ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.7, 4721–4727
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.7.38

AVERAGE DETOUR D-DISTANCE IN GRAPHS

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ABSTRACT. The average distance is one of the important parameters in graph theory. This article deals with the average distance between vertices using detour D-distance. We obtain some results by comparing the average detour D-distance of two graphs. Further, we work out the average detour D-distance of some families of graphs.

1. INTRODUCTION

The average distance is most significant parameters in graph theory. The applications of average distance are networks or in general good connection of networks. It is used as a tool of analytic network where the performance time is proportional to the distance between two points. In [2], Reddy Babu and Varma have introduced the representation of D-distance and extended to average D-distance between vertices in [3]. In previous article, the first two authors have introduced the idea of detour D-distance and some work related, see [4,5]. In present article we investigate the average D-distance between vertices using detour distance. Next, in section 2, we obtain some results by comparing the average detour D-distances of two graphs. In section 3, we work out the average detour D-distance of various families of graphs. In any connected graph contains n vertices, the total detour D-distance (abbreviated as

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2010 *Mathematics Subject Classification.* 05C12.

Key words and phrases. Detour distance, Average Detour distance, Detour diameter.

TDDD)is the addition of the detour distances among all possible pairs of vertices, i.e.,$TDDD = \sum_{u,v} D^D(u,v)$. The average detour D-distance is given by $\mu_D^D = \frac{TDDD}{n(n-1)} = \frac{1}{2}(\frac{TDDD}{n_{C_2}})$. The detour D-distance representation as a matrix of of the graph representation by $M_D^D(G)$, as $M_D^D(G) = [D^D(u_i,u_j)]_{n\times n}$ where $D^D(u_i,u_j)$ is the detour D-distance between the vertices u_i and u_j. Clearly, the detour D-distance matrix is symmetric matrix with order $n \times n$.All diagonal entices being zero. The average degree of the graph is given by $d(G) = \frac{1}{|V|} \sum\limits_{v\epsilon V} deg(v)$ where means degree of the in graph . Throughout this article, graph mean simple, connected, finite and undirected graph. For any inexplicable notations and terms we refer [1].

2. RESULTS ON AVERAGE DETOUR D-DISTANCE

We begin with a theorem and later give some consequences.

Theorem 2.1. *Let two graphs G_1 and G_2 having the same number of vertices and $diam_D^D(G_1) < diam_D^D(G_2)$. If $|E_1| < |E_2|$ then $\mu_D^D(G_1) < \mu_D^D(G_2)$.*

Proof. Since $diam_D^D(G_1) < diam_D^D(G_2)$, the biggest entry in the detour D-matrix of G_1 is less than G_2 and this causes total detour D-distance is to increase. Because the same order and the number of edges of G_1 is less than the number of edges of G_2, hence $\mu_D^D(G_1) < \mu_D^D(G_2)$. □

Theorem 2.2. *Let two graphs G_1 and G_2 having the same number of vertices and $diam_D^D(G_1) < diam_D^D(G_2)$.if $\delta(G_1) < \delta(G_2)$ then $\mu_D^D(G_1) < \mu_D^D(G_2)$*

Proof. Let G_1 and G_2 be two graphs having the same number of vertices and $diam_D^D(G_1) < diam_D^D(G_2)$. Then clearly $\delta(G_1) < \delta(G_2) \Rightarrow |E_1| < |E_2|$ then from Theorem 2.1, $\mu_D^D(G_1) < \mu_D^D(G_2)$. □

Theorem 2.3. *Let two graphs G_1 and G_2 having the same number of vertices and $diam_D^D(G_1) < diam_D^D(G_2)$. If the mean degree of G_1 is less than mean degree of G_2 then $\mu_D^D(G_1) < \mu_D^D(G_2)$*

Proof. Let G_1 and G_2 be two graphs having the same number of vertices and $diam_D^D(G_1) < diam_D^D(G_2)$.We have by definition, $|E| = \frac{1}{2}\sum deg(v) = \frac{1}{2}\, deg(G)$.As the graphs have same number of vertices and mean degree of G_1 is less than mean degree of G_2 we have $deg_D^D(G_1) < deg_D^D(G_2)$ and hence $|E_1| < |E_2|$. Thus by Th 2.1, hence we conclude that $\mu_D^D(G_1) < \mu_D^D(G_2)$. □

Theorem 2.4. *Let S is a spanning subgraph of G,$\mu_D^D(S) < \mu_D^D(G)$.*

Proof. Consider S be a spanning subgraph of G,Then S and G are same order and $|E(S)| < |E(G)|$. Thus by Theorem 2.1, hence we conclude that $\mu_D^D(S) < \mu_D^D(G)$. □

3. Average detour D-distance of various families of graphs

In this section we compute the average detour D-distance of various families of graphs. We start on with complete graph.

Theorem 3.1. *The average detour D-distance of K_n is $n^2 - 1$.*

Proof. In a complete graph, every vertex has $n - 1$ adjacent vertices. The detour D-distance between every pair of vertices is $n^2 - 1$. Thus the total detour D-distance (TDDD) is $2(n_{C_2})(n^2 - 1)$. Hence the average detour D-distance, $\mu_D^D(K_n) = \frac{1}{2}(\frac{TDDD}{n_{C_2}}) = n^2 - 1$. □

Next we compute the detour D-distance of a wheel graph.

Theorem 3.2. *In a wheel graph, the average detour D-distance is $5n$.*

Proof. Consider the wheel graph, $W_{1,n}$, on $n+1$ vertices $\{v_0, v_1, v_2, ..., v_n\}$.Assume that, without loss of generality, v_0 is adjacent to all other vertices. Then degree of and degree of $v_0 = n$ and degree of all other vertices is 3. The detour D-distance between any pair of vertices is $5n$. Thus the total detour D-distance (TDDD) is $2(n + 1_{C_2})(5n)$. Hence the average detour D-distance, $\mu_D^D(W_{1,n}) = \frac{1}{2}(\frac{TDDD}{n+1_{C_2}}) = 5n$. □

Next we consider cyclic graph.

Theorem 3.3. *In cyclic graph, the average detour D-distance is*

$$\mu_D^D(C_n) = \begin{cases} \frac{9n^2-4n+8}{4(n-1)} & if\ n\ is\ even \\ \frac{9n+5}{4} & if\ n\ is\ odd \end{cases}$$

Proof. In a cyclic graph C_n, with vertices, each vertex has two adjacent vertices. We consider independently cases if $n$even and odd.

Case 1: n is even Detour D-distances between pairs of vertices are as shown below:

	v_1	v_2	v_3	...	$v_{\frac{n}{2}-1}$	$v_{\frac{n}{2}}$	$v_{\frac{n}{2}+1}$	$v_{\frac{n}{2}+2}$	...	v_{n-1}	v_n
v_1	0	$3n-1$	$3n-4$	...	$\frac{3n+16}{2}$	$\frac{3n+10}{2}$	$\frac{3n+4}{2}$	$\frac{3n+10}{2}$	...	$3n-4$	$3n-1$
v_2	$3n-1$	0	$3n-1$	...	$\frac{3n+22}{2}$	$\frac{3n+16}{2}$	$\frac{3n+4}{2}$	$\frac{3n+4}{2}$	...	$3n-7$	$3n-4$
v_3	$3n-4$	$3n-1$	0	...	$\frac{3n+28}{2}$	$\frac{3n+22}{2}$	$\frac{3n+16}{2}$	$\frac{3n+10}{2}$	...	$3n-10$	$3n-7$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$v_{\frac{n}{2}-1}$	$\frac{3n+16}{2}$	$\frac{3n+22}{2}$	$\frac{3n+28}{2}$	...	0	$3n-1$	$3n-4$	$3n-7$	...	$\frac{3n+4}{2}$	$\frac{3n+10}{2}$
$v_{\frac{n}{2}}$	$\frac{3n+10}{2}$	$\frac{3n+16}{2}$	$\frac{3n+22}{2}$	...	$3n-1$	0	$3n-1$	$3n-4$	...	$\frac{3n+10}{2}$	$\frac{3n+4}{2}$
$v_{\frac{n}{2}+1}$	$\frac{3n+4}{2}$	$\frac{3n+10}{2}$	$\frac{3n+16}{2}$	...	$3n-4$	$3n-1$	0	$3n-1$	...	$\frac{3n+16}{2}$	$\frac{3n+10}{2}$
$v_{\frac{n}{2}+2}$	$\frac{3n+10}{2}$	$\frac{3n+4}{2}$	$\frac{3n+10}{2}$	...	$3n-7$	$3n-4$	$3n-1$	0	...	$\frac{3n+22}{2}$	$\frac{3n+16}{2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{n-1}	$3n-4$	$3n-7$	$3n-10$	...	$\frac{3n+4}{2}$	$\frac{3n+10}{2}$	$\frac{3n+16}{2}$	$\frac{3n+22}{2}$	...	0	$3n-1$
v_n	$3n-1$	$3n-4$	$3n-7$	...	$\frac{3n+10}{2}$	$\frac{3n+4}{2}$	$\frac{3n+10}{2}$	$\frac{3n+16}{2}$	...	$3n-1$	0

Table: 1 Detour D-distance of cyclic graphs (n is even)

	v_1	v_2	v_3	...	$v_{\frac{n-3}{2}}$	$v_{\frac{n-1}{2}}$	$v_{\frac{n+1}{2}}$	$v_{\frac{n+3}{2}}$	...	v_{n-1}	v_n
v_1	0	$3n-1$	$3n-4$	...	$\frac{3n+13}{2}$	$\frac{3n+7}{2}$	$\frac{3n+7}{2}$	$\frac{3n+13}{2}$	...	$3n-4$	$3n-1$
v_2	$3n-1$	0	$3n-1$	...	$\frac{3n+19}{2}$	$\frac{3n+13}{2}$	$\frac{3n+7}{2}$	$\frac{3n+7}{2}$	...	$3n-7$	$3n-4$
v_3	$3n-4$	$3n-1$	0	...	$\frac{3n+25}{2}$	$\frac{3n+19}{2}$	$\frac{3n+13}{2}$	$\frac{3n+7}{2}$	...	$3n-10$	$3n-7$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$v_{\frac{n-3}{2}}$	$\frac{3n+13}{2}$	$\frac{3n+19}{2}$	$\frac{3n+25}{2}$	...	0	$3n-1$	$3n-4$	$3n-7$	...	$\frac{3n+7}{2}$	$\frac{3n+7}{2}$
$v_{\frac{n-1}{2}}$	$\frac{3n+7}{2}$	$\frac{3n+13}{2}$	$\frac{3n+19}{2}$	...	$3n-1$	0	$3n-1$	$3n-4$	...	$\frac{3n+13}{2}$	$\frac{3n+7}{2}$
$v_{\frac{n+1}{2}}$	$\frac{3n+7}{2}$	$\frac{3n+7}{2}$	$\frac{3n+13}{2}$	...	$3n-4$	$3n-1$	0	$3n-1$	...	$\frac{3n+19}{2}$	$\frac{3n+13}{2}$
$v_{\frac{n+3}{2}}$	$\frac{3n+13}{2}$	$\frac{3n+7}{2}$	$\frac{3n+7}{2}$	...	$3n-7$	$3n-4$	$3n-1$	0	...	$\frac{3n+25}{2}$	$\frac{3n+19}{2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{n-1}	$3n-4$	$3n-7$	$3n-10$	...	$\frac{3n+7}{2}$	$\frac{3n+13}{2}$	$\frac{3n+19}{2}$	$\frac{3n+25}{2}$	...	$3n-1$	0
v_n	$3n-1$	$3n-1$	$3n-7$	...	$\frac{3n+7}{2}$	$\frac{3n+7}{2}$	$\frac{3n+13}{2}$		...	$3n-1$	0

Table: 2 Detour D-distance of cyclic graphs (n is odd)

The sum of the elements in each row is

$$S_n = 2\frac{(\frac{n}{2}-1)}{2}[2(3n-1)+(\frac{n}{2}-1-1)(-3)]+(\frac{3n+4}{2})$$

$= \frac{n-2}{4}(9n+8) + (\frac{3n+4}{2}) = \frac{9n^2-4n-8}{4}$. There are n number of rows. Thus the total detour D-distance is $TDDD = \frac{n(9n^2-4n-8)}{4}$.
Hence the average detour D-distance $\mu_D^D(C_n) = \frac{1}{2}(\frac{TDDD}{n_{C_2}}) = \frac{(9n^2-4n+8)}{4(n-1)}$.
Case 2: n is odd
Detours D-distance between pairs of vertices are as shown in table 2.
The sum of the elements in each row is

$$S_n = [\frac{(\frac{n}{2}-1)}{2}[2(3n-1)+(\frac{n}{2}-1-1)(-3)]] = [\frac{n-1}{4}\frac{(9n+5)}{2}] = \frac{(9n+5)(n-1)}{8}.$$

There are n number of rows. Thus total detour D-distance, TDDD, is $n \times S_n = \frac{n(n-1)(9n+5)}{4}$. Hence the average detour D-distance $\mu_D^D(C_n) = \frac{1}{2}(\frac{TDDD}{n_{C_2}}) = \frac{(9n+5)}{4}$. □

Next we go through complete bipartite graph.

Theorem 3.4. *Let the graph be a complete bipartite graph,* $K_{m,n}(m < n)$*, the average D-distance is*

$$\mu_D^D(K_{m,n}) = \frac{n(n-1)(m^2+mn+3n)+2mn(m^2+mn+2m-1)+m(m-1)(m^2+mn+m-2)}{(m+n)(m+n-1)}.$$

Proof. The partition of the two vertex set of $K_{m,n}$ be able representation as $A \cup B$, where $A = \{v_1, v_2, v_3, ..., v_m\}$,$B = \{w_1, w_2, w_3, ..., w_n\}$. Then $D^D(v_i, v_j) = m^2 + mn + 3m$,$D^D(w_i, w_j) = m^2 + mn + m - 2$,$D^D(v_i, w_j) = m^2 + mn + 2m - 1$, see [4]. Thus the total detour D-distance is $TDDD = n_{C_2}(m^2 + mn + 3n) + mn(m^2 + 2m + mn - 1) + m_{C_2}(m^2 + mn + m - 2)$.
Hence the average detour D-distance

$$\mu_D^D(K_{m,n}) = \frac{1}{2}(\frac{TDDD}{(m+n)_{C_2}}) = \frac{(TDDD)}{(m+n)(m+n-1)}$$
$$= \frac{n(n-1)(m^2+mn+3n)+2mn(m^2+mn+2m-1)+m(m-1)(m^2+mn+m-2)}{(m+n)(m+n-1)}.$$

□

Theorem 3.5. *The average detour D-distance of,* $K_{m,m}$*, is*

$$\mu_D^D(K_{m,m}) = \frac{4m^3+m^2-4m+2}{2m-1}.$$

Proof. Let $K_{m,m}$ be a complete bipartite graph. The vertex set of $K_{m,m}$,can be written as $A \cup B$, where $A = \{v_1, v_2, v_3, ..., v_m\}$,$B = \{w_1, w_2, w_3, ..., w_m\}$.In the complete bipartite graph the detour D-distances between different pairs are $D^D(v_i, v_j) = 2m^2+m-2$,$D^D(v_i, w_j) = 2m^2+2m-1$. The total detour D-distance (TDDD) is twice the $m_{C_2}(2m^2 + m - 2) + m^2(2m^2 + 2m - 1)$.
Hence

$$\begin{aligned} \mu_D^D(K_{m,n}) &= \frac{1}{2}(\frac{TDDD}{2m_{C_2}}) \\ &= \frac{m_{C_2}(2m^2 + m - 2) + m^2(2m^2 + m - 1)}{(2m)(2m-1)} = \frac{4m^3 + m^2 - 4m + 2}{(2m-1)}. \end{aligned}$$

□

Now we consider graphs which are trees.

Theorem 3.6. *The average detour D-distance of the path graph is* $\mu_D^D(P_n) = \dfrac{2a_n}{n}$*, where* $a_n = a_{n-1} + n + 1$ *with* $a_1 = 0$ *.*

Proof. Let P_n,the detour D-distance between two vertices is same as the D-distance as there is a single pathway connecting any two vertices. Thus the outcome from Theorem 4.4 in [3]. □

Theorem 3.7. *In a star graph, the average detour D-distance is* $\mu_D^D(St_{1,n}) = \dfrac{2(n+2) + (n-1)(n+4)}{n-1}$.

Proof. In a star graph $St_{1,n}$,the detour D-distance is same as the D-distance. Thus the result follows from theorem 4.5 of [3]. □

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