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ANALYSIS OF FRACTIONAL ORDER SOIL MOISTURE DIFFUSION EQUATION FOR HETEROGENEOUS AND HOMOGENEOUS SYSTEM

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ABSTRACT. The aim of this paper is to examine time fractional soil moisture diffusion equation of Caputo sense with initial and boundary conditions by using homotopy perturbation method. Here we give appropriate solution to the practical interest in the form of graphical results for homogeneous and heterogeneous system of diffusion for various values of coefficient of diffusion.

1. INTRODUCTION

Fractional Calculus has extended the analysis of mathematical models in wide sense [1–3] on account of its scope in science and technology. Scientists and researchers are looking forward to get the most appropriate and fruitful results in the form of fractional calculus models for the practical results in the field of science, technology, finance, biology, hydrology etc [4] [5]. Fractional calculus models has created the platform to analyze the phenomenon of science and technology more precisely and adequately [6].

There is a great need of implementing effective and easy methods for solving fractional non linear differential equations. For this purpose many scientists are applying various methods to solve the fractional order differential equations

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like collocation method, homotopy perturbation method(HPM) [7], [8], Adomian Decomposition method [9], variational iteration method [10], homotopy analysis method proposed by Lio [11], finite Difference method [12] etc.

In this work, we have solved the mathematical model of time fractional soil moisture diffusion equation with boundary condition by using homotopy perturbation [HPM]. He has proposed the new technique homotopy perturbation method to solve non linear partial differential equation [11]. The conventional perturbation theory develops strong and efficient platform for homotopy perturbation method [HPM] [10]. This method gives rapidly convergent successive approximate solutions to large number of linear and non-linear differential equations. Here, we have considered the effective diffusion coefficient which carries the effects of free diffusion (D), volumetric water content, tortuous pathway, effects of geometry, fluidity, soil porosity, degree of saturation of soil and anion exclusion [14]. The results are explained and analysed graphically for homogeneous system (when diffusion coefficient takes the positive value) and heterogeneous diffusion(when diffusion coefficient takes the negative value) in time fractional soil-moisture diffusion equation.

1.1. **Fractional Integral and Fractional Derivative.** In this section, we have presented some of the type of definitions of fractional order derivatives and integrations [1–4].

Definition 1.1. *[1–4] **Riemann-Liouville fractional integral.***
If $f(t) \in C[a,b]$ and $a < t < b$ then

$$ {}_aD_t^{-\alpha} f(t) = {}_a I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, $$

where $\alpha \in \mathcal{R}^+$ is called the Riemann-Liouville fractional integral of order α.

Definition 1.2. ***Riemann-Liouville Fractional Derivative.***
If $f(t) \in C[a,b]$ and $a < t < b$ then

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dt} \int_a^t \frac{f(\tau)}{(t-\tau)^{(\alpha-n+1)}} d\tau, \tag{1.1} $$

where α is a positive integer and n is a positive integer such that $n-1 < \alpha < n$ then (1.1) *is called the Riemann-Liouville fractional derivative of order α.*

Definition 1.3. ***Caputo Sense Fractional Derivative.***
If $f(t) \in C[a,b]$ and $a < t < b$ then the caputo fractional derivative of order α is defined as follows

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{1.2} $$

where α is a positive integer and n is a positive integer such that $n-1 < \alpha < n$ then (1.2) is called the Caputo fractional derivative of order α.

Mittag Leffler function of one variable $E_\alpha(z)$ was defined and studied by Mittag-Leffler in the year 1903.

Definition 1.4. ***Mittag-Leffler function of one parameter:*** *[1–4]*
Mittag-Leffler function of one parameter is denoted by $E_\alpha(z)$ and defined as

$$ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, $$

where $\alpha \in \mathbb{R}$, such that $\alpha > 0$ and $z \in \mathbb{C}$.

1.2. **Analysis of Homotopy Perturbation Method(HPM).** [7]- [8]

In this section, we give the brief introduction of the method by considering the following non linear differential equation as

$$ L(u) + N(u) = f(r), \qquad r \in \Omega, $$

subject to boundary conditions

$$ B(u, \frac{\partial u}{\partial n}) = 0 \qquad r \in \Gamma, $$

where 'L' gives a linear operator, 'N' is a non -linear operator and 'f(r)' defines a known analytic function present in the working problem, 'B' gives the boundary conditions along with 'Γ' defines the boundary in the domain 'Ω'.
By using the technique of homotopy perturbation method[HPM] ,
we can construct Homotopy as $U(r, p) : \Omega \times [0, 1] \to R$, which satisfies,

$$ H(U, p) = (1-p)[L(U) - L(u_0)] + p[L(U) + N(U) - f(r)] = 0, $$

where $p \in [0, 1]$ is an embedding parameter and u_0 is the initial approximation of (1.2) which satisfies the boundary conditions. So we may have

$$ \begin{aligned} H[U, 0] &= L(U) - L(u_0) = 0 \\ H[U, 1] &= L(U) - N(U) - f(r) = 0. \end{aligned} $$

The deformation of the parameter p from 0 to 1 gives rise to the change from u_0 to $u(r)$ is called as homotopy in topology. Assuming a small embedding parameter, the solution may be considered in the form of power series as

$$U = u_0 + p\,u_1 + p^2 u_2 + p^3 u_3 + \dots . \tag{1.3}$$

Replacing p=1, gets the approximate solution of the differential equation as

$$U = u_0 + u_1 + u_2 + u_3 + \dots .$$

2. Mathematical formulation

In this section, we are giving the solution of time fractional soil moisture diffusion equation by homotopy perturbation method. The approximate solutions in the form of series have been effectively analysed graphically .

Let's take the time fractional soil moisture diffusion equation by using second Frick's law in one dimensional, as it provides the standard text of diffusion equation [14] with flux controlled initial and boundary conditions as follows:

$$\frac{\partial^\alpha u(x,\, t)}{\partial t^\alpha} = D\,\frac{\partial^2 u(x,t)}{\partial x^2}, \quad t > 0, \quad 0 \le x \le 1, \quad 0 < \alpha < 1 .$$

Boundary Conditions

$$u(0,t) = f(t) \quad \text{and} \qquad \frac{\partial u(x,\, t)}{\partial x} = 0 \quad \text{for} \quad x \to \infty, t \ge 0 ,$$

where u(x, t)= Volumetric water content, D = Effective diffusion coefficient i.e. $D = D_0 \tau \alpha_1 \gamma \theta$. Here $D_0 = \dfrac{uRT}{N}$ = Free solution diffusion coefficient R= Universal gas constant, T= Absolute temperature, N = Avagadro's number, u= Absolute mobility of the particle, τ = Effects of geometry, α_1 = Fluidity, γ = Anion exclusion in soil diffusion, θ = Tortuosity or Transmission factor. Here we have taken boundary condition taking time domain(t) in $(0,\ \infty)$ and solved them by using HPM.

3. Working Problem

Let's consider the time fractional soil moisture diffusion equation with initial and boundary conditions of space co-ordinate x tends to ∞

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D\,\frac{\partial^2 u(x,t)}{\partial x^2}, \quad t > 0, \quad 0 \le x \le 1, \quad 0 < \alpha < 1\,.$$

Boundary conditions

$$u(0,t) = 1 \quad and \qquad \frac{\partial u(x,t)}{\partial x} = 0 \quad for \quad x \to \infty, t \ge 0\,.$$

3.1. **Solution of the problem.** According to the homotopy perturbation method, homotopy can be constructed as

$$H\,(U,p) = (1-p)\,\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + p\left[\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - D\,\frac{\partial^2 u(x,t)}{\partial x^2}\right] = 0\,, \tag{3.1}$$

where $p \in [0,\, 1]$.
Let's choose the initial guess

$$u_0(x,t) = e^{-x}\,.$$

Equating the coefficients of powers of p in (3.1), we get $u_0,\, u_1,\, u_2,\; and\;\; u_3$.
By putting values of $u_0,\, u_1,\, u_2,\; and\;\; u_3$ respectively in (1.3), we get the approximate solution in the form of series as;

$$\begin{aligned} u &= e^{-x} + p\,D\,e^{-x}\,\frac{t^\alpha}{\Gamma\alpha+1} + p^2\,D^2\,e^{-x}\,\frac{t^{2\alpha}}{\Gamma 2\,\alpha+1} + p^3\,D^3\,e^{-x}\,\frac{t^{3\alpha}}{\Gamma 3\,\alpha+1} + \dots \\ u &= e^{-x}\left[1 + p\,D\,\frac{t^\alpha}{\Gamma\alpha+1} + p^2\,D^2\,\frac{t^{2\alpha}}{\Gamma 2\,\alpha+1} + p^3\,D^3\,\frac{t^{3\alpha}}{\Gamma 3\,\alpha+1} + \dots\right] \end{aligned}$$

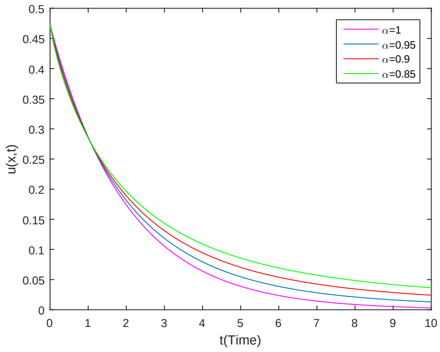
Putting $p = 1$, we get the approximate solution in the form of series as follows:

$$u = e^{-x}\left[1 + D\,\frac{t^\alpha}{\Gamma\alpha+1} + D^2\,\frac{t^{2\alpha}}{\Gamma 2\,\alpha+1} + D^3\,\frac{t^{3\alpha}}{\Gamma 3\,\alpha+1} + \dots\right].$$

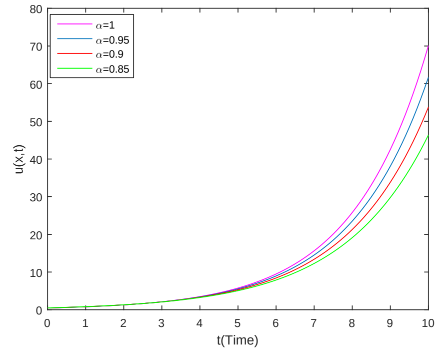
In Mittag-Leffler form, the solution is expressed as

$$u = e^{-x}E_\alpha\,(D\,t^\alpha)\;.$$

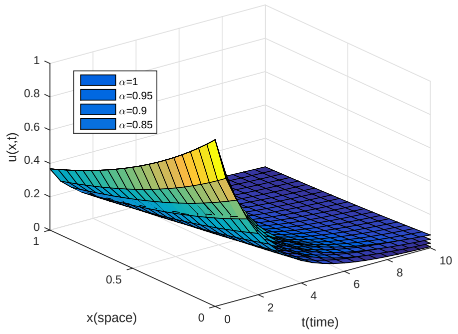
Since the solution is in the form of Mittag-Leffler function, it implies that the series solution is convergent. The graphs of homogeneous system and heterogeneous system have explained the nature of change of concentration on account of diffusion coefficient with respect to time.



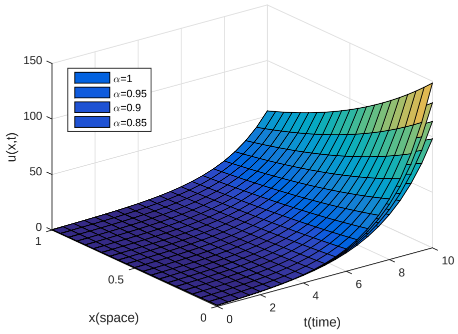
(a) 2-D plot for heterogeneous system for $D = -0.5$ and x=0.75



(b) 2-D plot for homogeneous system at $D = 0.5$ and x=0.75



(c) 3-D plot for heterogeneous system at $D = -0.5$



(d) 3-D plot for homogeneous system at $D = 0.5$

FIGURE 1. Graphical representation for working problem

4. CONCLUSIONS

In this paper, we have proposed some guidelines to analyse the solutions of some types of time fractional soil moisture diffusion equations. Fractional differential equations express the more generalised results of the physical models. Subsequently graphical representation deals with the results for various values of fractional order which highlights the possible outcomes. Homotopy perturbation method have been applied to get the solution of the time fractional differential equations and the solution matches suitably to the exact solution. It is necessary to state that the series solution obtained by homotopy perturbation method precisely converges.

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