

**EXISTENCE AND UNIQUENESS OF SOLUTION OF FIRST ORDER
DIFFERENCE EQUATION OF FINITE DELAY IN CONE METRIC SPACE**G. C. DONE, K. L. BONDAR¹, AND P. U. CHOPADE

ABSTRACT. In this paper, we investigate the existence and uniqueness of solution of First Order Difference equation of finite delay with nonlocal condition in cone metric space. The result is obtained by using the some extensions of Banach contraction principle in complete cone metric space and also provide example to demonstrate of main result.

1. INTRODUCTION

The aim of this paper is study the existence and uniqueness of solution of First order Difference equation with nonlocal condition in cone metric space of the form:

$$(1.1) \quad \Delta x(t) = A(t)x(t) + f(t, x(t), x(t-1)), t \in J = [0, b]$$

$$(1.2) \quad x(t-1) = \psi(t), t \in [0, 1]$$

$$(1.3) \quad x(0) + g(x) = x_0,$$

where $A(t)$ is a bounded linear operator on a Banach space X with domain $D(A(t))$, the unknown $x(\cdot)$ takes values in the Banach space X ; $f : J \times X \times X \rightarrow X$, $g : C(J, X) \rightarrow X$ are appropriate continuous functions and x_0 is given element of X .

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$\psi(t)$ is a continuous function for $[0, 1]$ $\lim_{t \rightarrow (1-0)} \psi(t)$ exists, for which we denote by $\psi(1-0) = c_0$. If we observed a function $x(t-1)$ which is unable to define as solution for $[0, 1]$. Hence, we have to impose some condition, for example the condition (1.2). We note that, if $[0, 1]$, the problem is reduced to Difference equation

$$\Delta x(t) = A(t)x(t) + f(t, x(t), \psi(t)), \quad t \in J = [0, b],$$

with initial condition $x(0) + g(x) = x_0$. Here, it is essential to obtain the solutions of (1.1)-(1.3) for $[0, b]$.

Many authors have been studied the problems of existence, uniqueness, continuation and other properties of solutions of Difference equation, R. Agrawal [2], W G Kelley and A C Peterson [13] had developed theory of difference equation and their inequalities. Some Existence, Uniqueness and Comparison results on difference equation and summation equation are obtained by K.L. Bondar et al. [3–6, 8, 9].

In Section 2, we discuss the preliminaries. Section 3, deals with study of existence and uniqueness of solution of Difference equation of Finite delay with nonlocal condition in cone metric space. Finally in Section 4, we give example to illustrate the application of our result.

2. DEFINITIONS AND PRELIMINARIES

Let us recall the concepts of the cone metric space and we refer the reader to [1, 10–12, 14, 16] for the more details.

Definition 2.1. Let E be a real Banach space and P is a subset of E . Then P is called a cone if and only if,

- (1) P is closed, nonempty and $P \neq 0$.
- (2) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$.
- (3) $x \in P$ and $-x \in P \Rightarrow x = 0$.

For a given cone $P \in E$, we define a partial ordering relation $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$. Where $\text{int}P$ denotes the interior of P . The cone P is called normal if there is a number $K > 0$ such that $\leq x \leq y$

implies $\|x\| \leq k\|y\|$, for every $x, y \in E$. The least positive number satisfying above is called the normal constant of P .

In the following way, we always suppose E is a real Banach space, P is cone in E with $\text{int } P \neq \emptyset$, and $\leq$ is partial ordering with respect to P .

Definition 2.2. Let X a nonempty set. Suppose that the mapping $d : X \times X \rightarrow E$ satisfies:

- (d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
- (d2) $d(x, y) = d(y, x)$, for all $x, y \in X$;
- (d3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space. The concept of cone metric space is more general than that of metric space. The following example is a cone metric space.

Example 1. Let $E = \mathbb{R}^2, p = \{(x, y) \in E : x, y \geq 0\}, x = \mathbb{R}$, and $d : X \times X \rightarrow E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constat and then (X, d) is cone metric space.

Definition 2.3. Let X be an ordered space. A function $\Phi : X \rightarrow X$ is said to a comparison function if every $x, y \in X, x \leq y$, implies that $\Phi(x) \leq \Phi(y), \Phi(x) \leq x$ and $\lim_{n \rightarrow \infty} \|\Phi^n(x)\| = 0$, for every $x \in X$.

Example 2. Let $E = \mathbb{R}^2, p = \{(x, y) \in E : x, y \geq 0\}$, it is easy to check that $\Phi : E \rightarrow E$ with $\Phi(x, y) = (ax, ay)$, for some $a \in (0, 1)$ is a comparison function. also if Φ_1, Φ_2 are two comparison function over $\mathbb{R}$. then $\Phi(x, y) = (\Phi_1(x), \Phi_2(y))$ is also a comparison function over E .

3. MAIN RESULT

Let X is a Banach space with norm $\|\cdot\|$. $B = C(J, X)$ be the Banach space of all continuous function from J into X endowed with supremum norm

$$\|x\|_{\infty} = \sup\{\|x(t)\| : t \in [0, b]\}.$$

Let $P = (x, y) : x, y \geq 0 \subset E = \mathbb{R}^2$, and define

$$d(f, g) = (\|f - g\|_{\infty}, \alpha\|f - g\|_{\infty}),$$

for every $f, g \in B$, then it is easily seen that (B, d) is a cone metric space.

Definition 3.1. *The function $x \in B$ satisfies the summation equation*

case I: for $t \in [0, 1]$

$$x(t) = x_0 - g(x) + \sum_{s=0}^t A(s)f(s, x(s), x(s-1)),$$

case II: for $t \in [1, b]$

$$x(t) = x_0 - g(x) + \sum_{s=0}^1 A(s)f(s, x(s), x(s-1)) \\ + \sum_{s=1}^t A(s)f(s, x(s), x(s-1)),$$

is called the mild solution of the equation (1.1)-(1.3).

We need the following theorem for further discussion:

Lemma 3.1. [15] *Let (X, d) be a complete cone metric space, where P is a normal cone with normal constant K . Let $f : X \rightarrow X$ be a function such that there exists a comparison function $\Phi : P \rightarrow P$ such that*

$$d(f(x), f(y)) \leq \Phi(d(x, y))$$

for very $x, y \in X$. Then f has unique fixed point.

We list the following hypothesis for our convenience:

(H1) $A(t)$ is a bounded linear operator on X for each $t \in J$, the function $t \rightarrow A(t)$ is continuous in the uniform operator topology and hence there exists a constant K such that

$$K = \sup_{t \in J} \|A(t)\|.$$

(H2) Let $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a comparison function

(i) There exist continuous function $p_1, p_2 : J \rightarrow \mathbb{R}^+$ such that

case I: for $t \in [0, 1]$

$$(\|f(t, x(t), \psi(t)) - f(t, y(t), \psi(t))\|, \alpha \|f(t, x(t), \psi(t)) - f(t, y(t), \psi(t))\|) \\ \leq p_1(t)\Phi(d(x, y)),$$

case II: for $t \in [1, b]$

$$(\|f(t, x(t), x(t-1)) - f(t, y(t), y(t-1))\|, \alpha \|f(t, x(t), x(t-1)) - f(t, y(t), y(t-1))\|)$$

$$\leq p_2(t)\Phi(d(x, y)),$$

for every $t \in J$ and $x, y \in X$.

(ii) There exists a positive constant G such that

$$(\|g(x) - g(y)\|, \alpha\|g(x) - g(y)\|) \leq G\Phi(d(x, y)),$$

for every $x, y \in X$

(H3)

$$\sup_{t \in J} \left\{ G + K \sum_{s=0}^t [p_1(s) + p_2(s)] \right\} = 1.$$

Our main result are given in the following theorem:

Theorem 3.1. *Assume that hypotheses $(H_1) - (H_3)$ hold. Then the difference equation (1.1)-(1.3) has a unique solution x on J .*

Proof. The operator $F : B \rightarrow B$ is defined by

case I: for $t \in [0, 1]$

$$Fx(t) = x_0 - g(x) + \sum_{s=0}^t A(s)f(s, x(s), x(s-1)),$$

case II: for $t \in [1, b]$

$$\begin{aligned} Fx(t) &= x_0 - g(x) + \sum_{s=0}^1 A(s)f(s, x(s), x(s-1)) \\ &\quad + \sum_{s=1}^t A(s)f(s, x(s), x(s-1)). \end{aligned}$$

By using the hypothesis $(H_1) - (H_3)$, We have

case I: for $t \in [0, 1]$

$$\begin{aligned} &(\|Fx(t) - Fy(t)\|, \alpha\|Fx(t) - Fy(t)\|) \\ &\leq \left(\|g(x) - g(y)\| + \sum_{s=0}^t \|A(s)\| \|f(s, x(s), \psi(s)) - f(s, y(s), \psi(s))\|, \right. \\ &\quad \left. \alpha\|g(x) - g(y)\| + \alpha \sum_{s=0}^t \|A(s)\| \|f(s, x(s), \psi(s)) - f(s, y(s), \psi(s))\| \right) \\ &\leq (\|g(x) - g(y)\|, \alpha\|g(x) - g(y)\|) \end{aligned}$$

$$\begin{aligned}
& + \sum_{s=0}^t K(\|f(s, x(s), \psi(s)) - f(s, y(s), \psi(s))\|, \alpha\|f(s, x(s), \psi(s)) - f(s, y(s), \psi(s))\|) \\
& \leq G\Phi(\|x - y\|, \alpha\|x - y\|) + \sum_{s=0}^t Kp_1(s)\Phi(\|x(s) - y(s)\|, \alpha\|x(s) - y(s)\|) \\
& \leq G\Phi(\|x - y\|_\infty, \alpha\|x - y\|_\infty) + \Phi(\|x - y\|_\infty, \alpha\|x - y\|_\infty) \sum_{s=0}^t Kp_1(s) \\
& \leq G\Phi(d(x, y)) + \Phi(d(x, y)) \sum_{s=0}^t Kp_1(s) \\
& \leq \Phi(d(x, y)) \left\{ G + \sum_{s=0}^t Kp_1(s) \right\} \\
& \leq \Phi(d(x, y)) \left\{ G + \sum_{s=0}^t K[p_1(s) + p_2(s)] \right\} \\
& \leq \Phi(d(x, y)).
\end{aligned}$$

case II: for $t \in [1, b]$

$$\begin{aligned}
& (\|Fx(t) - Fy(t)\|, \alpha\|Fx(t) - Fy(t)\|) \\
& \leq \left(\|g(x) - g(y)\| + \sum_{s=0}^1 \|A(s)\| \|f(s, x(s), \psi(s)) - f(s, y(s), \psi(s))\| \right. \\
& \quad \left. + \sum_{s=1}^t \|A(s)\| \|f(s, x(s), x(s-1)) - f(s, y(s), y(s-1))\|, \right. \\
& \quad \left. \alpha\|g(x) - g(y)\| + \alpha \sum_{s=0}^1 \|A(s)\| \|f(s, x(s), \psi(s)) - f(s, y(s), \psi(s))\| \right. \\
& \quad \left. + \alpha \sum_{s=1}^t \|A(s)\| \|f(s, x(s), x(s-1)) - f(s, y(s), y(s-1))\| \right) \\
& \leq G\Phi(d(x, y)) + \sum_{s=0}^1 Kp_1(s)\Phi(d(x, y)) + \sum_{s=1}^t Kp_2(s)\Phi(d(x, y)) \\
& \leq G\Phi(d(x, y)) + \sum_{s=0}^1 K[p_1(s) + p_2(s)]\Phi(d(x, y)) + \sum_{s=1}^t K[p_1(s) + p_2(s)]\Phi(d(x, y))
\end{aligned}$$

$$\leq \Phi(d(x, y)) \left\{ G + \sum_{s=0}^t K[p_1(s) + p_2(s)] \right\} \\ \leq \Phi(d(x, y)),$$

for every $x, y \in B$. This implies that $d(Fx, Fy) \leq \Phi(d(x, y))$, for every $x, y \in B$. Now an application of Lemma 3.1., the operator has a unique point in B . This means that the equation (1.1)-(1.3) has unique solution. $\square$

4. APPLICATION

In this section, we give an example to illustrate the usefulness of our result discussed in previous section. Let us consider the following Difference equation:

$$(4.1) \quad \Delta x(t) = \frac{35}{36}e^{-t}x(t) + f(t, x(t), x(t-1)), \quad t \in J = [0, 2], \quad x \in X$$

$$(4.2) \quad x(0) + \frac{x}{8+x} = x_0,$$

where,

$$f(t, x(t), x(t-1)) = \frac{te^{-t}x(t)}{(9+e^t)(1+x(t))}, \quad \text{for } t \in [0, 1] \\ f(t, x(t), x(t-1)) = \frac{2te^{-(t-1)}x(t-1)}{(9+e^{t-1})(1+x(t-1))}, \quad \text{for } t \in [1, 2].$$

Therefore, we have

$$A(t) = \frac{35}{36}e^{-t}, \quad t \in J \\ f(x, x(t), \psi(t)) = \frac{te^{-t}x(t)}{(9+e^t)(1+x(t))}, \quad (t, x) \in J \times X \\ f(t, x(t), x(t-1)) = \frac{2te^{-(t-1)}x(t-1)}{(9+e^{t-1})(1+x(t-1))}, \quad (t, x) \in J \times X \\ g(x) = \frac{x}{8+x}, \quad x \in X.$$

Now for $x, y \in C(J, X)$ and $t \in J$, we have

case I: for $t \in [0, 1]$

$$(\|f(t, x(t), x(t-1)) - f(t, y(t), y(t-1))\|, \alpha \|f(t, x(t), x(t-1)) - f(t, y(t), y(t-1))\|)$$

$$\begin{aligned}
&= \frac{te^{-t}}{9+e^t} (\| \frac{x(t)}{1+x(t)} - \frac{y(t)}{1+y(t)} \|, \alpha \| \frac{x(t)}{1+x(t)} - \frac{y(t)}{1+y(t)} \|) \\
&= \frac{te^{-t}}{9+e^t} (\| \frac{x(t)-y(t)}{(1+x(t))(1+y(t))} \|, \alpha \| \frac{x(t)-y(t)}{(1+x(t))(1+y(t))} \|) \\
&\leq \frac{te^{-t}}{9+e^t} (\|x(t)-y(t)\|, \alpha \|x(t)-y(t)\|) \\
&\leq \frac{t}{10} \Phi(d(x, y)),
\end{aligned}$$

where $p_1(t) = \frac{t}{10}$, which is continuous function of J into $\mathbb{R}^+$ and a comparison function $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\Phi(d(x, y)) = d(x, y)$.

case II: for $t \in [1, 2]$

$$\begin{aligned}
&(\|f(t, x(t), x(t-1)) - f(t, y(t), y(t-1))\|, \alpha \|f(t, x(t), x(t-1)) - f(t, y(t), y(t-1))\|) \\
&= \frac{2te^{-(t-1)}}{9+e^{t-1}} (\| \frac{x(t-1)}{1+x(t-1)} - \frac{y(t-1)}{1+y(t-1)} \|, \alpha \| \frac{x(t-1)}{1+x(t-1)} - \frac{y(t-1)}{1+y(t-1)} \|) \\
&\leq \frac{2te^{-(t-1)}}{9+e^{t-1}} (\|x(t-1)-y(t-1)\|, \alpha \|x(t-1)-y(t-1)\|) \\
&\leq \frac{2te^{-(t-1)}}{9+e^{t-1}} (\|x-y\|_\infty, \alpha \|x-y\|_\infty) \\
&\leq \frac{2te^{-(t-1)}}{9+e^{t-1}} d(x, y) \\
&\leq \frac{t}{5} \Phi(d(x, y)),
\end{aligned}$$

where $p_2(t) = \frac{t}{5}$, which is continuous function of J into $\mathbb{R}^+$ and a comparison function $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\Phi(d(x, y)) = d(x, y)$.

Similarly, we can have

$$\begin{aligned}
(\|g(x) - g(y)\|, \alpha \|g(x) - g(y)\|) &\leq 8 (\frac{\|x-y\|}{(8+\|x\|)(8+\|y\|)}, \alpha \frac{\|x-y\|}{(8+\|x\|)(8+\|y\|)}) \\
&\leq \frac{8}{64} (\|x-y\|, \alpha \|x-y\|) \\
&\leq \frac{1}{8} \Phi(d(x, y)),
\end{aligned}$$

where $G = \frac{1}{8}$, and the comparison function Φ defined as above. Hence the condition (H_1) holds with $K = \frac{35}{36}$.

Moreover,

$$\begin{aligned}
 \sup_{t \in J} \left\{ G + K \sum_{s=0}^t (p_1(s) + p_2(s)) \right\} &= \sup_{t \in J} \left\{ \frac{1}{8} + \frac{35}{36} \sum_{s=0}^t \left(\frac{s}{10} + \frac{s}{5} \right) \right\} \\
 &= \sup_{t \in J} \left\{ \frac{1}{8} + \frac{35}{36} \sum_{s=0}^t \left(\frac{3s}{10} \right) \right\} \\
 &= \sup_{t \in J} \left\{ \frac{1}{8} + \frac{35}{36} \left[\frac{3s^2}{20} \right]_0^{t+1} \right\} \\
 &= \sup_{t \in J} \left\{ \frac{1}{8} + \frac{35}{36} \left[\frac{3(t+1)^2}{20} \right] \right\} \\
 &= \left[\frac{1}{8} + \frac{35}{36} \times \frac{18}{20} \right] = \left[\frac{1}{8} + \frac{7}{8} \right] = 1.
 \end{aligned}$$

Since all the conditions of Theorem 3.1 are satisfied, the problem (4.1)-(4.2) has a unique solution x on J .

5. CONCLUSION

In this paper, we studied the existence for First order Difference equations of finite delay in cone metric spaces and proved that solution of this result is unique. We proved this result by using the some extensions of Banach contraction principle in complete cone metric space. Moreover we also gave application of above result.

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