertices of G. By Theorem 2.1, S is the subset of every geodetic set, geodetic dominating set, restrained geodetic dominating set and upper restrained geodetic dominating set and clearly it is not a geodetic set of G. Obviously $S_1 = S \cup \{x\}$ is a geodetic set and geodetic dominating set and restrained geodetic dominating set of G. Also S_1 is the minimal restrained geodetic dominating set of G so that $\gamma^+_{gr}(G) = \{z_1, z_2, z_3, \ldots, z_{a-1}, x\} = a - 1 + 1 = a$. □

Theorem 2.14. *For every pair a, b of integers with $3 \leq a \leq b$, there exists a connected graph G such that $\gamma_{gr}(G) = a$ and $\gamma^+_{gr}(G) = b$.*

Proof. Let $P_1 : v_1, v_2, v_3, v_4$ be a path of order 4. Let H be the graph obtained from P_1 by adding the new vertices z_i $(1 \leq i \leq a-2)$ and joining each z_i $(1 \leq i \leq a-2)$ to v_4. Let G be the graph obtained from H by adding the new vertices u_i $(1 \leq i \leq b-a)$ and join each u_i $(1 \leq i \leq b-a)$ to both v_1 and v_4. The resulting graph G is given in Figure 3.

Let $S = \{z_1, z_2, z_3, \ldots, z_{a-2}\}$ be the set of all extreme vertices of G. By Theorem 2.1, S is the subset of every geodetic set, geodetic dominating set, restrained geodetic dominating set and upper restrained geodetic dominating set and clearly it is not a geodetic set of G. It is clear that $S_1 = S \cup \{v_1, v_2\}$ is a geodetic set and geodetic dominating set and restrained geodetic dominating set of G so that $\gamma_{gr}(G) = a - 2 + 2 = a$. Now, we have seen that $S_2 = S_1 \cup \{u_1, u_2, \ldots, u_{b-a}\}$ is the minimal restrained geodetic dominating set of G so that $\gamma^+_{gr}(G) = a - 2 + 2 + b - a = b$.

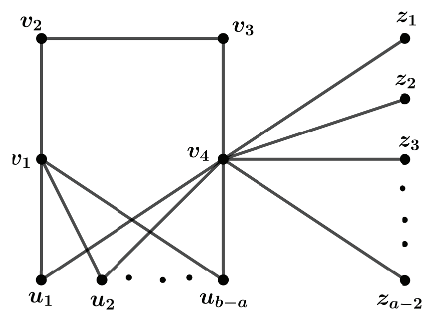

FIGURE 3

□

Theorem 2.15. *For any four integers a, b, c and d with $2 \leq a \leq b \leq c \leq d \leq p$, there exists a connected graph G such that $g(G) = a$, $\gamma_g(G) = b$, $\gamma_{gr}(G) = c$ and $\gamma^+_{gr}(G) = d$.*

Proof. Case 1. $2 < a < b + 1 = c < d$, Let $V(G) - k = d$.

Let $P_1 : v_1, v_2, v_3, v_4, v_5$ be a path of order 5. Let H be the graph obtained from P_1 by adding $a-1$ new vertices $\{z_1, z_2, z_3, \ldots, z_{a-1}\}$ and join each z_i $(1 \leq i \leq a-1)$ to v_1. Let G be the graph obtained from H by adding a cycle C of even order n. Take the vertices of C be $\{x_1, x_2, x_3, \ldots, x_n\}$. Now, we identify the vertex v_5 from a path P_1 with a vertex x_1 from the cycle C. The resulting graph G is given in Figure 4.

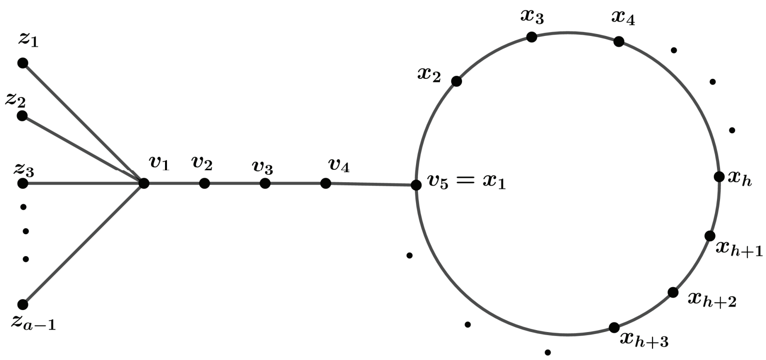

FIGURE 4

Let $S = \{z_1, z_2, z_3, \ldots, z_{a-1}\}$ be the set of all extreme vertices of G. By Theorem 2.1, S is the subset of every geodetic set, geodetic dominating set, restrained geodetic dominating set and upper restrained geodetic dominating set and clearly it is not a geodetic set of G. Clearly $S_1 = S \cup \{x_h\}$ is a minimal geodetic set of G so that $g(G) = a - 1 + 1 = a$. Obviously $S_2 = S_1 \cup \{v_2, v_5 = x_1, x_4, x_7, \ldots x_{b-a-1}\}$ is a minimal geodetic dominating set of G so that $\gamma_g(G) = a + b - a - 1 + 1 = b$. It is clear that $S_3 = S_2 \cup \{v_1\}$ is a minimal restrained geodetic dominating set of G so that $\gamma_{gr}(G) = b + 1 = c$. Also, clearly $S_4 = V(G) - \{v_3, v_4, x_2, x_3, x_5, x_6 \ldots x_k\}$ is a minimum upper restrained geodetic dominating set of G so that $\gamma_{gr}^{+}(G) = V(G) - k = d$.

Case 2. $2 \leq a + 1 = b = c < d$

Consider the cycle $C_6 : v_1, v_2, v_3, v_4, v_5, v_6$. Let H be the graph obtained from C_6 by adding new vertices $\{x_1, x_2, x_3, \ldots, x_{a-1}\}$ and join each x_i $(1 \leq i \leq a-1)$ to v_4. The resulting graph G is given in Figure 5.

Let $S = \{x_1, x_2, x_3, \ldots, x_{a-1}\}$ be the set of all extreme vertices of G. By Theorem 2.1, S is the subset of every geodetic set, geodetic dominating set, restrained geodetic dominating set and upper restrained geodetic dominating set

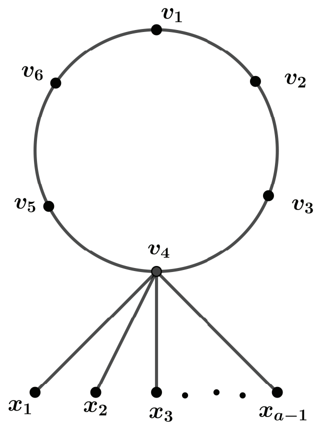

FIGURE 5

and clearly it is not a geodetic set of G. Clearly $S_1 = S \cup \{v_1\}$ is a minimal geodetic set of G so that $g(G) = a-1+1 = a$. Now, it is clear that $S_2 = S_1 \cup \{v_4\}$ is the geodetic dominating set of G so that $\gamma_g(G) = a+1 = b$ it is also a minimal restrained geodetic dominating set of G so that $\gamma_{gr}(G) = b = c$. Now, it is seen that $S_3 = V(G) - \{v_1, v_6\}$ is the minimum upper restrained geodetic dominating set of G so that $\gamma^+_{gr}(G) = |V(G) - \{v_1, v_6\}| = d$.

Case 3. $2 = a = b < c = d$

Let $P : v_1, v_2, v_3$ be a path of order 3. Let G be the graph obtained from P by adding a set of new vertices $\{w_1, w_2, w_3, \ldots, w_{c-b-1}\}$ and joining each w_i $(1 \leq i \leq c-b-1)$ to the vertices v_1 and v_3. The resulting graph G is given in Figure 6.

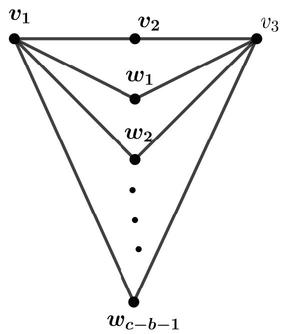

FIGURE 6

Let $S_1 = \{v_1, v_3\}$ be the minimum geodetic set and minimum geodetic dominating set of G so that $g(G) = 2 = a = \gamma_g(G) = b$. Now, we have seen that $S_2 = S_1 \cup \{w_1, w_2, w_3, \ldots, w_{c-b-1}, v_2\}$ is the minimum restrained geodetic dominating set of G so that $\gamma_{gr}(G) = b + c - b - 1 + 1 = c$, which is also a minimum upper restrained geodetic dominating set of G so that $\gamma^+_{gr}(G) = c = d$.

Case 4. $2 < a + 1 = b + 1 = c < d$

Let $P : v_1, v_2, v_3, v_4, v_5$ be a path of order 5. Let G be the graph obtained from P by adding a set of new vertices $\{z_1, z_2, z_3, \ldots, z_{a-1}\}$ and joining each z_i $(1 \leq i \leq a-1)$ to the vertex v_1. The resulting graph G is given in Figure 7.

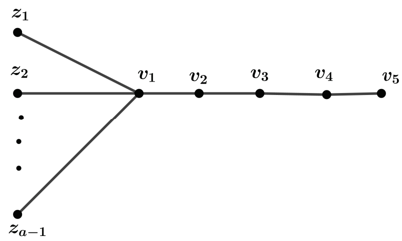

FIGURE 7

Let $S = \{z_1, z_2, z_3, \ldots, z_{a-1}\}$ be the set of all extreme vertices of G. By Theorem 2.1, S is the subset of every geodetic set, geodetic dominating set, restrained geodetic dominating set and upper restrained geodetic dominating set and clearly it is not a geodetic set of G. Clearly $S_1 = S \cup \{v_5\}$ is a minimum geodetic set of G so that $g(G) = a - 1 + 1 = a$. Now, we have seen that $S_2 = S_1 \cup \{v_2\}$ is a minimum geodetic dominating set of G so that $\gamma_g(G) = a + 1 = b$. Obviously $S_3 = S_2 \cup \{v_1\}$ is a minimum restrained geodetic dominating set of G so that $\gamma_{gr}(G) = b + 1 = c$. It is clear that $S_4 = V(G) - \{v_3, v_4\}$ is a minimum upper restrained geodetic dominating set of G so that $\gamma^+_{gr}(G) = |V(G) - \{v_3, v_4\}| = d$. $\square$

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