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ON A NEW SUBCLASS OF BI-PSEUDO-STARLIKE FUNCTIONS DEFINED BY FRASIN DIFFERENTIAL OPERATOR

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ABSTRACT. The main aim of this research is to introduce and examine new subclasses of functions class $\mathfrak{C}$ of bi-univalent functions defined in Δ associating with γ-pseudo-starlike functions with sakaguchi type functions $\mathfrak{H}^{\beta}_{\mathfrak{C}}(\mu,\gamma,\nu,s,t)$ and $\mathfrak{H}^{\beta}_{\mathfrak{C}}(\mu,\gamma,\phi,s,t)$, which are defined by a differential operator of holomorphic functions with binomial series. Also, the estimate on the coefficient $|n_2|$ and $|n_3|$ for functions in these new subclasses are determined. Results acquired generalized some known consequences.

1. INTRODUCTION

We indicate by $\mathcal{V}$ the subclass of class of function $\mathcal{L}$ which is of the form

$$\psi(z)=z+\sum_{g=2}^{\infty}n_g z^g$$

consisting of functions which are holomorphic and univalent in the unit disk Δ. Let $\mathfrak{S}^*(\vartheta)$ and $\mathfrak{K}(\vartheta)$ indicate the familiar classes of starlike and convex function of order $\vartheta(0\leq\vartheta<1)$ respectively.
Let $\psi^{-1}(z)$ be the inverse of the function $\psi(z)$ then we have

$$\psi^{-1}(\psi(z))=z,$$

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$$\psi(\psi^{-1}(u)) = u, \quad |u| < r_0(\psi); r_0(\psi) \geq \frac{1}{4}$$

where

$$h(u) = \psi^{-1}(u) = u - n_2 u^2 + (2n_2^2 - n_3)u^3 - (5n_2^3 - 5n_2 n_3 + n_4)u^4 + \cdots .$$

A function $\psi(z) \in \mathcal{L}$ denoted by $\mathfrak{E}$ is said to be bi-univalent in Δ, considering that $\psi(z)$ and $\psi^{-1}(z)$ are univalent in Δ. For more details see: [7], [4], [15], [5], [10].

Definition 1.1. *[3] Let $\psi(z) \in \mathcal{L}$, suppose $0 \leq \vartheta < 1$ and $\gamma \geq 1$ is real. Then $\psi(z) \in L_\gamma(\vartheta)$ of γ-pseodu-starlike function of order ϑ in Δ if and only if*

$$\Re\left(\frac{z[\psi'(z)]^\gamma}{\psi(z)}\right) > \vartheta.$$

Babalola [3] verified that, all pseodu-starlike function are Bazilevic of type $\left(1 - \frac{1}{\gamma}\right)$, order $\vartheta^{\frac{1}{\gamma}}$ and univalent in Δ.

A function $\psi(z) \in \mathcal{L}$ satisfying the condition

$$\Re\left(\frac{z\psi'(z)}{\psi(z) - \psi(-z)}\right) > 0.$$

is required to be a starlike functions with respect to symmetric point, which was investigated by Sakaguchi [13]. Many other authors examine bounds for numerous subclasses of bi-univalent functions, (for more details see; [9], [11], [16]).

Frasin [6] introduced the differential operator $D^\beta_{k,\mu}\psi(z)$ defined as follows:

$$D^0\psi(z) = \psi(z)$$

$$D^1_{k,\mu}\psi(z) = (1-\mu)^k\psi(z) + (1-(1-\mu)^k)z\psi'(z) = D_{k,\mu}\psi(z)$$

$$D^\beta_{k,\mu}\psi(z) = D_{k,\mu}(D^{\beta-1}\psi(z))$$

where $\beta \in \mathbb{N}$, then we have

$$D^\beta_{k,\mu}\psi(z) = z + \sum_{g=2}^{\infty}\left(1 + (g-1)\sum_{d=1}^{k}\binom{k}{d}(-1)^{d+1}\mu^d\right)^\beta n_g z^g . \tag{1.1}$$

Using (1.1), we have

$$C^k_d(\mu)z(D^\beta_{k,\mu}\psi(z))' = D^{\beta+1}_{k,\mu}\psi(z) - (1 - C^k_d(\mu))D^\beta_{k,\mu}\psi(z)$$

where $\mu > 0$, $k \in \mathbb{N}$, $\beta \in \mathbb{N}_0$ and $C^k_d(\mu) := \sum_{d=1}^{k}\binom{k}{d}(-1)^{d+1}\mu^d$.

Remark 1.1. *We observe that*

(1) *When $k = 1$, we obtain the Al-Oboudi differential operator [2].*
(2) *When $k = \mu = 1$, we obtain the Salagean operator [14].*

Motivated by the earlier works of [8], [1], we introduced new subclasses $\mathfrak{H}^{\beta}_{\mathfrak{E}}(\mu, \gamma, \nu, s, t)$ and $\mathfrak{H}^{\beta}_{\mathfrak{E}}(\mu, \gamma, \phi, s, t)$ of the function class $\mathfrak{E}$ which are defined by a differential operator of holomorphic functions comprising of binomial series in Δ. Hence, the estimate on the coefficient $|n_2|$ and $|n_3|$ for functions in these new subclasses are determined.

Lemma 1.1. *[12] If $r(z) \in \mathcal{P}$ and $z \in \Delta$, then $|w_n| \leq 2$ for each n. where $\mathcal{P}$ is the family of all function r holomorphic in Δ for which $\Re(r(z)) > 0$,*

$$r(z) = 1 + w_1 z + w_2 z^2 + \cdots$$

2. Coefficient Bounds for the Function Class $\mathfrak{H}^{\beta}_{\mathfrak{E}}(\mu, \gamma, \nu, s, t)$

Definition 2.1. *A function $\psi(z) \in \mathcal{L}$ which gratify the condition below:*

$$\left|\arg\left[\frac{(s-t)z[(D^{\beta}_{k,\mu}\psi(z))']^{\gamma}}{D^{\beta}_{k,\mu}\psi(sz) - D^{\beta}_{k,\mu}\psi(tz)}\right]\right| < \frac{\nu\pi}{2}, \tag{2.1}$$

and

$$\left|\arg\left[\frac{(s-t)u[(D^{\beta}_{k,\mu}h(u))']^{\gamma}}{D^{\beta}_{k,\mu}h(su) - D^{\beta}_{k,\mu}h(tu)}\right]\right| < \frac{\nu\pi}{2} \tag{2.2}$$

where $\psi(z) \in \mathfrak{E}$, $\gamma \geq 1$, $0 < \nu \leq 1$, $s, t \in \mathbb{C}$, $z, u \in \Delta$ with $|s| \leq 1$, $|t| \leq 1$; $s \neq t$ and

$$h(u) = u - n_2 u^2 + (2n_2^2 - n_3)u^3 - (5n_2^3 - 5n_2 n_3 + n_4)u^4 + \cdots$$

is said to be in the class $\mathfrak{H}^{\beta}_{\mathfrak{E}}(\mu, \gamma, \nu, s, t)$.

Theorem 2.1. *Suppose $\psi(z) \in \mathcal{L}$ is in the class $\mathfrak{H}^{\beta}_{\mathfrak{E}}(\mu, \gamma, \nu, s, t)$, then*

$$|n_2| \leq \frac{2\nu}{\sqrt{\left|\nu(6\gamma - 2s^2 - 2t^2 - 2st)\left(1 + 2C^k_d(\mu)\right)^{\beta} - \left[2\nu(2\gamma(s+t+1-\gamma) - s^2 - t^2 - 2st) + (\nu-1)(2\gamma - s - t)^2\right]\left(1 + C^k_d(\mu)\right)^{2\beta}\right|}} \tag{2.3}$$

and

$$|n_3| \leq \frac{4\nu^2}{|(2\gamma - s - t)^2|\left(1 + C_d^k(\mu)\right)^{2\beta}} + \frac{2\nu}{|(3\gamma - s^2 - t^2 - st)|\left(1 + 2C_d^k(\mu)\right)^{\beta}}.$$

Proof. Let $\psi(z) \in \mathfrak{H}_{\mathfrak{C}}^{\beta}(\mu, \gamma, \nu, s, t)$, then it follows from (2.1) and (2.2) that

$$\frac{(s-t)z[(D_{k,\mu}^{\beta}\psi(z))']^{\gamma}}{D_{k,\mu}^{\beta}\psi(sz) - D_{k,\mu}^{\beta}\psi(tz)} = [y(z)]^{\nu} \tag{2.4}$$

and

$$\frac{(s-t)u[(D_{k,\mu}^{\beta}h(u))']^{\gamma}}{D_{k,\mu}^{\beta}h(su) - D_{k,\mu}^{\beta}h(tu)} = [x(u)]^{\nu} \tag{2.5}$$

where $y(z)$ and $x(u)$ are in the class $\mathcal{P}$ which is of the form

$$y(z) = 1 + y_1 z + y_2 z^2 + y_3 z^3 + \cdots \tag{2.6}$$

$$x(u) = 1 + x_1 u + x_2 u^2 + x_3 u^3 + \cdots \tag{2.7}$$

Hence,

$$[y(z)]^{\nu} = 1 + \nu y_1 z + \left(\nu y_2 + \frac{\nu(\nu-1)y_1^2}{2!}\right) z^2 + \cdots$$

$$[x(u)]^{\nu} = 1 + \nu x_1 u + \left(\nu x_2 + \frac{\nu(\nu-1)x_1^2}{2!}\right) u^2 + \cdots$$

Now, equating the coefficient in (2.4) and (2.5) we get

$$(2\gamma - s - t)\left(1 + C_d^k(\mu)\right)^{\beta} n_2 = \nu y_1 \tag{2.8}$$

$$(3\gamma - s^2 - st - t^2)\left(1 + 2C_d^k(\mu)\right)^{\beta} n_3 - (2\gamma(s + t - \gamma + 1) - s^2 - 2st - t^2) \left(1 + C_d^k(\mu)\right)^{2\beta} n_2^2 = \nu y_2 + \frac{\nu(\nu-1)}{2!} y_1^2 \tag{2.9}$$

$$-(2\gamma - s - t)\left(1 + C_d^k(\mu)\right)^{\beta} n_2 = \nu x_1 \tag{2.10}$$

(2.11)

$$\Bigg[(6\gamma - 2s^2 - 2t^2 - 2st)\left(1 + 2C_d^k(\mu)\right)^{\beta} - (2\gamma(s + t - \gamma + 1) - s^2 - 2st - t^2) \left(1 + C_d^k(\mu)\right)^{2\beta}\Bigg] n_2^2 - \left[(3\gamma - s^2 - t^2 - st)\left(1 + 2C_d^k(\mu)\right)^{\beta}\right] n_3 = \nu x_2 + \frac{\nu(\nu-1)}{2!} x_1^2.$$

From (2.8) and (2.10) we get

$$y_1 = -x_1 \tag{2.12}$$

and

$$2(2\gamma - s - t)^2 \left(1 + C_d^k(\mu)\right)^{2\beta} n_2^2 = \nu^2(y_1^2 + x_1^2)\,. \tag{2.13}$$

Also from (2.9) and (2.11) we have

(2.14)
$$\Bigg[(6\gamma - 2s^2 - 2t^2 - 2st)\left(1 + 2C_d^k(\mu)\right)^{\beta} - 2(2\gamma(s + t + 1 - \gamma) - s^2 - t^2 - 2st)$$
$$\left(1 + C_d^k(\mu)\right)^{2\beta}\Bigg] n_2^2 = \nu(y_2 + x_2) + \frac{\nu(\nu - 1)}{2!}(y_1^2 + x_1^2)\,.$$

From (2.14) and (2.13), we have

$$\Bigg[\nu(6\gamma - 2s^2 - 2t^2 - 2st)\left(1 + 2C_d^k(\mu)\right)^{\beta} - 2\nu(2\gamma(s + t + 1 - \gamma) - s^2 - t^2 - 2st)$$
$$\left(1 + C_d^k(\mu)\right)^{2\beta} - (\nu - 1)(2\gamma - s - t)^2\left(1 + C_d^k(\mu)\right)^{2\beta}\Bigg] n_2^2 = \nu^2(y_2 + x_2)\,.$$

Therefore, we have

$$n_2^2 = \frac{\nu^2(y_2 + x_2)}{\left|\begin{array}{l}\nu(6\gamma - 2s^2 - 2t^2 - 2st)\left(1 + 2C_d^k(\mu)\right)^{\beta} - 2\nu(2\gamma(s + t + 1 - \gamma) - s^2 - t^2 - 2st) \\ \qquad \left(1 + C_d^k(\mu)\right)^{2\beta} - (\nu - 1)(2\gamma - s - t)^2\left(1 + C_d^k(\mu)\right)^{2\beta}\end{array}\right|}\,.$$

From Lemma 1.1, we have

$$|n_2| \le \frac{2\nu}{\sqrt{\left|\begin{array}{l}\nu(6\gamma - 2s^2 - 2t^2 - 2st)\left(1 + 2C_d^k(\mu)\right)^{\beta} - \Big[2\nu(2\gamma(s + t + 1 - \gamma) - s^2 - t^2 - 2st) \\ \qquad + (\nu - 1)(2\gamma - s - t)^2\Big]\left(1 + C_d^k(\mu)\right)^{2\beta}\end{array}\right|}}\,.$$

Also, subtracting (2.11) from (2.9), we get

(2.15)

$$2(3\gamma - s^2 - st - t^2)\left(1 + 2C_d^k(\mu)\right)^\beta n_3 - 2(3\gamma - s^2 - t^2 - st)\left(1 + 2C_d^k(\mu)\right)^\beta n_2^2 = \nu(y_2 - x_2) + \frac{\nu(\nu-1)}{2!}(y_1^2 - x_1^2),$$

it follows from (2.12), (2.13) and (2.15) that

$$2(3\gamma - s^2 - st - t^2)\left(1 + 2C_d^k(\mu)\right)^\beta n_3 = 2(3\gamma - s^2 - t^2 - st)\left(1 + 2C_d^k(\mu)\right)^\beta \frac{\nu^2(y_1^2 + x_1^2)}{2(2\gamma - s - t)^2\left(1 + C_d^k(\mu)\right)^{2\beta}} + \nu(y_2 - x_2)$$

which is equivalent to,

$$n_3 = \frac{\nu^2(y_1^2 + x_1^2)}{2(2\gamma - s - t)^2\left(1 + C_d^k(\mu)\right)^{2\beta}} + \frac{\nu(y_2 - x_2)}{2(3\gamma - s^2 - t^2 - st)\left(1 + 2C_d^k(\mu)\right)^\beta}.$$

Applying Lemma 1.1 for the coefficients y_1, y_2, x_1 and x_2, we have

$$|n_3| \leq \frac{4\nu^2}{|(2\gamma - s - t)^2|\left(1 + C_d^k(\mu)\right)^{2\beta}} + \frac{2\nu}{|(3\gamma - s^2 - t^2 - st)|\left(1 + 2C_d^k(\mu)\right)^\beta}.$$

We get the desired estimate $|n_3|$ as asserted in (2.4).

□

Putting $\beta = 0$ in Theorem 2.1, we have;

Corollary 2.1. *Suppose $\psi(z) \in \mathcal{L}$ is in the class $\mathfrak{H}_{\mathfrak{C}}^0(\gamma, \nu, s, t)$, then*

$$|n_2| \leq \frac{2\nu}{\sqrt{\left|(6\gamma - 4\gamma(s + t + 1 - \gamma) + 2st)\nu - (\nu - 1)(2\gamma - s - t)^2\right|}}$$

and

$$|n_3| \leq \frac{4\nu^2}{|(2\gamma - s - t)^2|} + \frac{2\nu}{|(3\gamma - s^2 - t^2 - st)|}.$$

which is the results obtain by Emeka and Opoola [8].

Putting $\gamma = 1$ in Theorem 2.1, we have;

Corollary 2.2. *Suppose $\psi(z) \in \mathcal{L}$ is in the class $\mathfrak{H}_{\mathfrak{E}}^{\beta}(\mu, 1, \nu, s, t)$, then*

$$|n_2| \leq \frac{2\nu}{\sqrt{\left|\nu(6-2s^2-2t^2-2st)\left(1+2C_d^k(\mu)\right)^{\beta} - \left[2\nu(2s+2t-s^2-t^2-2st) + (\nu-1)(2\gamma-s-t)^2\right]\left(1+C_d^k(\mu)\right)^{2\beta}\right|}}$$

and

$$|n_3| \leq \frac{4\nu^2}{|(2-s-t)^2|\left(1+C_d^k(\mu)\right)^{2\beta}} + \frac{2\nu}{|(3-s^2-t^2-st)|\left(1+2C_d^k(\mu)\right)^{\beta}}.$$

which is the results obtain by Aldawish, Al-Hawary and Frasin [1].

Putting $t=0$, $s=1$ and $\gamma=1$ in Corollary 2.1, we have;

Corollary 2.3. *Suppose $\psi(z) \in \mathcal{L}$ is in the class $\mathfrak{H}_{\mathfrak{E}}^{0}(1, \nu, 1, 0)$, then*

$$|n_2| \leq \frac{2\nu}{\sqrt{1+\nu}}$$

and

$$|n_3| \leq \nu(4\nu+1)\,.$$

Putting $t=-1$, $s=1$ and $\gamma=1$ in Corollary 2.1, we have:

Corollary 2.4. *Suppose $\psi(z) \in \mathcal{L}$ is in the class $\mathfrak{H}_{\mathfrak{E}}^{0}(1, \nu, 1, -1)$, then*

$$|n_2| \leq \nu$$

and

$$|n_3| \leq \nu(\nu+1)\,.$$

3. Coefficient Bounds for the Function Class $\mathfrak{H}_{\mathfrak{E}}^{\beta}(\mu, \gamma, \phi, s, t)$

Definition 3.1. *A function $\psi(z) \in \mathcal{L}$ which gratify the condition below:*

(3.1)
$$\Re\left[\frac{(s-t)z[(D_{k,\mu}^{\beta}\psi(z))']^{\gamma}}{D_{k,\mu}^{\beta}\psi(sz) - D_{k,\mu}^{\beta}\psi(tz)}\right] > \phi,$$

and

(3.2)
$$\Re\left[\frac{(s-t)u[(D_{k,\mu}^{\beta}h(u))']^{\gamma}}{D_{k,\mu}^{\beta}h(su) - D_{k,\mu}^{\beta}h(tu)}\right] > \phi$$

where $\psi(z) \in \mathfrak{E}$, $\gamma \geq 1$, $0 \leq \phi \leq 1$, $s, t \in \mathbb{C}$, $z, u \in \Delta$ *with* $|s| \leq 1$, $|t| \leq 1$; $s \neq t$ *and*

$$h(u) = u - n_2 u^2 + (2n_2^2 - n_3)u^3 - (5n_2^3 - 5n_2 n_3 + n_4)u^4 + \cdots$$

is said to be in the class $\mathfrak{H}_{\mathfrak{E}}^{\beta}(\mu, \gamma, \phi, s, t)$.

Theorem 3.1. *Suppose* $\psi(z) \in \mathcal{L}$ *is in the class* $\mathfrak{H}_{\mathfrak{E}}^{\beta}(\mu, \gamma, \phi, s, t)$, *then*

$$|n_2| \leq \sqrt{\frac{2(1-\phi)}{|(3\gamma - s^2 - t^2 - st)\left(1 + 2C_d^k(\mu)\right)^{\beta} - (2\gamma(s+t-\gamma+1) - s^2 - 2st - t^2)\left(1 + C_d^k(\mu)\right)^{2\beta}|}}$$

and

(3.3)
$$|n_3| \leq \frac{4(1-\phi)^2}{|(2\gamma - s - t)^2|\left(1 + C_d^k(\mu)\right)^{2\beta}} + \frac{2(1-\phi)}{|(3\gamma - s^2 - st - t^2)|\left(1 + 2C_d^k(\mu)\right)^{\beta}}.$$

Proof. From equation (3.1) and (3.2) we get:

$$\frac{(s-t)z[(D_{k,\mu}^{\beta}\psi(z))']^{\gamma}}{D_{k,\mu}^{\beta}\psi(sz) - D_{k,\mu}^{\beta}\psi(tz)} = \phi + (1-\phi)y(z) \tag{3.4}$$

and

$$\frac{(s-t)u[(D_{k,\mu}^{\beta}h(u))']^{\gamma}}{D_{k,\mu}^{\beta}h(su) - D_{k,\mu}^{\beta}h(tu)} = \phi + (1-\phi)x(u) \tag{3.5}$$

where $y(z)$ and $x(u)$ in $\mathcal{P}$ given by (2.6) and (2.7), that is

$$\phi + (1-\phi)y(z) = 1 + (1-\phi)y_1 z + \phi + (1-\phi)y_2 z^2 + \cdots$$

and

$$\phi + (1-\phi)x(u) = 1 + (1-\phi)x_1 u + \phi + (1-\phi)x_2 u^2 + \cdots$$

Equating the coefficients of (3.4) and (3.5) we get

$$(2\gamma - s - t)\left(1 + C_d^k(\mu)\right)^{\beta} n_2 = (1-\phi)y_1, \tag{3.6}$$

$$(3\gamma - s^2 - st - t^2)\left(1 + 2C_d^k(\mu)\right)^{\beta} n_3 - (2\gamma(s+t-\gamma+1) - s^2 - 2st - t^2)\left(1 + C_d^k(\mu)\right)^{2\beta} n_2^2 = (1-\phi)y_2, \tag{3.7}$$

$$-(2\gamma - s - t)\left(1 + C_d^k(\mu)\right)^{\beta} n_2 = (1-\phi)x_1, \tag{3.8}$$

(3.9)

$$\left[(6\gamma - 2s^2 - 2t^2 - 2st)\left(1 + 2C_d^k(\mu)\right)^{\beta} - (2\gamma(s + t - \gamma + 1) - s^2 - 2st - t^2)\left(1 + C_d^k(\mu)\right)^{2\beta}\right] n_2^2 - \left[(3\gamma - s^2 - t^2 - st)\left(1 + 2C_d^k(\mu)\right)^{\beta}\right] n_3 = (1 - \phi) x_2 \,.$$

From (3.6) and (3.8) we get

$$y_1 = -x_1$$

and

$$2(2\gamma - s - t)^2 \left(1 + C_d^k(\mu)\right)^{2\beta} n_2^2 = (1 - \phi)^2 (y_1^2 + x_1^2) \,. \tag{3.10}$$

Now adding (3.7) and (3.9), we deduce that

$$\left[(6\gamma - 2s^2 - 2t^2 - 2st)\left(1 + 2C_d^k(\mu)\right)^{\beta} - (4\gamma(s + t - \gamma + 1) - s^2 - 2st - t^2)\left(1 + C_d^k(\mu)\right)^{2\beta}\right] n_2^2 = (1 - \phi)(y_2 + x_2) \,.$$

Thus, we have

$$n_2^2 = \frac{(1 - \phi)(y_2 + x_2)}{(6\gamma - 2s^2 - 2t^2 - 2st)\left(1 + 2C_d^k(\mu)\right)^{\beta} - (4\gamma(s + t - \gamma + 1) - s^2 - 2st - t^2)\left(1 + C_d^k(\mu)\right)^{2\beta}} \,,$$

$$|n_2^2| \leq \frac{(1 - \phi)(|y_2| + |x_2|)}{|(6\gamma - 2s^2 - 2t^2 - 2st)\left(1 + 2C_d^k(\mu)\right)^{\beta} - (4\gamma(s + t - \gamma + 1) - s^2 - 2st - t^2)\left(1 + C_d^k(\mu)\right)^{2\beta}|} \,.$$

Applying Lemma 1.1 we have:

$$|n_2^2| \leq \frac{2(1 - \phi)}{|(3\gamma - s^2 - t^2 - st)\left(1 + 2C_d^k(\mu)\right)^{\beta} - (2\gamma(s + t - \gamma + 1) - s^2 - 2st - t^2)\left(1 + C_d^k(\mu)\right)^{2\beta}|}$$

$$|n_2| \leq \sqrt{\frac{2(1-\phi)}{|(3\gamma-s^2-t^2-st)\left(1+2C_d^k(\mu)\right)^\beta-(2\gamma(s+t-\gamma+1) -s^2-2st-t^2)\left(1+C_d^k(\mu)\right)^{2\beta}|}}.$$

Also, subtracting (3.9) from (3.7), we get

$$2(3\gamma-s^2-st-t^2)\left(1+2C_d^k(\mu)\right)^\beta n_3-2(3\gamma-s^2-t^2-st)\left(1+2C_d^k(\mu)\right)^\beta n_2^2 = (1-\phi)(y_2-x_2)$$

$$n_3=n_2^2+\frac{(1-\phi)(y_2-x_2)}{2(3\gamma-s^2-st-t^2)\left(1+2C_d^k(\mu)\right)^\beta}.$$

Then from (3.10), we have

$$n_3=\frac{(1-\phi)^2(y_1^2+x_1^2)}{2(2\gamma-s-t)^2\left(1+C_d^k(\mu)\right)^{2\beta}}+\frac{(1-\phi)(y_2-x_2)}{2(3\gamma-s^2-st-t^2)\left(1+2C_d^k(\mu)\right)^\beta}.$$

Applying Lemma 1.1 for the coefficients y_1, y_2, x_1 and x_2, we have

$$n_3\leq\frac{4(1-\phi)^2}{|(2\gamma-s-t)^2|\left(1+C_d^k(\mu)\right)^{2\beta}}+\frac{2(1-\phi)}{|(3\gamma-s^2-st-t^2)|\left(1+2C_d^k(\mu)\right)^\beta}.$$

We get desired estimate on $|n_3|$ as asserted in (3.3). □

Putting $\beta=0$ in Theorem 3.1, we have:

Corollary 3.1. *Suppose $\psi(z)\in\mathcal{L}$ is in the class $\mathfrak{H}^0_{\mathfrak{C}}(\gamma,\phi,s,t)$, then*

$$|n_2|\leq\sqrt{\frac{2(1-\phi)}{|3\gamma-2\gamma(s+t-\gamma+1)+st|}}$$

and

$$|n_3|\leq\frac{4(1-\phi)^2}{|(2\gamma-s-t)^2|}+\frac{2(1-\phi)}{|(3\gamma-s^2-st-t^2)|}$$

where $0\leq\phi<1$.

which is the results obtain by Emeka and Opoola [8].
Setting $\gamma=1$ in Theorem 3.1, we have:

Corollary 3.2. *Suppose $\psi(z)\in\mathcal{L}$ is in the class $\mathfrak{H}^\beta_{\mathfrak{C}}(\mu,1,\phi,s,t)$, then*

$$|n_2| \leq \sqrt{\frac{2(1-\phi)}{|(3-s^2-t^2-st)\left(1+2C_d^k(\mu)\right)^\beta-(2s+2t-s^2-2st-t^2)\left(1+C_d^k(\mu)\right)^{2\beta}|}}$$

and

$$|n_3| \leq \frac{4(1-\phi)^2}{|(2-s-t)^2|\left(1+C_d^k(\mu)\right)^{2\beta}} + \frac{2(1-\phi)}{|(3-s^2-st-t^2)|\left(1+2C_d^k(\mu)\right)^{\beta}}$$

where $0 \leq \phi < 1$.

which is the results obtain by Aldawish et. al. [1].

Putting $\gamma = 1$ and $\beta = 0$ in Corollary 3.2, we have:

Corollary 3.3. *Suppose* $\psi(z) \in \mathcal{L}$ *is in the class* $\mathfrak{H}_{\mathfrak{E}}^0(1, \phi, s, t)$*, then*

$$|n_2| \leq \sqrt{\frac{2(1-\phi)}{|3-2(s+t)+st|}}$$

and

$$|n_3| \leq \frac{4(1-\phi)^2}{|(2-s-t)^2|} + \frac{2(1-\phi)}{|(3-s^2-st-t^2)|},$$

where $0 \leq \phi < 1$.

Putting $s = 1$ and $t = -1$ in Corollary 3.3, we have:

Corollary 3.4. *Suppose* $\psi(z) \in \mathcal{L}$ *is in the class* $\mathfrak{H}_{\mathfrak{E}}^0(1, \phi, 1, -1)$*, then*

$$|n_2| \leq \sqrt{1-\phi}$$

and

$$|n_3| \leq (1-\phi)(2-\phi),$$

where $0 \leq \phi < 1$.

Taking $t = 0$ and $s = 1$ in Corollary 3.3, we have:

Corollary 3.5. *Suppose* $\psi(z) \in \mathcal{L}$ *is in the class* $\mathfrak{H}_{\mathfrak{E}}^0(1, \phi, 1, 0)$*, then*

$$|n_2| \leq \sqrt{2(1-\phi)}$$

and

$$|n_3| \leq (1-\phi)(5-4\phi),$$

where $0 \leq \phi < 1$.

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