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BILATERAL RELATION FOR GENERALIZED HYPERGEOMETRIC FUNCTION WITH SYLVESTER POLYNOMIAL

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ABSTRACT. In the field of special functions, bilateral generating relations play a vital role. In the present investigation, it is to obtain a bilateral generating relation of a Generalized Hypergeometric function with modified Generalized Sylvester polynomial. Some of the applications of it as special cases also discussed.

1. INTRODUCTION

Many of the special functions have been generalized by different authors. Namely, Cesaro polynomials, Laguerre polynomials, Sylvester polynomials etc. has been generalized recently and their generating relations also discussed. In [2, 3, 10, 11] authors defined a class of generalized hypergeometric function and obtained bilateral generating relations, extended linear generating relations and some integral results associated with it. In the same way [4, 9, 12] authors defined a modified Konhauser Polynomial and derived bilateral generating relations, also investigated a few integral results of it. Lahiri and Satyanarayana [6, 7] also generalized hypergeometric function using a difference operator and investigated bilateral generating relations.

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We define generalized hypergeometric function $B_n^{(\alpha,\beta)}(x,y,v)$ in the following manner, see [2, 3, 10, 11]:

(1.1)
$$B_n^{(\alpha,\beta)}(x,y,v) = \frac{\Gamma(n+\beta+1)\Gamma(1+n+\alpha)}{n!}\sum_{k=0}^{\infty}\frac{(-1)^k y^{[kv]} J_{n-k}^{(n)}(x,v)}{k!\;\Gamma(n-k+\alpha+1)\,\Gamma(k+\beta+1)}.$$

Here $J_n^{(n)}(x,v)$ is the modified-Jacobi polynomial.

The function in (1.1) can be cast in the form of a double sum using the definition of $J_n^{(\alpha)}(x,v)$ and considering the equations (1.3), (1.4), (1.5) as follows:

$$\begin{aligned} B_n^{(\alpha,\beta)}(x,y,v) &= \frac{(1+\beta)_n(1+\alpha)_n}{(n!)^2}\sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{(-n)_{k+l}\left(\frac{y}{v}\right)_k\left(\frac{x}{v}\right)_l(-w)^k(w)^l}{k!\;l!\;(1+\alpha)_l\;(1+\beta)_k} \\ &= \frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2}F^{1:1;1}_{-:1;1}\left[\begin{matrix}-n & : & -\frac{y}{v} & , & \frac{x}{v} & ; \\ - & : & 1+\beta & , & 1+\alpha & ;\end{matrix} -v,v\right] \end{aligned} \tag{1.2}$$

where $F^{p:k;u}_{q:l;v}$ is a double hypergeometric function. By assuming the limits $v\to 0$; $v\to 0,\ \beta=0, y=0$; $v\to 0,\ \alpha=0, x=0$ in (1.2), the following interesting cases will arise:

$$L_n^{(\alpha,\beta)}(x,y) = \frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2}\sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{(-n)_{k+l}(y)^k(x)^l}{k!\;l!\;(1+\alpha)_l\;(1+\beta)_k},$$

where $L_n^{(\alpha,\beta)}(x,y)$ is a Laguerre polynomial of two variables.

$$\lim_{v\to o} B_n^{(\alpha,0)}(x,0,v) = L_n^{(\alpha)}(x)$$

and

$$\lim_{v\to o} B_n^{(0,\beta)}(0,y,v) = L_n^{(\beta)}(y)$$

where $L_n^{(\beta)}(y)$ is the well-known Laguerre polynomial [5].

The familiar notations are adopted here are:

$$(k)_m = \frac{\Gamma(k+m)}{\Gamma(k)} \tag{1.3}$$

$$= \begin{cases} 1 & \text{if } m=0 \\ k(k+1).......(k+m-1) & \text{if } m=1,2,.....\ \text{ and } k\neq 0\end{cases},$$

(1.4) $$(k)_{l-t} = \frac{(-1)^t (k)_l}{(1-k-l)_t},$$

(1.5) $$(-k)_t = \frac{(-1)^t k!}{(k-t)!}.$$

The generalized Sylvester polynomial defined as, see [1], [15],p.450:

$$f_n(x;c) = \frac{(cx)^n}{n!} 2F_0 \left[-n, x; - ; -\frac{1}{ax} \right].$$

Also $f_n(x;1) = \phi_n(x)$, where $\phi_n(x)$ is the Sylvester polynomial [5] and the extended linear generating function [1, 15] is given by

$$\sum_{n=0}^{\infty} \binom{k+n}{n} f_{n+k}(x;c)\, t^n = (1-t)^{-x-k} e^{cxt} f_k(x; a(1-t)),$$

where k is a positive integer. M. A. Malik introduced modified generalized Sylvester polynomial [8, 13] as:

(1.6) $$f_n(x;c,d) = \frac{(dx)^n}{n!} {}_2F_0 \left[-n, cx; -; (-dx)^{-1} \right],$$

and its extended linear generating function [8] is given by

(1.7) $$\sum_{n=0}^{\infty} \binom{k+n}{n} f_{n+k}(x;c,d)\, t^n = (1-t)^{-cx-k} e^{dxt} f_k(x;c,d(1-t)).$$

In the coming sections, we are going to establish bilateral generating function of $B_n^{(\alpha,\beta)}(x,y,v)$ and its special cases as applications.

2. Bilateral Generating Function

In this section, we derive bilateral generating function for the class of generalized hypergeometric function $B_n^{(\alpha,\beta)}(x,y,v)$ with modified generalized Sylvester polynomial, $f_n(x;c,d)$.

Theorem 2.1.

(2.1) $$\sum_{n=0}^{\infty} \frac{(n!)^2}{(1+\alpha)_n (1+\beta)_n} B_n^{(\alpha,\beta)}(x,y,v) f_n(x;c,d) t^n$$

$$= (1-t)^{-cx} e^{dxt} F^{2:0;0;1}_{1:0;0;1}\left[\begin{matrix} \left[\frac{-y}{v}:1,1,1\right],[cx:1,0,1]\ :-;-;\frac{x}{v}; \\ [1+\beta:1,1,1],\quad - \quad :-;\ -;\ 1+\alpha; \end{matrix} t_1,\ t_2,\ t_3\right]$$

where $t_1=\left(\frac{vt}{1-t}\right)$, $t_2=(vtdx)$, $t_3=\left(\frac{-v^2t}{1-t}\right)$ *and* $F^{p:r,u}_{q:s,v}(x,y)$ *is a double hypergeometric function by Srivastava and Karlsson [16].*

Proof. Taking left hand side of (2.1) and on making use of the (1.2), we obtain
$\sum\limits_{n=0}^{\infty}\sum\limits_{r=0}^{n}\sum\limits_{s=0}^{n-r}\frac{\left(\frac{-y}{v}\right)_r\left(\frac{x}{v}\right)_s(-v)^rv^s(-n+r)_s(-n)_r}{(1+\alpha)_s(1+\beta)_r r!s!}f_n(x;c,d)t^n$
on replacing n by $n+r$, we get

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{n}\frac{(-n)_s\left(\frac{-y}{v}\right)_r\left(\frac{x}{v}\right)_s(vt)^rv^s}{(1+\alpha)_s(1+\beta)_r\ s!}\sum_{n=0}^{\infty}\binom{n+r}{r}f_{n+r}(x;c,d)\,t^n\,.$$

On making use of the (1.7), we have
$=\sum\limits_{r=0}^{\infty}\sum\limits_{s=0}^{n}\frac{(-n)_s\left(\frac{-y}{v}\right)_r\left(\frac{x}{v}\right)_s\left(\frac{vt}{1-t}\right)^rv^s}{(1+\alpha)_s(1+\beta)_r\ s!}(1-t)^{-cx-r}e^{dxt}f_r\left(x;c,d(1-t)\right)\,.$

By using the (1.6), we obtain
$=(1-t)^{-cx}\sum\limits_{r=0}^{\infty}\sum\limits_{s=0}^{n}\frac{(-n)_s\left(\frac{-y}{v}\right)_r\left(\frac{x}{v}\right)_s\left(\frac{vt}{1-t}\right)^rv^s}{(1+\alpha)_s(1+\beta)_r s!}$
$\times e^{dxt}\left(\frac{(d(1-t)x)^r}{r!}\right)\ {}_2F_0\left[\begin{matrix}-r,cx; \\ -;\end{matrix}\ \frac{-1}{d(1-t)x}\right]$
$=(1-t)^{-cx}e^{dxt}\sum\limits_{r=0}^{\infty}\sum\limits_{s=0}^{n}\frac{(-n)_s\left(\frac{-y}{v}\right)_r\left(\frac{x}{v}\right)_s(vt)^rv^s}{(1+\alpha)_s(1+\beta)_r s!}\left(\frac{(dx)^r}{r!}\right)\sum\limits_{n=0}^{r}\frac{(-r)_n(cx)_n}{n!}\left(-\frac{1}{d(1-t)x}\right)^n$
$=(1-t)^{-cx}e^{dxt}\sum\limits_{n=0}^{r}\sum\limits_{s=0}^{n}\sum\limits_{r=0}^{\infty}\frac{(-n)_s\left(\frac{-y}{v}\right)_r\left(\frac{x}{v}\right)_s(vt)^rv^s(dx)^r(cx)_n}{(1+\alpha)_s(1+\beta)_r s!\,(r-n)!\,n!}\left(\frac{1}{d(1-t)x}\right)^n,$
and replacing r by $r+n$, we have:

$$=(1-t)^{-cx}e^{dxt}\sum_{n=0}^{\infty}\sum_{s=0}^{n}\sum_{r=0}^{\infty}\frac{(-n)_s\left(\frac{-y}{v}\right)_{r+n}\left(\frac{x}{v}\right)_s(vtdx)^{r+n}v^s(cx)_n}{(1+\alpha)_s(1+\beta)_{r+n}s!\ r!\ n!}\left(\frac{1}{d(1-t)x}\right)^n.$$

Replacing n by $n+s$, we obtain

$$= (1-t)^{-cx}e^{dxt}\sum_{n=0}^{\infty}\sum_{s=0}^{n}\sum_{r=0}^{\infty}\frac{\left(\frac{-y}{v}\right)_{r+n+s}\left(\frac{x}{v}\right)_s(vtdx)^{r+n+s}(-v)^s(cx)_{n+s}}{(1+\alpha)_s(1+\beta)_{r+n+s}}$$

$$\times\frac{\left(\frac{1}{d(1-t)x}\right)^{n+s}}{s!\ r!\ n!} \tag{2.2}$$

$$= (1-t)^{-cx}\ e^{dxt}F^{2:0;0;1}_{1:0;0;1}\left[\begin{matrix}\left[\frac{-y}{v}:1,1,1\right],[cx:1,0,1]:-;-;\frac{x}{v}; \\ [1+\beta:1,1,1],\ -\ :\ -\ ;\ -\ ;\ \ 1+\alpha\ ;\end{matrix} t_1,t_2,t_3\right]$$

where $t_1 = \left(\frac{vt}{1-t}\right)$, $t_2 = vtdx$, $t_3 = \left(\frac{-v^2t}{1-t}\right)$. □

3. Applications

Theorem 3.1.

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(1+\alpha)_n(1+\beta)_n}L_n^{(\alpha,\beta)}(x,y)f_n(x;c,d)t^n \tag{3.1}$$

$$= (1-t)^{-cx}e^{dxt}F^{1:-;-;-}_{1:-;-;1}\left[\begin{matrix}[cx:1,0,1]:-;-;-;\\ [1+\beta:1,1,1]:-;-;1+\alpha;\end{matrix} t_1,t_2,t_3\right]$$

where $t_1 = -\frac{yt}{1-t}$, $t_2 = -ytdx$, $t_3 = -\frac{ytx}{1-t}$ *and* $L_n^{(\alpha,\beta)}(x,y)$ *is a Laguerre polynomial of two variables [3, 14].*

Proof. On assuming $v \to 0$ in the above result (2.2), the left side of it becomes

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(1+\alpha)_n(1+\beta)_n}L_n^{(\alpha,\beta)}(x,y)f_n(x;c,d)t^n .$$

by using the definition of $L_n^{(\alpha,\beta)}(x,y)$ from [14] we have:

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n+r)_s(-n)_r(y)^r(x)^s}{(1+\alpha)_s(1+\beta)_r r!s!}f_n(x;c,d)t^n .$$

On replacing n by $n+r$, we obtain

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{n}\frac{(-n)_s(-yt)^r x^s}{(1+\alpha)_s(1+\beta)_r s!}\ \sum_{n=0}^{\infty}\binom{n+r}{r} f_{n+r}(x;c,d)t^n ,$$

and with use of the equation (1.7) and later on (1.6), we obtain

$$= (1-t)^{-cx-r}e^{dxt}\ \sum_{r=0}^{\infty}\sum_{s=0}^{n}\frac{(-n)_s(-yt)^r x^s}{(1+\alpha)_s(1+\beta)_r s!}f_r\left(x;c,d(1-t)\right)$$

$$= (1-t)^{-cx}e^{dxt}\ \sum_{r=0}^{\infty}\sum_{s=0}^{n}\frac{(-n)_s\left(-\frac{yt}{1-t}\right)^r x^s(d(1-t)x)^r}{(1+\alpha)_s(1+\beta)_r s!\ r!}\ {}_2F_0\left[\begin{matrix}-r\ ,\ cx\ ;\\ -\ \ ;\end{matrix}\ -\frac{1}{d(1-t)x}\right]$$

$$= (1-t)^{-cx}e^{dxt}\sum_{n=0}^{r}\sum_{r=0}^{\infty}\sum_{s=0}^{n}\frac{(-n)_s(-ytdx)^r x^s(cx)_n}{(1+\alpha)_s(1+\beta)_r\ s!\ n!\ (r-n)!}\left(\frac{1}{d(1-t)x}\right)^n .$$

Replacing r by $r+n$, we get

$$= (1-t)^{-cx}e^{dxt}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{n}\frac{(-x)^s(-ytdx)^{r+n}(cx)_n}{(n-s)!(1+\alpha)_s(1+\beta)_{r+n}\ s!\ r!}\left(\frac{1}{d(1-t)x}\right)^n .$$

Finally, on replacing n by $n+s$, we get

$$= (1-t)^{-cx} e^{dxt} F^{1:-;-;-}_{1:-;-;1}\left[\begin{matrix}[cx:1,0,1]:-;-;-; \\ [1+\beta:1,1,1]:-;-;1+\alpha;\end{matrix} t_1, t_2, t_3\right]$$

where $t_1 = -\frac{yt}{1-t}$, $t_2 = -ytdx$, $t_3 = -\frac{ytx}{1-t}$. □

Theorem 3.2.
(3.2)

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(1+\alpha)_n} L_n^{(\alpha)}(x) f_n(x;c,d) t^n = (1-t)^{-cx} e^{dxt} \phi_3[cx\,;1+\alpha\,;-\frac{tx}{1-t}, -dtx^2]$$

where $\phi_3[\beta\,;\gamma\,;x,y] = \sum_{m,n=0}^{\infty} \frac{(\beta)_m\, x^m y^n}{(\gamma)_{m+n}\, m!\, n!}$ *is the confluent hypergeometric functions of two variables([16], p.58,59).*

Proof. On assuming $v \to 0; \beta = 0$ and $y = 0$ in the above result (2.2), the left side of it becomes [2]

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(1+\alpha)_n} L_n^{(\alpha)}(x) f_n(x;c,d) t^n = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(-n)_r x^r}{(1+\alpha)_r\, r!} f_n(x;c,d)\ t^n\,.$$

On replacing n by $n+r$ in the second summation, we get

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{\binom{n+r}{r} (-tx)^r}{(1+\alpha)_r\, r!} f_{n+r}(x;c,d)\ t^n\,.$$

Using equation (1.7) and later on applying equation (1.6), we get

$$= (1-t)^{-cx} e^{dxt} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(cx)_n (-dtx^2)^r}{(1+\alpha)_r\, (r-n)!\ n!} \left(\frac{1}{d(1-t)x}\right)^n.$$

On replacing r by $r+n$, we get

$$= (1-t)^{-cx} e^{dxt} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(cx)_n (-dtx^2)^r}{(1+\alpha)_{r+n}\, r!\ n!} \left(-\frac{t}{(1-t)x}\right)^n$$

$$= (1-t)^{-cx} e^{dxt}\ \phi_3[cx;1+\alpha;-\frac{tx}{1-t}, -d\,t\,x^2]\,.$$

□

4. Conclusion

(3.1) and (3.2) are the new bilateral generating relations of Laguerre polynomial of one and two variables with modified generalized Sylvester polynomial as the special cases of (2.1). One may get multilateral generating functions using these polynomials and are useful in obtaining the solutions of BVP which arises in mathematical physics and in many engineering problems.

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References

[1] A. K. Agarwal, H. L. Manocha: *On some new generating functions*, Indian J.Pure and Appl. Math., **13**(11) (1982), 1369 – 1375.

[2] B. Satyanarayana, N. Srimannarayana, Y. Pragathi Kumar: *A class of generalized hypergeometric functions of two discrete variables*, Annals of the 'Constantin Brancusi' of TarguJiu, Engineering series, **3** (2010), 260 – 269.

[3] B. Satyanarayana, N. Srimannarayana, Y. Pragathi Kumar: *Certain bilateral generating relations for a class of Generalized Hypergeometric functions of two variables*, Unniversal Journal of Applied Mathematics, **2**(1) (2014), 5 – 9.

[4] B. Satyanarayana, N. Srimannarayana, Y. Pragathi Kumar: *Bilateral generating relations for modified Konhauser Polynomial with discrete variable*, Mathematical Theory and Modelling, **5**(6) (2015), ISSN: 2225-0522.

[5] E. D. Rainville: *Special Functions*, Macmillan, New York, 1960.

[6] M. Lahiri , B. Satyanarayana:*A class of generalized hypergeometric functions defined by using a difference operator*, Soochow J.Math., **19** (1993), 163 – 171.

[7] M. Lahiri, B. Satyanarayana:*Certain bilateral generating relations for generalized hypergeometric functions*, Proc. Indian Acad.Sci.Math.Sci. I, **105**(3) (1995), 297 – 301.

[8] N. Ozmen:*Bilateral and Bilinear generating functions for the modified generalized Sylvester Polynomials*, Facta Universitatis (NIS), **33**(2) (2018), 279 – 293.

[9] N. Srimannarayana, B. Satyanarayana, Y. Pragathi Kumar: *On modified Konhauser Polynomial with discrete variable*, Apllied Mathematical Sciences, **7**(105) (2013), 5235 – 5241.

[10] N. Srimannarayana, B. Satyanarayana, D. Ramesh: *Some Integral results associated with generalized hypergeometric function*, International Journal of Recent Technology and Engineering (IJRTE), **8**(2) (2019), 1067–1071.

[11] N. SRIMANNARAYANA, B. SATYANARAYANA, D. RAMESH: *Convergence and extended linear generating function for generalized hypergeometric function*, International Journal of Recent Technology and Engineering(IJRTE), **8**(2S11) (2019), 3577 – 3579.

[12] N. SRIMANNARAYANA, B. SATYANARAYANA, D. RAMESH, M. RADHA MADHAVI: *Integral representation for modified Konhauser's polynomial*, International Journal of Advanced Science and Technology, **29**(3) (2020), 6473–6481.

[13] M. A. MALIK: *A Note on Generatring Functions and Summation Formulae for Modified Generalized Sylvester Polynomials*, International Jornal on Recent and Innovation Trends in Computing and Communication, **2** (2014), 2162 – 2165.

[14] S. F. RAGAB: *On Laguerre polynomial of two variables* $L_n^{(\alpha,\beta)}(x,y)$, Bull.cal.Math.Soc., **83**(3) (1991), 253 – 262.

[15] H. M. SRIVASTAVA, H. L. MANOCHA: *A Treatise on generating functions*, John Wiley and Sons (Halsted press) New York, Harwood Chichester, 1984.

[16] H. M. SRIVASTAVA, W. KARLSSON: *Multiple Gaussian Hypergeometric Series*, John Wiley and Sons (Halsted press) New York, Harwood Chichester, 1985.

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