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(a,d)-TOTAL NEIGHBORHOOD-ANTIMAGIC LABELING

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ABSTRACT. In this paper, we introduce a new variant of (a,d)-antimagic total labeling for a graph $G = (V, E)$ called (a,d)-total neighborhood-antimagic labeling (TNAL). A total labeling is said to be an (a,d)-total neighborhood-antimagic labeling if the set of total weights of vertices form an arithmetic progression with initial value a and difference d. We give some necessary conditions for the existence of this labeling for a graph G. We investigate (a,d)-total neighborhood-antimagic labeling of cycle C_n. We also discuss the construction of dual labeling from the existing labeling.

1. INTRODUCTION

All graphs $G = (V, E)$ considered here are finite, simple, connected and undirected. Throughout this paper, we use notations v and e for the cardinality of vertex set V and edge set E respectively, $N(x)$ for the neighborhood of a vertex x, $\deg(x)$ for the degree of a vertex x, Δ and δ for maximum degree and minimum degree of vertex in a graph G respectively.

A labeling of graph whose domain is $V \cup E$ and co-domain is the set of numbers (positive or non-negative integers) is called total labeling. In [1] N. Hartsfield and G. Ringel introduced antimagic labeling of graphs. In [2],

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R. Bodendiek and G. Walther firstly introduced $(a,d)-$arithmetic antimagic labeling. They have established the necessary condition for a graph to become $(a,d)-$arithmetic antimagic graphs through the Diophantine equation. They have proved positive results for cycle C_{2n}, path P_{2n}, star $S_n(n \geq 3)$, Cube Q_3 and several complete graphs K_n admits $(a,d)-$arithmetic labeling. They have also proved negative results for binary tree, n-ary tree, complete graph K_4, Kuratowski Graph and Peterson graph. The variant of this labeling was introduced by M. Bača *et al.* in [3] as (a,d)-vertex antimagic total labeling. A total labeling λ is called (a,d)-vertex antimagic total labeling if the sum $\lambda(x)+\lambda(xy)$ for every vertex $x \in V$ forms (a,d)-arithmetic progression. In [4], Simanjuntak *et al.* defined (a,d)-edge magic total labeling. The details of research work about these labelings are available in [5]. Motivated by the study of (a,d)-vertex antimagic total labeling, we introduce a new variant of (a,d)-antimagic total labeling and we call it (a,d)-total neighborhood-antimagic labeling (See Definition 1.2). By total neighborhood of a vertex x we mean the set of adjacent vertices and incident edges to the vertex x. In general context, the term weight of the vertex was earlier defined by many authors as the sum of appropriate labels at a vertex and it is denoted by $w(x)$ or $wt(x)$. Here, we define the weight of a vertex as follows. To avoid ambiguity we call it total weight(See Definition 1.1) of the vertex.

Definition 1.1. *Let $G=(V,E)$ be a graph. For any labeling λ, the total weight of vertex $x \in V$ is defined as $\sum_{y \in N(x)} [\lambda(y)+\lambda(xy)]$, where $N(x)=\{z|z$ is adjacent to $x\},\ xy \in E$. We use notation $WT(x)$ for total weight at vertex x.*

Definition 1.2. *Let $G=(V,E)$ be a graph with v vertices and e edges. A bijection $\lambda: V(G)\cup E(G) \to \{1,2,\cdots,v+e\}$ is called an (a,d)-total neighborhood-antimagic labeling (TNAL) of G if the set of total weights of vertices form an arithmetic progression with initial value a and difference d. A graph G admits an (a,d)-total neighborhood-antimagic labeling (TNAL) is called an (a,d)-total neighborhood-antimagic graph.*

2. Basic Counting

Let $G=(V,E)$ be any graph. Let us denote $|V|=v$ and $|E|=e$ and $M=v+e$. Let S_v, S_e denotes the sum of vertex labels and sum of edge labels respectively.

Note that the sum of all the labels is :

$$S_v + S_e = \sum_{i=1}^{M} i = \binom{M+1}{2} \tag{2.1}$$

If $\lambda : V \cup E \to \{1, 2, \cdots, v+e\}$ is an (a, d)-total neighborhood-antimagic labeling with total weights $WT(x_i) = a + id$, then the summing of total weights over all vertices adds each vertex label $\deg(x)$ times and each edge label twice, so we have:

$$\sum_{i=1}^{v} deg(x_i)\lambda(x_i) + 2S_e = \frac{v}{2}(2a + (v-1)d).$$

If δ denotes the smallest degree in G, then the minimum possible total weight on a vertex is at least $1 + 2 + \cdots + 2\delta$,

$$a \geq \delta(2\delta + 1).$$

Similarly, if Δ denotes the largest degree in G, then the maximum total weight of vertex can not exceed the sum of the 2Δ largest labels. Therefore,

$$\begin{aligned} a + (v-1)d &\leq \sum_{M-(2\Delta-1)}^{M} i \\ &= \Delta(2M - 2\Delta + 1). \end{aligned}$$

We summarize above discussion in Theorem 2.1. This theorem gives necessary conditions for the existence of an (a, d)-total neighborhood-antimagic labeling for a graph G.

Theorem 2.1. *Let $G = (V, E)$ be any graph with $v \geq 2$ and $e \geq 1$. If $\lambda : V \cup E \to \{1, 2, \cdots, v+e\}$ is an (a, d)-total neighborhood-antimagic labeling of G then G must satisfy following conditions:*

(a) *If S_e denotes the sum of all edge labels then*

$$\sum_{i=1}^{v} deg(x_i)\lambda(x_i) + 2S_e = va + \frac{v(v-1)}{2}\, d, \tag{2.2}$$

where $x_i \in V$, $1 \leq i \leq v$ and $\deg(x_i) \geq 1$.

(b) *If δ is the smallest degree in G, then $a \geq \delta(2\delta + 1)$.*

(c) *If Δ is the largest degree in G, then $d \leq \frac{\Delta(2(v+e)-2\Delta+1)-\delta(2\delta+1)}{v-1}$.*

Corollary 2.1. *If $G = (V, E)$ with $v = n \geq 2$ and $e \geq 1$ is an r-regular graph and λ is an (a, d)-total neighborhood-antimagic labeling, then*

$$rS_v + 2S_e = na + \frac{n(n-1)}{2}\, d. \tag{2.3}$$

Here S_v and S_e denotes the sum of vertex labels and sum of edge labels respectively.

Proof. Substitute $\deg(x_i) = r$ in Equation (2.2), we get required result. □

Corollary 2.2. *If $G = (V, E)$ with $v = n \geq 2$ and $e \geq 1$ is an 2-regular graph and λ is an (a, d)-total neighborhood-antimagic labeling, then*

$$2na + n(n-1)\, d = 2(n+e)(n+e+1). \tag{2.4}$$

Proof. Substitute $r = 2$ in Equation (2.3) and using Equation (2.1), we get desired result. □

3. CYCLES WITH ODD LENGTH

In this section, we provide (a, d)-total neighborhood-antimagic labeling for cycles having odd length. For odd cycle C_n where $n = 2k + 1, k \in \mathbb{N}$, we have $v = e = 2k + 1$ and $\delta = \Delta = 2$. We substitute all these values in Equation (2.4), we have linear Diophantine equation

$$a + k\, d = 8k + 6. \tag{3.1}$$

The particular solution of the above equation is $a_0 = 6$ and $d_0 = 8$. The set of all possible solutions is given by $a = 6 + kt$ and $b = 8 - t$, for $t \in \mathbb{Z}$. The positive solutions of Equation (3.1) can be obtained by taking feasible value of t. (See [6])

Theorem 3.1. *For odd $n \geq 3$, the cycle C_n has $(\frac{7n+5}{2}, 1)$-total neighorhood-antimagic labeling.*

Proof. Let $(v_1, v_2, \cdots, v_n)$ be a cycle of length $n \geq 3$. Let e_i be an edge corresponds to consecutive vertices v_i and v_{i+1} of cycle C_n, where the subscripts of vertices and edges are calculated under modulo n. We label the vertices and edges of C_n by

$$\lambda(v_i) = 2n - i,\ 1 \leq i < n$$
$$\lambda(v_n) = 2n$$
$$\lambda(e_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd} \\ \frac{n+1+i}{2} & \text{if } i \text{ is even} \end{cases}$$

where $1 \leq i \leq n$.

Note that the total weight of each vertices are as follows:

$$WT(v_i) = \begin{cases} \frac{9n+1}{2} & \text{if } i = n \\ \frac{9n+3}{2} & \text{if } i = n-1 \\ \frac{9n+1}{2} - i & \text{otherwise} \end{cases}$$

Apparently, the total weights of vertices are $\frac{9n-1}{2}, \frac{9n-3}{2}, \cdots, \frac{7n+5}{2}, \frac{9n+3}{2}, \frac{9n+1}{2}$. Which form arithmetic progression with initial value $a = \frac{7n+5}{2}$ (achieved at vertex v_{n-2}) and difference $d = 1$. □

Theorem 3.2. *For odd $n \geq 3$, the cycle C_n has $(3n+3, 2)$-total neighborhood-antimagic labeling.*

Proof. Let $(v_1, v_2, \cdots, v_n)$ be a cycle of length $n \geq 3$. Let e_i denote the edge between consecutive vertices v_i and v_{i+1} of cycle C_n, where the subscripts of vertices and edges are calculated under modulo n. If we label the vertices and edges of C_n by

$$\lambda(v_1) = 2,$$
$$\lambda(v_i) = (2n+4) - 2i,\ 2 \leq i \leq n$$
$$\lambda(e_i) = \begin{cases} i & \text{if } i \text{ is odd} \\ n+i & \text{if } i \text{ is even} \end{cases}$$

where $1 \leq i \leq n$. Then the total weight of each vertices are

$$WT(v_i) = \begin{cases} 3n+5 & \text{if } i = 1 \\ 3n+3 & \text{if } i = 2 \\ 5n+7-2i & \text{otherwise.} \end{cases}$$

Evidently, the total weights of vertices are $3n+5, 3n+3, 5n+1, \cdots, 3n+7$. Which form arithmetic progression with initial value $a = 3n+3$ (attained at vertex v_2) and difference $d = 2$. $\square$

Theorem 3.3. *For odd $n \geq 3$, the cycle C_n has $(2n+4, 4)$-total neighborhood-antimagic labeling.*

Proof. Let $(v_1, v_2, \cdots, v_n)$ be a cycle of length $n \geq 3$. Let e_i denote the edge between consecutive vertices v_i and v_{i+1} of cycle C_n, where the subscripts of vertices and edges are calculated under modulo n. We define a labeling $\lambda : V(C_n) \cup E(C_n) \rightarrow \{1, 2, \cdots, n\}$ by considering the following two cases:

Case-I: When $n \equiv 3 \ (mod\ 4)$, we assign the labels to vertices and edges of C_n as follows:

$$\lambda(v_i) = \begin{cases} 2i & \text{if } i = 1, 5, 9 \cdots, n-2 \\ 2(i-1) & \text{if } i = 3, 7, 11 \cdots, n \end{cases}$$

$$\lambda(v_i) = \begin{cases} 2(i+1) & \text{if } i = 2, 6, 10, \cdots, n-1 \\ 2i & \text{if } i = 4, 8, 12, \cdots, n-3 \end{cases}$$

$$\lambda(e_i) = \begin{cases} (2n-1) - 2i & \text{if } i \text{ is odd and } i < n \\ 2i-3 & \text{if } i \text{ is even} \\ 2n-1 & \text{if } i = n \end{cases}$$

Case-II: When $n \equiv 1 \ (mod\ 4)$, we assign the labels to vertices and edges of C_n as follows:

$$\lambda(v_i) = \begin{cases} 2i & \text{if } i = 1, 5, 9 \cdots, n \\ 2(i-1) & \text{if } i = 3, 7, 11 \cdots, n-2 \end{cases}$$

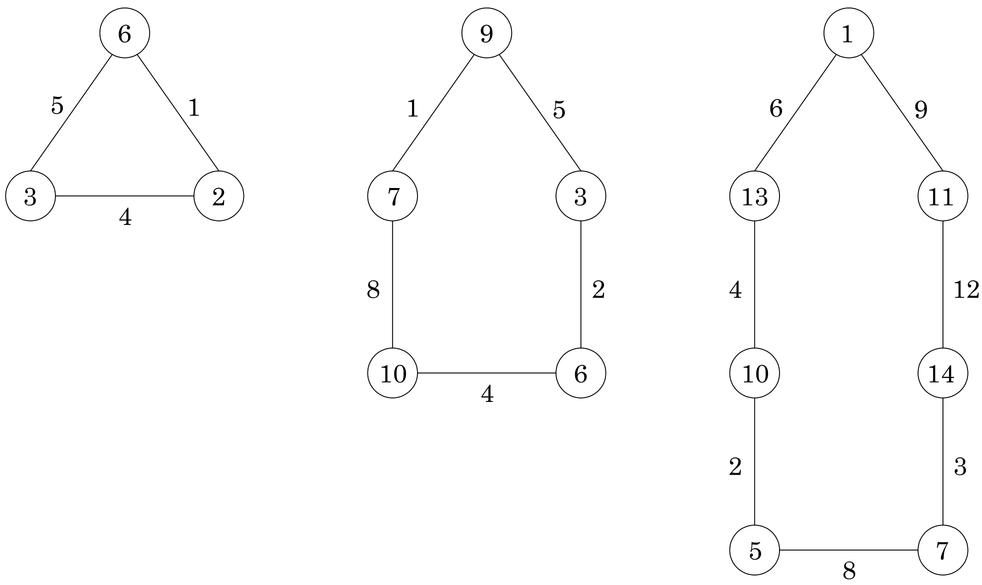
$$\lambda(v_i) = \begin{cases} 2(i+1) & \text{if } i = 2, 6, 10, \cdots, n-3 \\ 2i & \text{if } i = 4, 8, 12, \cdots, n-1 \end{cases}$$

$$\lambda(e_i) = \begin{cases} (2n-3) - 2i & \text{if } i \text{ is odd and } i < n \\ 2i-1 & \text{if } i \text{ is even} \\ 2n-1 & \text{if } i = n \end{cases}$$

In both the cases, the total weights of vertices are given by

$$WT(v_i) = \begin{cases} 6n & \text{if } i = 1 \\ (2n+4) + 4(i-2) & \text{if } 2 \le i \le n. \end{cases}$$

Obviously, the total weights of vertices of C_n are $6n, 2n+4, 2n+8, \cdots, 6n-4$ respectively. Which form arithmetic progression with initial value $a = 2n+4$ (attained at v_2) and difference $d = 4$. □



(A) (11,3)-TNAL of C_3 (B) (16,3)-TNAL of C_5 (C) (21,3)-TNAL of C_7

FIGURE 1. $(\frac{5n+7}{2}, 3)$-TNAL of cycle C_n for $n = 3, 5, 7$

We have shown $(11, 3)$, $(16, 3)$ and $(21, 3)$-total neighborhood-antimagic labeling of cycles C_3, C_5 and C_7 in Figure 1 $(a), (b)$ and (c) respectively. However, it is difficult to determine a unique labeling pattern which admits $(\frac{5n+7}{2}, 3)$-total neighborhood-antimagic labeling for general cycle C_n(odd length). To determine the general pattern of $(\frac{5n+7}{2}, 3)$-total neighborhood-antimagic labeling of odd cycles, we state as an open problem.

Open Problem 1. *For odd $n \geq 3$, the cycle C_n has $(\frac{5n+7}{2}, 3)$-total neighborhood-antimagic labeling.*

4. Cycles with even length

In this section, we investigate (a, d)-total neighborhood-antimagic labeling of cycle having even length. In particular, we calculate initial value a and difference d for even cycles. Further, we discuss $(12, 4)$ and $(15, 2)$-TNAL for a cycle C_4. Later on, we show that there does not exist any (a, d)-total neighborhood-antimagic labeling where $d = 1, 3$ and 5 for even cycle C_n.

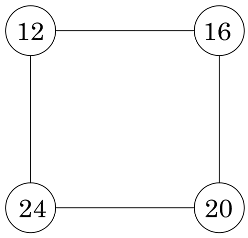
Consider the even cycle C_n where $n = 2k, k \in \mathbb{N}$, we have $v = e = 2k$. Substituting these values in Equation (2.4), we get

$$2a + (2k-1)d = 16k + 4. \tag{4.1}$$

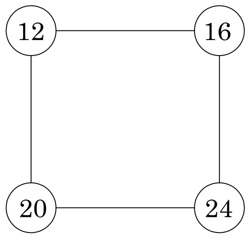
Clearly, the particular solution of above equation is $a = 6$ and $d = 8$. The set of all solutions is given by $a = 6 + (2k-1)t$ and $d = 8 - 2t$. The set of all positive solutions can be obtained by taking $t \in \{-1, 0, 1, 2, 3\}$. From Theorem 2.1 it follows that, the only possible positive solutions are (i) $a = 4k + 4, d = 4$ (ii) $a = 6k + 3, d = 2$.

Theorem 4.1. *The cycle C_4 does not admit a $(12, 4)$-total neighborhood-antimagic labeling.*

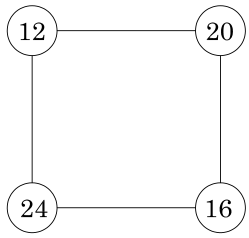
Proof. Let us denote the vertex set and edge set of cycle C_4 as $V(C_4) = \{v_i | i = 1, 2, 3, 4\}$ and $E(C_4) = \{e_i | i = 1, 2, 3, 4\}$. If we think about one-one map from $f : V(C_4) \cup E(C_4) \to \{1, 2, \cdots, 8\}$ then there are $8! = 40,320$ such maps, which may or may not satisfy our requirement. That means, out of these we are interested in those maps for which total weights of vertices form $(12, 4)$-arithmetic progression. The theory of permutation tells us the following six distinct arrangements, which form $(12, 4)$-arithmetic progression according to its total weights:



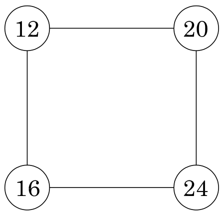
(A) Arrangement A



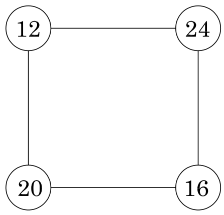
(B) Arrangement B



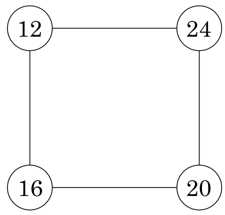
(C) Arrangement C



(D) Arrangement D



(E) Arrangement E



(F) Arrangement F

If there exists $(12, 4)$-total neighborhood-antimagic labeling of cycle C_4 then it must follow one of the above six arrangement. By selecting an arrangement A, we assign the labels to vertices and edges. For the sake of convenience, let the total weights of vertices be $WT(v_1) = 12, WT(v_2) = 16, WT(v_3) = 20$ and $WT(v_4) = 24$. The choices of partitions for these weights are shown in Table 1. Using the choices of partitions, we assign the labels to the vertices v_2 and v_4. There are six choices to select these two labels from the selected partitions. Let us select first partition of 12 from the table 1, then we have six choices namely (1) $1, 2$ (2) $1, 3$ (3) $1, 6$ (4) $2, 3$ (5) $2, 6$ and (6) $3, 6$ to label the vertices v_2 and v_4. Among these six choices, we can easily eliminate two choices namely (1) and (2) as there is no partition of $WT(v_3) = 20$ consisting $1, 2$ and $1, 3$. In the remaining choices, if we try to assign the labels to the other vertices and edges using the choices of partitions then any of the one label will be repeated. Thus, such an arrangement is not possible for first partition (i) $1 + 2 + 3 + 6$. Now, we repeat the same process for second partition (ii) $1 + 2 + 4 + 5$ of 12. One may also agree that in partition (ii) such an arrangement is again not possible. Hence, it eliminate the arrangement A. So, under this arrangement A there is no $(12, 4)$-TNAL exist.

TABLE 1. Choices of Partition for total weights

Sr. No.	Total Weights $WT(v)$	Choices of Partitions
1	12	(i) $1+2+3+6$ (ii) $1+2+4+5$
2	16	(i) $1+2+5+8$ (ii) $1+2+6+7$ (iii) $1+3+4+8$ (iv) $1+3+5+7$ (v) $1+4+5+6$ (vi) $2+3+4+7$ (vii) $2+3+5+6$
3	20	(i) $1+4+7+8$ (ii) $1+5+6+8$ (iii) $2+3+7+8$ (iv) $2+4+6+8$ (v) $2+5+6+7$ (vi) $3+4+5+8$ (vii) $3+4+6+7$
4	24	(i) $3+6+7+8$ (ii) $4+5+7+8$

Similarly, one can eliminate arrangement B and C. Note that arrangement A & F, B & D and C & E are similar. Since, we are failed to arrange the weight 16 in arrangement A, then because of symmetry of C_4 arrangement F is also not possible. Similarly, the arrangements D and E are also eliminated. Thus, C_4 does not admit $(12, 4)$-total neighborhood-antimagic labeling. □

Remark 4.1. *Note that the Theorem 4.1 shows that the conditions given Theorem 2.1 is necessary but not sufficient.*

Using the above methodology described in the proof of Theorem 4.1, the $(15, 2)$-total neighborhood-antimagic labeling of C_4 has shown in Figure 2.2.

Theorem 4.2. *For even n, there does not exist any integer a such that the cycle C_n admit $(a, 1)$-total neighborhood-antimagic labeling.*

Proof. Let $n = 2k, k \in \mathbb{N}$. Taking $d = 1$ in Equation (4.1), we have $a = \frac{14k+5}{2}$, which is not an integer for any value of k. □

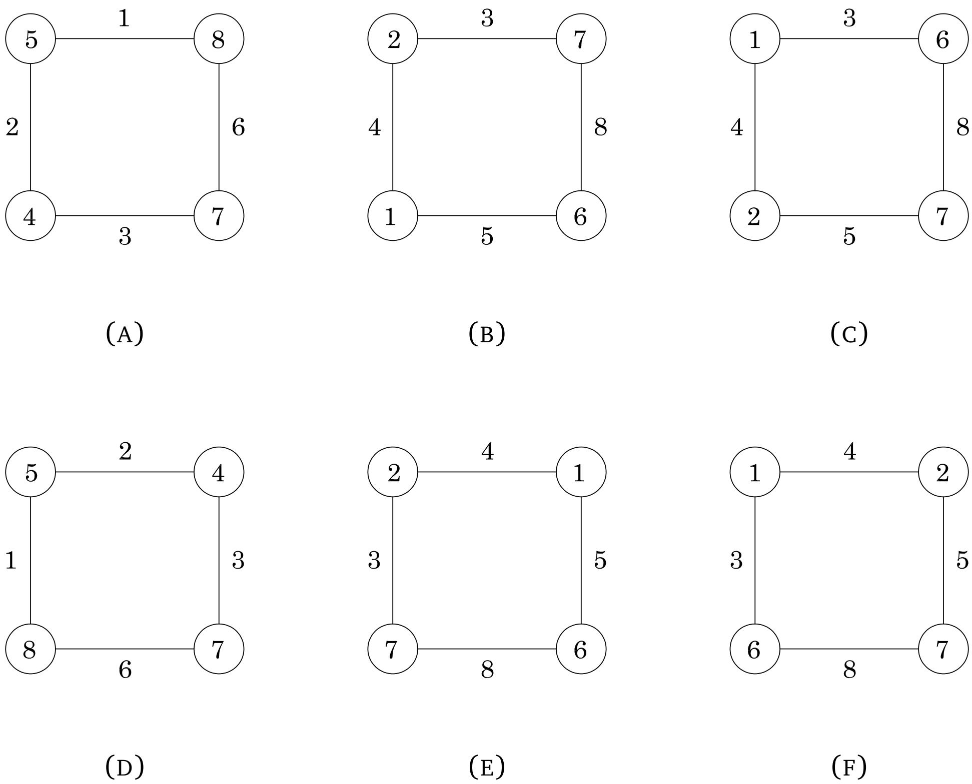


FIGURE 3. Different $(15, 2)$-TNAL of cycle C_4

Theorem 4.3. *For even n, there does not exist any integer a such that the cycle C_n admit $(a, 3)$-total neighborhood-antimagic labeling.*

Proof. Let $n = 2k, k \in \mathbb{N}$. Taking $d = 3$ in Equation (4.1), we have $a = \frac{10k+7}{2}$, which is not an integer for any value of k. □

Theorem 4.4. *For even n, there does not exist any integer a such that the cycle C_n admit $(a, 5)$-total neighborhood-antimagic labeling.*

Proof. Let $n = 2k, k \in \mathbb{N}$. Taking $d = 5$ in Equation (4.1), we have $a = \frac{6k+9}{2}$, which is not an integer for any value of k. □

Open Problem 2. *For the cycle C_n $(n > 4)$, determine if there is an (a, d)-total neighborhood-antimagic labeling for (i) $a = 3n+3, d = 2$ and (ii) $a = 2n+4, d = 4$.*

5. Dual Labeling

For given TNAL on a graph $G = (V, E)$, one can create another TNALs from it. If $\lambda : V \cup E \to \{1, 2, \cdots v + e\}$ is a injective then the dual λ' of λ on $V \cup E$ is defined as follows:

$$\lambda(x) = v + e + 1 - \lambda(x),\ x \in V$$

$$\lambda(xy) = v + e + 1 - \lambda(xy),\ xy \in E$$

In general, the concept of finding dual labeling with the help of existing labeling is only true for regular graphs, which was introduced in [7] for edge-magic total labeling. The same concept also holds for TNAL. Let us see the following theorem, which provide us different labeling for the same graph.

Theorem 5.1. *The dual of (a, d)-total neighborhood-antimagic labeling for a graph G is an (a', d)-total neighborhood-antimagic labeling for some a' if and only if G is regular.*

Proof. Let λ is an (a, d)-total neighborhood-antimagic labeling for $G = (V, E)$. The set $W_\lambda = \{WT_\lambda(x)|x \in V\} = \{a, a + d, \cdots, a + (v - 1)d\}$ denote the set of total weights at each vertex x in a graph G, which is in arithematic progression. Now, for any vertex $x \in V$ we have

$$\begin{aligned} WT_{\lambda'}(x) &= \sum_{y \in N(x)} \lambda'(y) + \sum_{xy \in E} \lambda'(xy) \\ &= \sum_{y \in N(x)} [(v + e + 1) - \lambda(y)] + \sum_{xy \in E} [(v + e + 1) - \lambda(xy)] \\ &= \sum_{y \in N(x)} (v + e + 1) + \sum_{xy \in E} (v + e + 1) - \left(\sum_{y \in N(x)} \lambda(y) + \sum_{xy \in E} \lambda(xy) \right) \\ &= (v_x + e_x)(v + e + 1) - WT_\lambda(x) \end{aligned}$$

where v_x is the number of vertices adjacent to x and e_x is the edges incident to vertex x. Obviously, the set $W_{\lambda'} = \{WT_{\lambda'}(x)|x \in V\}$ contains arithmetic progression with same difference d if and only if v_x and e_x is constant for every

x. That means this is possible only when G is regular graph. This completes the poof. □

Corollary 5.1. *Let G be a regular graph of degree s. Then G has a (a, d)-total neighborhood-antimagic labeling if and only if G has an (a', d)-total neighborhood-antimagic total labeling for $a' = 2s(v + e + 1) - a - (v - 1)d$*

Proof. The proof follows from the above theorem. □

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REFERENCES

[1] N. HARTSFIELD AND G. RINGEL: *Pearls in Graph Theory*, Academic Press, BostonSan Diego-New York-London, 1990.

[2] R. BODENDIEK AND G. WALTHER: *Arithmetisch antimagische graphen in K. Wagner and R. Bodendiek, Graphentheorie III*, BI-Wiss. Verl., MannheimLeipzig-Wien-Zürich, 1993.

[3] M. BAČA, F. BERTAULT, J. MACDOUGALL, M. MILLER, R. SIMANJUNTAK AND SLAMIN: *Vertex-antimagic total labelings of graphs,* Discuss. Math. Graph theory **23** (2003), 67–83.

[4] R. SIMANJUNTAK, F. BERTAULT AND M. MILLER: *Two new (a, d)-antimagic graph labelings* in Proc. of Eleventh Australasian Workshop on Combinatorial Algorithms, 2000, 179–189.

[5] J. A. GALLIAN: *A dynamic survey of graph labeling*, The Electronic Journal of Combinatorics, 2019, # DS6.

[6] D. M. BURTON: *Elementary Number Theory*, McGraw-Hill Publishers, Seventh Edition, 2010.

[7] W. D. WALLIS: *Magic Graphs*, 2001.

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