ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.6, 4175–4186
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.6.99 Spec. Issue on ICIGA-2020

SOME MORE RESULTS ON HARMONIC MEAN INDICES OF GRAPHS

G. SURESH SINGH[1] AND N. J. KOSHY

ABSTRACT. In graph-theoretical terms, a topological descriptor is a single number representing a chemical structure. If the topological descriptor is correlated with a molecular property, then it is called topological indices. The oldest topological index was introduced by Harold Weiner in 1947 and it is ordinary vertex version of Weiner index and is the sum of all distance between vertices of graph. Iranmanesh et al., in 2008, introduced its vertex version based on distance between edges. There are several topological indices on degree-based, distance based and eccentricity based. They have many applications in modeling pharmaceutical, chemical and in other properties of molecules. Here we study some more properties of Harmonic Mean Indices of a graph G denoted by $H_{MI}(G)$ introduced by G.Suresh Singh and N.J.Koshy in 2019.

1. INTRODUCTION

All graphs under our consideration are finite, simple and undirected. A graph $G = (V, E)$ is an ordered pair where V is a non empty set and E is a set of unordered pairs of elements of V. Elements of V are called the vertices and the set is known as a vertex set of G denoted by $V(G)$. Similarly elements of E are called edges of G and the set is called as an edge set of G denoted by $E(G)$. The cardinality of $V(G)$ is called the order of G and that of $E(G)$ is called the size of

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2010 *Mathematics Subject Classification.* 05C99.
Key words and phrases. Harmonic mean topological indices, line graphs, helm graph, sunflower graph, graph operations.

the graph. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ or simply $d(v)$. The degree $d_G(e)$ of an edge $e = uv$ of G is given by $d_G(e) = d_G(u) + d_G(v) - 2$.

The topological indices are numerical values associated with molecular graphs. They play a vital role in chemical documentation, isomer discrimination, relationship analysis like QSAR (quantitative structure activity relationship) and QSPR (quantitative structure property relationship) and in modeling pharmaceutical, chemical and in studying other properties of molecules such as entropy, heat formation, boiling points and enthalpy of vaporization of different chemical compounds.

A topological index [2] can be defined as a function T from the set of all graphs Ω to R^+ if for any pair of isomorphic graphs G and $H, T(G) = T(H)$. That is, if $G \cong H$, then $T : \Omega \to R^+$ satisfies $T(G) = T(H)$.

Motivated by various topological indices and harmonic mean labeling of graphs, even though topological indices not related to labeling, G. Suresh Singh and N.J. Koshy introduced a new topological indices of a graph G called harmonic mean topological indices [7] denoted by $H_{MI}(G)$ and is defined as

$$H_{MI}(G) = \sum_{uv \in E(G)} \frac{2d_G(u)d_G(v)}{d_G(u) + d_G(v)} .$$

In the second section of this paper we evaluate some more results for harmonic mean indices of some special graphs like helm graph, and sun-flower graph, and that of their line graphs. The third section deals with the evaluation of harmonic mean indices $H_{MI}(G)$ of some graph operations, lexicographic product and corona product and some upper bounds for $H_{MI}(G)$ of these graph operations.

The line graph [1], [3] of a graph G denoted as $L(G)$ is obtained from G by considering the edges of G as vertices of $L(G)$ and two vertices are adjacent in $L(G)$ if the corresponding edges are adjacent in G.

A helm graph [1] denoted by H_n is a graph of order $2n + 1$ obtained from a wheel W_n having a pendent edge attached to each rim vertices of W_n. The central vertex v_c has degree n. The vertices of $H_n \setminus \{v_c\}$ are of two kinds, vertices of degree 4 and that of one. The vertices of degree one are referred as minor vertices and that of degree four are referred as major vertices. There are $2n + 1$ vertices and $3n$ edges for a helm graph.

The sunflower graph [1] denoted as SF_n consists of a wheel with central vertex v_c and an $n-$cycle $v_0, v_1, \cdots, v_{n-1}, v_0$ and additional n vertices $w_0, w_1, \cdots, w_{n-1}$ where w_i is joined to the vertices v_i, and v_{i+1} for $i = 0, 1, \cdots, n-1$, where $i+1$ is taken modulo n. Then for the central vertex $v_c, d_{SF_n}(v_c) = n$. The vertices of $SF_n \setminus \{v_c\}$ are of two kinds, v_i of degree 5 and w_i of degree 2. Vertices of degree 2 are called minor vertices and that of 5 are called major vertices. SF_n has $2n+1$ vertices and $4n$ edges.

The composition or lexicographic product of two disjoint graphs G_1 and G_2 denoted by $G_1[G_2]$ is a graph with vertex set $V(G_1) \times V(G_2)$ and (u_i, v_j) is adjacent with (u_k, v_l) whenever u_i is adjacent with u_k in G_1 or $u_i = u_k$ and v_j is adjacent with v_l in G_2. By this definition one can see that [4]:

(i) $|E(G_1[G_2])| = |E(G_1)||V(G_2)^2| + |E(G_2)||V(G_1)|$;
(ii) $d_{G_1[G_2]}(u, v) = |V(G_2)|d_{G_1}(u) + d_{G_2}(v)$.

The Corona product of two graphs G_1 and G_2 denoted by $G_1 \odot G_2$ is a graph obtained by taking $|V(G_1)|$ copies of G_2 and joining each vertex of the i-th copy with vertex $v_i \in V(G_1)$. Let G_1 and G_2 be two connected graphs with orders n_1, n_2 and size m_1, m_2 respectively. Then, by [6], we have $|V(G_1 \odot G_2)| = |V(G_1)|(1+|V(G_2)| = n_1(1+n_2)$ and $|E(G_1 \odot G_2)| = |E(G_1)| + |V(G_1)|(|V(G_2)| + |E(G_2)|) = m_1 + n_1(n_2 + m_2)$. Also for a vertex:

$$u \in V(G_1 \odot G_2), \quad d_{G_1 \odot G_2}(u) = \begin{cases} d_{G_1}(u) + |V(G_2)| \; if \; u \in V(G_1) \\ d_{G_2}(u) + 1 \; if \; u \in V(G_2) \end{cases} .$$

The eccentricity [6] $e_G(v)$ of a vertex v in a connected graph G is the greatest geodesic distance between v and any other vertex. The diameter $d(G)$ of G is defined as $d(G) = \max\{e_G(v)|v \in V(G)\}$. Also the radius $r(G)$ is defined as the $r(G) = \min\{e_G(v)|v \in V(G)\}$. Further, note that $r(G_i) \leq n_i - e_{G_i}(u)$, where $n_i = |V(G_i)|$ and $d_{G_i}(u_i) \leq n_i - e_{G_i}(u)$.

Computational techniques and inductive-deductive approaches are used in deducing and proving various results.

2. Harmonic mean indices of some special graphs

Theorem 2.1. *For the helm graph H_n of order $2n+1$, the harmonic mean index is given by* $H_{MI}(H_n) = \dfrac{4n(17n+28)}{5(n+4)}$.

Proof. The vertices and edges of the helm graph H_n can be partitioned as follows.

$$V(H_n) = V_1 \cup V_2 \cup V_3 \text{ and } E(H_n) = E_1 \cup E_2 \cup E_3$$

$$V_1 = \{v_c \in V(H_n) | d_{H_n}(v_c) = n\} \qquad V_2 = \{v_i \in V(H_n) | d_{H_n}(v_i) = 4\}$$

$$V_3 = \{v_i \in V(H_n) | d_{H_n}(v_i) = 1\} \qquad E_1 = \{uv | d(u) = n \text{ and } d(v) = 4\}$$

$$E_2 = \{uv | d(u) = 4 \text{ and } d(v) = 4\} \qquad E_3 = \{uv | d(u) = 4 \text{ and } d(v) = 1\}$$

Now, $|E_1| = |E_2| = |E_3| = n$,

$$\begin{aligned}
H_{MI}(H_n) &= \sum_{uv \in E(H_n)} \frac{2d_{H_n}(u)d_{H_n}(v)}{d_{H_n}(u) + d_{H_n}(v)} \\
&= \sum_{uv \in E_1} \frac{2d_{H_n}(u)d_{H_n}(v)}{d_{H_n}(u) + d_{H_n}(v)} + \sum_{uv \in E_2} \frac{2d_{H_n}(u)d_{H_n}(v)}{d_{H_n}(u) + d_{H_n}(v)} \\
&\quad + \sum_{uv \in E_3} \frac{2d_{H_n}(u)d_{H_n}(v)}{d_{H_n}(u) + d_{H_n}(v)} \\
&= \sum_{uv \in E_1} \frac{2 \times n \times 4}{n+4} + \sum_{uv \in E_2} \frac{2 \times 4 \times 4}{4+4} + \sum_{uv \in E_3} \frac{2 \times 1 \times 4}{1+4} \\
&= \frac{8n}{n+4} \sum_{uv \in E_1} 1 + 4 \sum_{uv \in E_2} 1 + \frac{8}{5} \sum_{uv \in E_3 1} = \frac{8n}{n+4} \times n + 4 \times n + \frac{8}{5}n \\
&= n\left(\frac{8n}{n+4} + 4 + \frac{8}{5}\right) = \frac{4n(17n+28)}{5(n+4)}.
\end{aligned}$$

□

Theorem 2.2. *Let H_n be the helm graph of order $2n+1$, and $L(H_n)$ represents its line graph then the harmonic mean index of the line graph is*

$$H_{MI}(L(H_n)) = (n+2)n\left\{\frac{(n-1)}{2} + \frac{24}{(n+8)} + \frac{6}{n+5}\right\} + 14n.$$

Proof. The line graph $L(H_n)$ of H_n is a graph with $3n$ vertices and $\frac{n^2+11}{2}$ edges. Depending on the degrees, there are 3 types of vertices.

$V_1 = \{v_i \in V(L(H_n)) | d_{L(H_n)}(v_i) = n+2, i = 1, 2, \cdots, n\}$

$V_2 = \{v_i \in V(L(H_n)) | d_{L(H_n)}(v_i) = 6, i = n+1, \cdots, 2n\}\}$ and

$V_3 = \{v_i \in V(L(H_n)) | d_{L(H_n)}(v_i) = 3, i = 2n+1, \cdots, 3n\}$

Similarly, we have 5 types of edges
$E_1 = \{v_iv_j \in E(L(H_n))|v_i, v_j \in V_1\}$ $E_2 = \{v_iv_j \in E(L(H_n))|v_i \in V_1, v_j \in V_2\}$
$E_3 = \{v_iv_j \in E(L(H_n))|v_i, v_j \in V_2\}$ $E_4 = \{v_iv_j \in E(L(H_n))|v_i \in V_2, v_j \in V_3\}$
$E_5 = \{v_iv_j \in E(L(H_n))|v_i \in V_1, v_j \in V_3\}$.

Moreover it can be seen that $|E_1| = \dfrac{n(n-1)}{2}, |E_2| = |E_4| = 2n$ and $|E_3| = |E_5| = n$,

$$
\begin{aligned}
H_{MI}(L(H_n)) &= \sum_{uv \in E(L(H_n))} \frac{2d_{L(H_n)}(u)d_{L(H_n)}(v)}{d_{L(H_n)}(u) + d_{L(H_n)}(v)} \\
&= \sum_{uv \in E_1} \frac{2d_{L(H_n)}(u)d_{L(H_n)}(v)}{d_{L(H_n)}(u) + d_{L(H_n)}(v)} + \sum_{uv \in E_2} \frac{2d_{L(H_n)}(u)d_{L(H_n)}(v)}{d_{L(H_n)}(u) + d_{L(H_n)}(v)} \\
&\quad + \sum_{uv \in E_3} \frac{2d_{L(H_n)}(u)d_{L(H_n)}(v)}{d_{L(H_n)}(u) + d_{L(H_n)}(v)} + \sum_{uv \in E_4} \frac{2d_{L(H_n)}(u)d_{L(H_n)}(v)}{d_{L(H_n)}(u) + d_{L(H_n)}(v)} \\
&\quad + \sum_{uv \in E_5} \frac{2d_{L(H_n)}(u)d_{L(H_n)}(v)}{d_{L(H_n)}(u) + d_{L(H_n)}(v)} \\
&= \sum_{uv \in E_1} \frac{(2(n+2)(n+2)}{n+2+n+2} + \sum_{uv \in E_2} \frac{2(n+2)\times 6}{n+2+6} + \sum_{uv \in E_3} \frac{2\times 6\times 6}{6+6} \\
&\quad + \sum_{uv \in E_4} \frac{2\times 6\times 3}{6+3} + \sum_{uv \in E_5} \frac{2(n+2)\times 3}{n+2+3} \\
&= (n+2)\sum_{uv \in E_1} 1 + \frac{12(n+2)}{n+8}\sum_{uv \in E_2} 1 + 6\sum_{uv \in E_3} 1 + 4\sum_{uv \in E_4} 1 \\
&\quad + \frac{6(n+2)}{n+5}\sum_{uv \in E_5} 1 \\
&= (n+2)\frac{n(n-1)}{2} + \frac{12(n+2)}{n+8}\times 2n + 6\times n + 4\times 2n + \frac{6(n+2)}{n+5}\times n \\
&= (n+2)n\left\{\frac{(n-1)}{2} + \frac{24}{n+8} + \frac{6}{n+5}\right\} + 14n\,.
\end{aligned}
$$

$\square$

Theorem 2.3. *For the sunflower graph SF_n, the harmonic mean index is given by* $H_{MI}(SF_n) = n\left(\dfrac{10n}{n+5} + \dfrac{75}{7}\right)$.

Proof. The vertex set of SF_n can be partitioned as follows.

$V_1 = \{v_c \in V(SF_n) | d_{SF_n}(v_c) = n\}$;

$V_2 = \{v_i \in V(SF_n) | d_{SF_n}(v_i) = 5, i = 1, 2, \cdots, n\}$;

$V_3 = \{v_i \in V(SF_n) | d_{SF_n}(v_i) = 2, i = n+1, n+2, \cdots, 2n\}$.

The edges of SF_n can be partitioned into E_1, E_2, E_3 as follows

$E_1 = \{uv | d_{SF_n}(u) = n \text{ and } d_{SF_n}(v) = 5\},\ |E_1| = n$;

$E_2 = \{uv | d_{SF_n}(u) = 5 \text{ and } d_{SF_n}(v) = 5\},\ |E_2| = n$;

$E_3 = \{uv | d_{SF_n}(u) = 5 \text{ and } d_{SF_n}(v) = 2\},\ |E_3| = 2n$;

$$\begin{aligned} H_{MI}(SF_n) &= \sum_{uv \in E(SF_n)} \frac{2d_{SF_n}(u)d_{SF_n}(v)}{d_{SF_n}(u)d_{SF_n}(v)} \\ &= \sum_{uv \in E_1} \frac{2d_{SF_n}(u)d_{SF_n}(v)}{d_{SF_n}(u)d_{SF_n}(v)} + \sum_{uv \in E_2} \frac{2d_{SF_n}(u)d_{SF_n}(v)}{d_{SF_n}(u)d_{SF_n}(v)} \\ &\quad + \sum_{uv \in E_3} \frac{2d_{SF_n}(u)d_{SF_n}(v)}{d_{SF_n}(u)d_{SF_n}(v)} \\ &= \sum_{uv \in E_1} \frac{2n \times 5}{n+5} + \sum_{uv \in E_2} \frac{2 \times 5 \times 5}{5+5} + \sum_{uv \in E_3} \frac{2 \times 5 \times 2}{5+2} \\ &= \frac{10n}{n+5} \sum_{uv \in E_1} 1 + 5 \sum_{uv \in E_1} 1 + \frac{20}{7} \sum_{uv \in E_1} 1 \\ &= \frac{10n}{n+5} \times n + 5 \times n + \frac{20}{7} \times 2n = n\left(\frac{10n}{n+5} + \frac{75}{7}\right) \end{aligned}$$

□

Proposition 2.1. *For the line graph of a sunflower graph SF_n, the harmonic mean index is given by*

$$H_{MI}(L(SF_n) = (n+3)n\left(\frac{n-1}{2} + \frac{32}{n+11} + \frac{20}{n+8}\right) + \frac{554n}{13}.$$

3. Harmonic Mean Indices of Some Graph Operations

In this section we discuss about the graph theoretic operations given in [3], [4], [5], [6], the composition and Corona product. Further we find some upper bounds for them.

Observations 3.1 The harmonic mean indices of the lexicographic product graphs $P_2[C_n], P_3[C_n], P_4[C_n], P_5[C_n]$, and $P_6[C_n]$ are given by:

(i) $H_{MI}(P_2[C_n]) = n(n+2)(2n-1)$;
(ii) $H_{MI}(P_3[C_n]) = (n+2)\times 2n + \dfrac{2(n+2)(2n+2)}{3n+4}\times 2n^2 + (2n+2)n$;
(iii) $H_{MI}(P_4[C_n]) = (n+2)\times 2n + \dfrac{2(n+2)(2n+2)}{3n+4}\times 2n^2 + (2n+2)(n^2+2n)$;
(iv) $H_{MI}(P_5[C_n]) = (n+2)\times 2n + \dfrac{2(n+2)(2n+2)}{3n+4}\times 2n^2 + (2n+2)(2n^2+3n)$;
(v) $H_{MI}(P_6[C_n]) = (n+2)\times 2n + \dfrac{2(n+2)(2n+2)}{3n+4}\times 2n^2 + (2n+2)(3n^2+4n)$.

Theorem 3.1. *The harmonic mean indices of $P_m[C_n]$ is given by*

(a) $H_{MI}(P_2[C_n]) = n(n+2)(2n-1)$ *for* $m = 2, n \geq 3$;
(b) $H_{MI}(P_m[C_n]) = 2n(n+2) + \dfrac{8n^2(n+2)(n+1)}{(3n+4)} + 2n(n+1)(mn-3n+m-2)$ *for* $m, n \geq 3$.

Proof. For $m = 2, n = 3$, the graph is $n+2$ regular with $\binom{2n}{2} = \dfrac{2n(2n-1)}{2} = n(2n-1)$ edges:

$$\begin{aligned} H_{MI}(P_2[C_n]) &= \sum_{uv\in E(P_2[C_n])} \frac{2dudv}{du+dv} \\ &= \sum_{uv\in E(P_2[C_n])} \frac{2(n+2)(n+2)}{n+2+n+2} \\ &= (n+2)\sum_{uv\in E(P_2[C_n])} 1 \\ &= (n+2)n(2n-1) \\ &= n(n+2)(2n-1) \end{aligned}$$

Next we prove the case for $m, n \geq 3$. From observations 3.1, it can be easily verify that the edge set of $P_m[C_n]$ can be partitioned as follows for $m, n \geq 3$.
$E_1 = \{uv|du = n+2, dv = n+2\},\ |E_1| = 2n$
$E_2 = \{uv|du = n+2, dv = 2n+2\},\ |E_2| = 2n^2$
$E_3 = \{uv|du = 2n+2, dv = 2n+2\},\ |E_3| = (m-3)n^2 + (m-2)n$.

Now,

$$\begin{aligned}
H_{MI}(P_m[C_n]) &= \sum_{uv\in E(P_m[C_n])} \frac{2dudv}{du+dv}\\
&= \sum_{uv\in E_1} \frac{2dudv}{du+dv} + \sum_{uv\in E_2} \frac{2dudv}{du+dv} + \sum_{uv\in E_3} \frac{2dudv}{du+dv}\\
&= \sum_{uv\in E_1} \frac{2(n+2)(n+2)}{n+2+n+2} + \sum_{uv\in E_2} \frac{2(n+2)(2n+2)}{n+2+2n+2}\\
&\quad + \sum_{uv\in E_3} \frac{2(2n+2)(2n+2)}{2n+2+2n+2}\\
&= (n+2)\sum_{uv\in E_1} 1 + \frac{2(n+2)(2n+2)}{n+2+2n+2}\sum_{uv\in E_2} 1 + (2n+2)\sum_{uv\in E_3} 1\\
&= (n+2)\times 2n + \frac{4(n+2)(n+1)}{3n+4}\times 2n^2\\
&\quad +2(n+1)[(m-3)n^2+(m-2)n]\\
&= 2n(n+2) + \frac{8n^2(n+2)(n+1)}{3n+4} + 2n(n+1)[(m-3)n+m-2]\,.
\end{aligned}$$

□

The next two theorems give some upper bounds of the harmonic mean under certain given conditions.

Theorem 3.2. *Let G_1 and G_2 be two connected graphs with order n_1, n_2, size m_1, m_2 respectively and δ_i and Δ_i are minimum and maximum degrees of the vertices of $G_i, i = 1, 2$. Then for the harmonic mean index of $G_1[G_2]$,*

$$H_{MI}(G_1[G_2]) \leq \frac{(n_2\Delta_1+\Delta_2)^2(m_1n_2^2+m_2n_1)}{n_2\delta_1+\delta_2}\,.$$

Proof. By definition of harmonic mean indices and the relation that $\delta_i \leq \delta_{G_i}(u_i) \leq \Delta_i$ one can compute the harmonic mean indices of the composition of graphs

G_1 and G_2 as follows.

$$\begin{aligned} H_{MI}(G_1[G_2]) &= \sum_{\substack{(u_i,v_j),(u_k,v_l)\in E(G_1[G_2])\\(u_i,v_j)\neq(u_k,v_l)}} \frac{2d_{G_1[G_2]}(u_i,v_j)d_{G_1[G_2]}(u_k,v_l)}{d_{G_1[G_2]}(u_i,v_j)+d_{G_1[G_2]}(u_k,v_l)} \\ &= \sum_{\substack{(u_i,v_j),(u_k,v_l)\in E(G_1[G_2])\\(u_i,v_j)\neq(u_k,v_l)}} \frac{2(n_2d_{G_1}(u_i)+d_{G_2}(v_j))(n_2d_{G_1}(u_k)+d_{G_2}(v_l))}{(n_2d_{G_1})(u_i)+d_{G_2}(v_j)+n_2d_{G_1}(u_k)+d_{G_2}(v_l)} \\ &\leq \sum_{\substack{(u_i,v_j),(u_k,v_l)\in E(G_1[G_2])\\(u_i,v_j)\neq(u_k,v_l)}} \frac{2(n_2\Delta_1+\Delta_2)(n_2\Delta_1+\Delta_2)}{n_2\delta_1+\delta_2+n_2\delta_1+\delta_2} \\ &\leq \frac{(n_2\Delta_1+\Delta_2)^2}{n_2\delta_1+\delta_2} \sum_{\substack{(u_i,v_j),(u_k,v_l)\in E(G_1[G_2])\\(u_i,v_j)\neq(u_k,v_l)}} 1 \\ &= \frac{(n_2\Delta_1+\Delta_2)^2}{n_2\delta_1+\delta_2}(m_1n_2^2+m_2n_1) \end{aligned}$$

□

Remark 3.1. *In the above theorem equality holds for $G_1 = K_m$, $G_2 = K_n$, for $m,n \geq 2$.*

Theorem 3.3. *Let G_1 and G_2 be two connected graphs with order n_1, n_2 , size m_1, m_2 respectively and minimum degrees of the vertices of $G_i, i = 1,2$ are δ_i and $r(G_i)$ their radii. Then for the harmonic mean indices of $G_1[G_2]$,*

$$H_{MI}(G_1[G_2]) \leq \frac{(n_2(n_1+n_2-r(G_1)-r(G_2))^2}{n_2\delta_1+\delta_2}(m_1n_2^2+m_2n_1)\,.$$

Proof. We have

$$\begin{aligned} H_{MI}(G_1[G_2]) &= \sum_{\substack{(u_i,v_j),(u_k,v_l)\in E(G_1[G_2])\\(u_i,v_j)\neq(u_k,v_l)}} \frac{2d_{G_1[G_2]}((u_i,v_j)d_{G_1[G_2]}(u_k,v_l)}{d_{G_1[G_2]}((u_i,v_j)+d_{G_1[G_2]}(u_k,v_l)} \\ &= \sum_{\substack{(u_i,v_j),(u_k,v_l)\in E(G_1[G_2])\\(u_i,v_j)\neq(u_k,v_l)}} \frac{2(n_2d_{G_1}(u_i)+d_{G_2}(v_j))(n_2d_{G_1}(u_k)+d_{G_2}(v_l))}{(n_2d_{G_1}(u_i)+d_{G_2}(v_j)+n_2d_{G_1}(u_k)+d_{G_2}(v_l)} \\ &= \sum_{\substack{(u_i,v_j),(u_k,v_l)\in E(G_1[G_2])\\(u_i,v_j)\neq(u_k,v_l)}} \frac{2pq}{n_2\delta_1+\delta_2+n_2\delta_1+\delta_2} \end{aligned}$$

where $p = (n_2(n_1 - e_{G_1}(u_i)) + n_2 - e_{G_2}(v_j)) \leq (n_2(n_1 - r(G_1) + n_2 - r(G_2))$ and $q = (n_2(n_1 - e_{G_1}(u_k)) + n_2 - e_{G_2}(v_l)) \leq (n_2(n_1 - r(G_1) + n_2 - r(G_2))$, since $e_{G_i}(u) \geq r(G_i)$ for $i = 1, 2$

$$\begin{aligned}
&\leq \sum_{\substack{(u_i,v_j),(u_k,v_l)\in\\ E(G_1[G_2])\\ (u_i,v_j)\neq(u_k,v_l)}} \frac{2(n_2(n_1 - r(G_1) + n_2 - r(G_2))(n_2(n_1 - r(G_1) + n_2 - r(G_2))}{n_2\delta_1 + \delta_2 + n_2\delta_1 + \delta_2} \\
&\leq \frac{(n_2(n_1 + n_2 - r(G_1) - r(G_2))^2}{n_2\delta_1 + \delta_2} \sum_{\substack{(u_i,v_j),(u_k,v_l)\in\\ E(G_1[G_2])\\ (u_i,v_j)\neq(u_k,v_l)}} 1 \\
&\leq \frac{(n_2(n_1 + n_2 - r(G_1) - r(G_2))^2}{n_2\delta_1 + \delta_2} \cdot (m_1 n_2^2 + m_2 n_1).
\end{aligned}$$

□

Proposition 3.1. *For a unicyclic graph of the form obtained by attaching a copy of mK_1 to each cyclic vertex of C_n denoted by $C_n \odot mK_1$, the harmonic mean index is given by* $H_{MI}(C_n \odot mK_1) = \dfrac{3n(m+1)(m+2)}{m+3}$.

Proposition 3.2. *For the line graph of $C_n \odot mK_1$,*

$$H_{MI}(L(C_(n) \odot mK_1)) = \frac{n(m+1)(m+3)(3m+4)}{6}.$$

Theorem 3.4. *For the corona product $G_1 \odot G_2$ of the given graphs G_1, G_2 with orders n_i and size m_i and minimum and maximum degrees of the vertices $\delta_(i)$ and Δ_i for $i = 1, 2$.*

$$H_{MI}(G_1 \odot G_2) \leq \frac{(\Delta_1 + n_2)^2}{\delta_1 + n_2} m_1 + \frac{(\Delta_2 + 1)^2}{\delta_2 + 1} n_1 m_2 + 2\frac{(\Delta_1 + n_2)\Delta_2 + 1)}{\delta_1 + \delta_2 + n_2 + 1)} n_1 n_2 .$$

Proof. The edge sets of $G_1 \odot G_2$ can be partitioned into three kinds

$$\begin{aligned}
E_1 &= \{e = uv \in E(G_1 \odot G_2), e \in E(G_1)\}, |E_1| = m_1 \\
E_2 &= \{e = uv \in E(G_1 \odot G_2), e \in E(G_{2i}), i = 1, 2, \cdots, |V(G_1)|\}, |E_2| = n_1 m_2 \\
E_3 &= \{e = uv \in E(G_1 \odot G_2), e \in E(G_{2i}), i = 1, 2, \cdots, |V(G_1)| \text{ and } v \in V(G_1)\},
\end{aligned}$$

$|E_2| = n_1 n_2$. Now,

$$\begin{aligned}
H_{MI}(G_1 \odot G_2) &= \sum_{uv \in E(G_1 \odot G_2)} \frac{2 d_{G_1 \odot G_2}(u) d_{G_1 \odot G_2}(v)}{d_{G_1 \odot G_2}(u) + d_{G_1 \odot G_2}(v)} \\
&= \sum_{uv \in E_1} \frac{2(d_{G_1}(u) + n_2)(d_{G_1}(v) + n_2)}{d_{G_1}(u) + n_2 + d_{G_1}(v) + n_2} \\
&\quad + \sum_{uv \in E_2} 2 \frac{(d_{G_2}(u) + 1)(d_{G_2}(v) + 1)}{d_{G_2}(u) + 1) + (d_{G_2}(v) + 1} \\
&\quad + \sum_{uv \in E_3} \frac{2(d_{G_1}(u) + n_2)(d_{G_2}(v) + 1)}{d_{G_1}(u) + n_2 + d_{G_2}(v) + 1} \\
&\leq 2 \sum_{uv \in E_1} \frac{(\Delta_1 + n_2)(\Delta_1 + n_2)}{\delta_1 + n_2 + \delta_1 + n_2} + \sum_{uv \in E_2} 2 \frac{(\Delta_2 + 1)(\Delta_2 + 1)}{\delta_2 + 1 + \delta_2 + 1} \\
&\quad +2 \sum_{uv \in E_3} \frac{(\Delta_1 + n_2)(\Delta_2 + 1)}{\delta_1 + n_2 + \delta_2 + 1} \\
&= \frac{(\Delta_1 + n_2)^2}{\delta_1 + n_2} \sum_{uv \in E_1} 1 + \frac{(\Delta_2 + 1)^2}{\delta_2 + 1} \sum_{uv \in E_2} 1 \\
&\quad +2 \frac{(\Delta_1 + n_2)(\Delta_2 + 1)}{\delta_1 + \delta_2 + n_2 + 1} \sum_{uv \in E_3} 1 \\
&= \frac{(\Delta_1 + n_2)^2}{\delta_1 + n_2} m_1 + \frac{(\Delta_2 + 1)^2}{\delta_2 + 1} n_1 m_2 + 2 \frac{(\Delta_1 + n_2)(\Delta_2 + 1)}{\delta_1 + \delta_2 + n_2 + 1} n_1 n_2
\end{aligned}$$

□

4. Conclusion and Scope

In this paper we have investigated harmonic mean indices of some special graphs and that of their line graphs and find upper bounds for the harmonic mean indices of certain graph operations. A comparative study of harmonic mean indices and other standard topological indices is beyond the scope of this paper.

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