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m-ZUMKELLER GRAPHS

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ABSTRACT. A positive integer n is an m-Zumkeller number if the positive divisors of n can be partitioned into two disjoint subsets of equal product. In this paper, we provide algorithms to label complete bipartite graphs and wheel graphs by m-Zumkeller numbers.

1. INTRODUCTION

A positive integer n is perfect if it is the sum of all of its proper positive divisors. Generalizing this concept, R.H. Zumkeller defined a new type of number called Zumkeller number, which is a positive integer such that we can partition the positive divisors of the integer into two disjoint subsets of equal sum. Various properties of Zumkeller numbers are studied in [3]. In [1] the authors had provided algorithms for Zumkeller labeling of complete bipartite graphs and wheel graphs.

A positive integer n is called an m-Zumkeller number [2] if the positive divisors of n can be partitioned into two disjoint subsets of equal product. In this paper, we provide algorithms to label complete bipartite graphs and wheel graphs by m-Zumkeller numbers.

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2. Properties Of m-Zumkeller Numbers

Various properties of m-Zumkeller numbers discussed in [2] are given below-

(1) If n is an m-Zumkeller number, then $\tau(n) \geq 4$, where $\tau(n)$ gives the number of positive divisors of n.

(2) The integer $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ (where $p_i^{'s}$ are distinct primes) is an m-Zumkeller number if and only if $4|\alpha_i \tau(n) \ \ \forall\, i = 1, 2, \ldots, r$.

(3) The product of distinct prime numbers i.e. $\prod_{i=1}^{r} p_i$ (where $p_i^{'s}$ are distinct primes, $r \geq 2$) are always m-Zumkeller numbers.

(4) If p is prime then p^α is an m-Zumkeller number if and only if $\alpha \equiv 0 \ (mod\, 4)$ or $\alpha \equiv 3 \ (mod\, 4)$.

Example 1. *The integers* $6, 8, 10, 14, 15, 16, 21, 22$ *are the first few* m*-Zumkeller numbers.*

For 16*, the positive divisors of* 16 *are* $1, 2, 4, 8, 16$*. This divisors of* 16 *can be partitioned into two subsets,* $P = \{1, 2, 16\}$ *and* $Q = \{4, 8\}$ *such that the product of all the elements in each subset is 32. Hence,* 16 *is an* m*-Zumkeller number.*

3. Main Results

In this section we define m-Zumkeller graph and provide algorithms to label complete bipartite graphs and wheel graphs by m-Zumkeller numbers.

Definition 3.1. *Let* $G = (V, E)$ *be a graph. A one-one function* $f : V \longrightarrow N$ *is said to be an* m*-Zumkeller labeling of the graph G, if the induced function* $f^* : E \longrightarrow N$ *defined as* $f^*(xy) = f(x) f(y)$ *is an* m*-Zumkeller number for all* $xy \in E$ *and* $x, y \in V$*.*

Definition 3.2. *A graph is said to be an* m*-Zumkeller graph if it admits an* m*-Zumkeller labeling.*

Example 2. *The* m*-Zumkeller labeling of a graph is given in the figure 1.*

Here, all the integers on each edges are m*-Zumkeller numbers.*

Proposition 3.1. *A non-totally disconnected subgraph of an* m*-Zumkeller graph is an* m*-Zumkeller graph.*

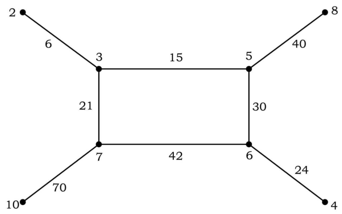

FIGURE 1. m-Zumkeller graph

Proof. Let G be an m-Zumkeller graph and $G^{'}$ be its subgraph such that $G^{'}$ is not a totally disconnected graph. Since the edges of $G^{'}$ are labeled by multiplying the labels of the end vertices of the edges in G, hence the non-totally disconnected subgraph $G^{'}$ is an m-Zumkeller graph. □

4. m-ZUMKELLER LABELING ALGORITHM FOR COMPLETE BIPARTITE GRAPH

Input: A complete bipartite graph $K_{p,q}$ having $p+q$ vertices and mn edges.
Output: m-Zumkeller complete bipartite graph.
Procedure: $mZum\_lab\_complete\ bipartite\ graph$.
$U = \{u_i|1 \leq i \leq p\}$ and $V = \{v_j|1 \leq j \leq q\}$ be the two disjoint vertex sets of the graph $K_{m,n}$.
$p_1 :=$ a prime number $\neq 2 < 10$
$p_2 :=$ a prime number $\neq 2 < 10$
$p_1 \neq p_2$
for $i := 1$ to p do
    $f(u_i) = p_1 2^i$
for $j := 1$ to q do
    $f(v_j) = p_2 2^j$
if $i \neq j$ then
begin
  for $i := 1$ to p do
    begin
      for $j := 1$ to q do
        $f^*(u_i v_j) = f(u_i) f(v_j)$
        end
    end

else

$f^*(u_i v_i) = f(u_i) f(v_i)$

end $Zum\_lab\_complete\ bipartite\ graph$.

Proposition 4.1. *The complete bipartite graph $K_{p,q}$ is an m-Zumkeller graph.*

Proof. Let $U = \{u_1, u_2, \ldots, u_p\}$ and $V = \{v_1, v_2, \ldots, v_q\}$ be the two disjoint vertex sets of the complete bipartite graph $K_{p,q}$ and $E = \{e_{ij} = u_i v_j | 1 \le i \le p, 1 \le j \le q\}$ be the edge set of $K_{p,q}$.

Let $f : W \longrightarrow N$ such that

$$\begin{aligned} f(u_i) &= p_1 2^i \quad \forall u_i \in U, \quad 1 \le i \le p \\ f(v_j) &= p_2 2^j \quad \forall v_j \in V, \quad 1 \le j \le q\,, \end{aligned}$$

where p_1 and p_2 are distinct prime numbers and $2 < p_1, p_2 \le 10$.

Define an induced function $f^* : E \longrightarrow N$ such that

$$\begin{aligned} f^*(e_{ij}) =& f^*(u_i v_j)\,, \qquad 1 \le i \le p, \quad 1 \le j \le q \\ =& f(u_i) f(v_j)\,. \end{aligned}$$

Now if $i = j$ then

$f^*(e_{ij}) = f^*(u_i v_j) = f(u_i) f(v_j) = p_1 p_2 2^{2i}$ is clearly an m-Zumkeller number since $\tau(p_1 p_2 2^{2i})$ is a multiple of 4.

Again if $i \ne j$ then

$f^*(e_{ij}) = f^*(u_i v_j) = f(u_i) f(v_j) = p_1 p_2 2^{i+j}$ is also clearly an m-Zumkeller number since $\tau(p_1 p_2 2^{i+j})$ is a multiple of 4.

Hence, $K_{p,q}$ is an m-Zumkeller graph. □

Corollary 4.1. *Every bipartite graph allows an m-Zumkeller labeling.*

Proof. Since every bipartite graph is a non-totally disconnected subgraph of a complete bipartite graph, the result follows. □

Example 3. *The complete bipartite graph $K_{6,3}$ is an m-Zumkeller graph for $p_1 = 3$ and $p_2 = 5$ which is given in the figure 2.*

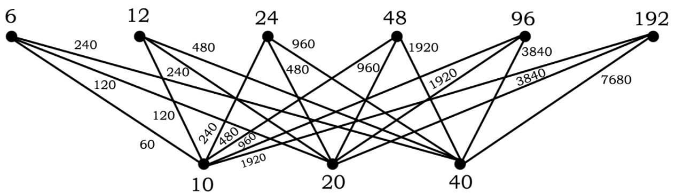

FIGURE 2. m-Zumkeller labeling of $K_{6,3}$

5. m-ZUMKELLER LABELING ALGORITHM FOR WHEEL GRAPH

Input: A wheel graph $W_n = K_1 + C_n$
Output: m-Zumkeller wheel graph
Procedure: $Zum\_lab\_wheel\ graph\, W_n = K_1 + C_n$.
Let w_0 be the central vertex of the wheel graph W_n
// $w_1, w_2, \ldots, w_n$ be the vertices of C_n in W_n
// $E = \{e_i = w_i w_{i+1} | 1 \le i \le n-1\} \bigcup \{e_n = w_n w_1\} \bigcup \{e_i^{'} = w_0 w_i | 1 \le i \le n\}$ be the edge set of the wheel graph W_n
$p_1 :=$ a prime number $\neq 2 < 12$
$p_2 :=$ a prime number $\neq 2 < 12$
$p_3 :=$ a prime number $\neq 2 < 12$
$p_4 :=$ a prime number $\neq 2 < 12$
$p_1 \neq p_2$, $p_1 \neq p_3$,$p_1 \neq p_4$,$p_2 \neq p_3$,$p_2 \neq p_4$,$p_3 \neq p_4$
$f(w_0) = 2p_1$// f is an injective function on the vertex set of W_n
If $n \equiv 1 \pmod 2$ then
  begin
    for $i = 1, 3, 5, \ldots, n-2$ do
      begin
        $f(w_i) = p_2 2^{\frac{i+1}{2}}$
        $f(w_{i+1}) = p_3 2^{\frac{i+1}{2}}$
      end
    if $i = n$ then
      $f(w_n) = 2p_4$
  end
else
  begin
    for $i = 1, 3, 5, \ldots, n-1$ do

begin
$f\left(w_i\right)=p_2 2^{\frac{i+1}{2}}$
$f\left(w_{i+1}\right)=p_3 2^{\frac{i+1}{2}}$
end
end
end $Zum\_lab\_wheel\ graph\ W_n=K_1+C_n$.

Proposition 5.1. *The wheel graph $W_n=K_1+C_n$ is an m-Zumkeller graph.*

Proof. Let w_0 be the central vertex and $w_1,w_2,\ldots w_n$ be the rim vertices of W_n. Let $E=\{e_i=w_iw_{i+1}|1\leq i\leq n-1\}\cup\{e_n=w_nw_1\}\cup\{e_i^{'}=w_0w_i|1\leq i\leq n\}$ be the edge set of the wheel graph W_n . Now we have the following two cases:

Case (1) If $n\equiv 1 \pmod 2$.

Define an injective function $f:V\longrightarrow N$ such that

$$\begin{aligned} f\left(w_0\right)=&2p_1\\ f\left(w_i\right)=&p_2 2^{\frac{i+1}{2}}\ , i=1,3,5,\ldots,n-2\\ f\left(w_{i+1}\right)=&p_3 2^{\frac{i+1}{2}},\ i=1,3,5,\ldots,n-2\\ f\left(w_n\right)=&2p_4\ , \end{aligned}$$

where p_1,p_2,p_3,p_4 are distinct primes such that $2<p_i<12,\ i=1,2,3,4$.

Define an induced function $f^*:E\longrightarrow N$ such that
$f^*\left(e_i\right)=f^*\left(w_iw_{i+1}\right)=f\left(w_i\right)f\left(w_{i+1}\right),\ \ 1\leq i\leq n-2$
$f^*\left(e_{n-1}\right)=f^*\left(w_{n-1}w_n\right)=f\left(w_{n-1}\right)f\left(w_n\right)$,
$f^*\left(e_n\right)=f^*\left(w_nw_1\right)=f\left(w_n\right)f\left(w_1\right)$ and
$f^*\left(e_i^{'}\right)=f^*\left(w_0w_i\right)=f\left(w_0\right)f\left(w_i\right),\ \ 1\leq i\leq n$.

Now, we have to show that the numbers labeled in each edge of W_n is an m-Zumkeller number.

(i) $f^*\left(e_i\right)=f^*\left(w_iw_{i+1}\right)=f\left(w_i\right)f\left(w_{i+1}\right)=p_2 2^{\frac{i+1}{2}}p_3 2^{\frac{i+1}{2}}=p_2p_3 2^{i+1}$, $1\leq i\leq n-2$, which is an m-Zumkeller number since $\tau(p_2p_32^{i+1})=4\left(i+2\right)$.

(ii) $f^*\left(e_{n-1}\right)=f^*\left(w_{n-1}w_n\right)=f\left(w_{n-1}\right)f\left(w_n\right)=p_3 2^{\frac{n-1}{2}}2p_4=p_3p_4 2^{\frac{n+1}{2}}$, which is an m-Zumkeller number since $\tau\left(p_3p_4 2^{\frac{n+1}{2}}\right)=4\times 2^{\frac{n+3}{2}}$.

(iii) $f^*\left(e_n\right)=f^*\left(w_nw_1\right)=f\left(w_n\right)f\left(w_1\right)=2p_4 2p_2=4p_2p_3$, which is also an m-Zumkeller number since $\tau(4p_2p_3)=12$.

(iv) If i is an odd number then

$f^*\left(e_i^{'}\right) = f^*\left(w_0w_i\right) = f\left(w_0\right)f\left(w_i\right) = p_1p_2 2^{\frac{i+3}{2}}$, which is also an m-Zumkeller number since $\tau\left(p_1p_2 2^{\frac{i+3}{2}}\right)$ is a multiple of 4.

And $f^*\left(e_{i+1}^{'}\right) = f\left(w_0\right)f\left(w_{i+1}\right) = p_1p_3 2^{\frac{i+3}{2}}$,which is also an m-Zumkeller number as $\tau\left(p_1p_3 2^{\frac{i+3}{2}}\right)$ is a multiple of 4.

(v) $f^*\left(e_n^{'}\right) = f\left(w_0\right)f\left(w_n\right) = 4p_1p_4$, which is also an m-Zumkeller number since $\tau(4p_1p_4) = 12$.

Case(2) If $n \equiv 1 \pmod 2$.

Define an injective function $f : V \longrightarrow N$ such that

$$f\left(w_0\right) = 2p_1$$
$$f\left(w_i\right) = p_2 2^{\frac{i+1}{2}}, i = 1, 3, 5, \ldots, n-1$$
$$f\left(w_{i+1}\right) = p_3 2^{\frac{i+1}{2}}, \ i = 1, 3, 5, \ldots, n-1$$

where p_1, p_2, p_3 are distinct primes such that $2 < p_i < 12, \ i = 1, 2, 3,$.

Define an induced function $f^* : E \longrightarrow N$ such that

$f^*\left(e_i\right) = f^*\left(w_iw_{i+1}\right) = f\left(w_i\right)f\left(w_{i+1}\right), \ 1 \le i \le n-1$

$f^*\left(e_n\right) = f^*\left(w_nw_1\right) = f\left(w_n\right)f\left(w_1\right)$ and

$f^*\left(e_i^{'}\right) = f^*\left(w_0w_i\right) = f\left(w_0\right)f\left(w_i\right), \ 1 \le i \le n-1$

From Case (1), we can say that for $1 \le i \le n-1$, $f^*\left(e_i\right)$ and for $1 \le i \le n$, $f^*\left(e_i^{'}\right)$ is an m-Zumkeller number.

Also, $f^*\left(e_n\right) = f\left(w_n\right)f\left(w_1\right) = p_3 2^{\frac{n}{2}} 2p_2 = 2^{\frac{n+2}{2}} p_2p_3$ is an m-Zumkeller number.

Hence, we can conclude that the wheel graph $W_n = K_1 + C_n$ is an m-Zumkeller graph.

□

Example 4. *For $n = 6$ the wheel graph W_6 is an m-Zumkeller graph, its m-Zumkeller labeling with $p_1 = 3$, $p_2 = 5$, $p_3 = 7$ is given in figure 3.*

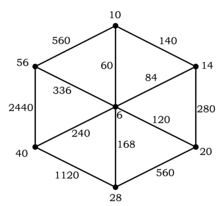

FIGURE 3. m-Zumkeller labeling of W_6

Example 5. *For $n = 7$ the wheel graph W_7 is an m-Zumkeller graph, its m-Zumkeller labeling with $p_1 = 3$, $p_2 = 5$, $p_3 = 7, p_4 = 11$ is given in figure 4.*

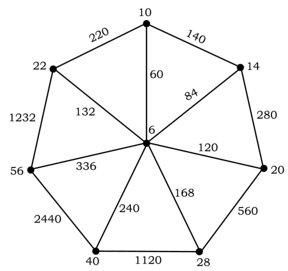

FIGURE 4. m-Zumkeller labeling of W_7

Corollary 5.1. *A cycle C_n admits an m-Zumkeller labeling.*

Proof. Since the cycle C_n obtained by removing the central vertex of W_n is a non-totally disconnected subgraph of a wheel graph W_n, the result follows. □

Corollary 5.2. *The path P_n with n vertices is an m-Zumkeller graph.*

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