ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.7, 4853–4862
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.7.51

EXISTENCE OF SOLUTION OF THE NONLINEAR DIFFERENTIAL EQUATION IN THE MODELLING OF EARDRUM BY USING HOMOTOPY PERTURBATION METHOD

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ABSTRACT. The eardrum model equation has non odd restoring force function. It is observed that it is asymmetrically loaded and undergoes asymmetric oscillations for positive and negative amplitudes In this study the nonlinear second order differential equation of an ear drum model is solved by using the homotopy perturbation method. The solution obtained by the homotopy perturbation method is compared with the analytical method.

1. INTRODUCTION

The ear is the body's main receiver system for acoustic wave information. The main objective of the ear is to receive the acoustic waves to amplify the intensity, to analyse the frequency and intensify the structure of the wave and to reject random background noise. The ear itself can be structured into three sections with the purpose to receive acoustical signals and to amplify these signals that is outer ear, middle ear and inner ear and these are separated by membrane windows eardrum between outer and the middle ear. The critical part of the outer ear is auditory canal which is approximately 2.5cm long. The canal is

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2010 *Mathematics Subject Classification.* 34B60.

Key words and phrases. Homotopy perturbation method(HPM), Phase diagram, truly nonlinear systems, Modified differential transform method (MDTM), Ear drum, Perturbation method, Closed form solution.

closed by the ear drum membrane and it represents a tube closed at one side. The eardrum is approximately 0.5 mm thick membrane with an area approximately $65mm^2$. It separates the outer ear canal from the middle ear cavity. The main purpose of the membrane is to absorb and transmit the pressure variations caused by the acoustical waves in the outer canal. The acoustical signal travels along the ear canal and hits the eardrum which causes the partial reflections and transmission of the signal. To behave the hearing sensitivity reflection should be increased and transmission decreased.

For further reference see [1]-[23].

2. General Model

By applying the Newton's second law to this vibrating eardrum and by treating the eardrum tympanic membrane as a mechanical system undergoing the one dimensional vibration about its equilibrium position, the displacement is x(t), and the restoring force for small displacement can be expressed as a Taylor's series expansion.

$$F(x) = F(x_0) + xF(x_0) + \frac{x^2}{2!}F(x_0) + \frac{x^3}{3!}F(x_0)\dots$$

In equilibrium, x=0 and the restoring force F(x) must also vanish. So that $F(x_0) = 0$. Suppose the term linear in x dominant so that $F(x) \approx F'(x_0)$ to be restoring force, we must have $F"(x_0) < 0$ for positive x. Setting $F"(x_0) = -k$, with the spring constant k,positive gives us the well-known Hooke's law $F(x) = -kx$ which is valid for small x. If $f(t)$ is the driving force on the eardrum produced by the periodically varying pressure of the incoming sound wave and m is the mass of the tympanic membrane, Newton's second law yields $m\ddot{x} = -kx + f(t)$ Which can be rewritten as $x + \omega^2 x = F(t)$ with $\omega_0 = \sqrt{\frac{k}{m}}$ and $F(t) = \frac{f(t)}{m}$. If we keep the quadratic term in Taylor's expansion and set $\frac{1}{2!}F"(x_0) = \beta m$ the equation becomes $\ddot{x} + \omega_0^2 x = \beta^2 x = F(t)$.

3. Solution of the Problem

The solution of the freely vibrating eardrum equation

$$\ddot{x} + \omega_0^2 x = \beta^2 x = F(t) \tag{3.1}$$

is given by assuming the eardrum is initially at rest that is at some positive value

$$x(0) = A, \dot{x}(0) = 0\,. \tag{3.2}$$

Applying the homotopy perturbation method to the above freely vibrating eardrum equation taking the values as $\omega_0 = 1$, $\beta = 0.1$ and $A = 1$, we construct a homotopy $\Omega \times [0,1] \to R$ which satisfies:

$$L(v) - L(x_0) + pL(x_0) + pv^2 = 0\,, \tag{3.3}$$

where

$$L(x) = \ddot{x} + x \quad \text{and} \quad N(x) = 0.1x^2\,.$$

Assuming the initial approximation of equation (3.1) is of the form

$$x_0(t) = A\cos(\alpha t)\,, \tag{3.4}$$

where $\alpha(\epsilon)$ is a nonzero unknown constant with $\alpha(0) = 1$. The approximate solution of the equation (3.3) has the form

$$v = v_0 + pv_1 + p^2 v_2 + \tag{3.5}$$

and

$$x = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \dots$$

Substituting equation (3.5) into (3.3) and equating the terms with identical powers of p, we have:

$$L(v_0) - L(x_0) = 0, v_0(0) = A\,,\ v_0'(0) = 0\,,$$

$$L(v_1) - L(x_0) + 0.1v_0^2 = 0, v_1(0) = v_1'(0) = 0\,. \tag{3.6}$$

From equation (3.4) we have

$$v_0 = x_0 = A\cos(\alpha(t))\,.$$

Then from the equation (3.6) we have:

$$\frac{d^2 v_1}{dt^2} + v_1 + (-\alpha^2 + 1)A\cos(\alpha t) + \frac{0.1A^2}{2} + \frac{0.1A^2}{2}\cos 2\alpha t = 0\,. \tag{3.7}$$

Solving the equation (3.7) one gets

$$v_1(t) = -\frac{0.1A^2}{2} + \frac{0.1A^2}{2(4\alpha^2 - 1)}\cos 2\alpha t\,. \tag{3.8}$$

In order to remove the secular term which may occurs coming iteration we get the coefficient of cost zero or $\alpha = 1$, the first order approximation for the equation (3.7) is $x(t) = v_0(t) + v_1(t)$. Then we have:

$$x(t) = A\cos t - \frac{0.1A^2}{2} + \frac{0.1A^2}{6}\cos 2t\,. \tag{3.9}$$

In the above equation (3.9) when the constants $A = 1$, then the solution becomes:

$$x(t) = -0.05 + \cos t + 0.0166\cos 2t\,. \tag{3.10}$$

4. Generation Of Phase Diagram

Form the Equation (3.7), we define the potential energy function, $I(x) = \int_0^x f\xi d\xi$ which implies that $\frac{dI}{dx} = f(x)$. Then the equation (3.1) can be rewritten in the form

$$\ddot{x} + f(x) = 0\,,$$

$$(\dot{x})^2 + 2I(x) - I(A) = 0\,. \tag{4.1}$$

The equation (4.1) for the equation of motion (3.7) with initial conditions (3.8) can be written in the form:

$$(\dot{x})^2 = (A - x)[(A + x)\frac{0.2\epsilon}{3}(A^2 + Ax + x^2)]\,. \tag{4.2}$$

Now the plot of $\dot{x}$ versus x represents a phase diagram for the differential equation (3.1) with initial conditions (3.2). In the equation if the value of A=1 and then the plot of versus x generated from equation (4.2) shows a closed boundary. The magnitude of positive and negative amplitudes of the nonlinear oscillations from equation (4.2) are 1 and -1.0717 respectively, which implies asymmetry of the phase diagram with respect to axis, whereas it is symmetric about the x-axis. Figure 1 show the phase diagram generated from equation (4.2) which indicates unequal magnitudes of the positive and negative amplitudes. The singular points are (0, 0) and (-10, 0). The point (0, 0) becomes the singular point whereas the other (-10, 0) becomes the saddle point. The range of amplitudes of to obtain the periodic solution is between -10 and 5. It is observed that the phase diagram corresponding to $x(0) = 5$ represents the separatrix, whereas for

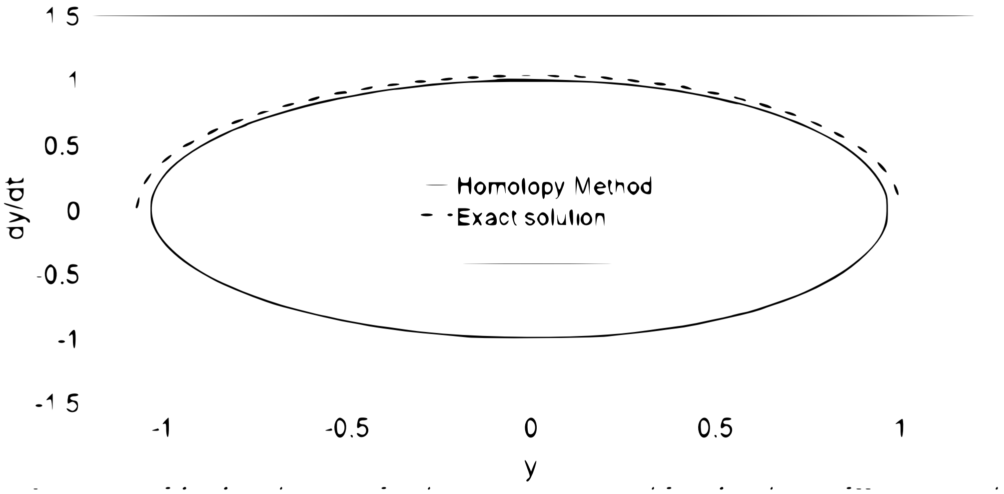

FIGURE 1. Comparison of the phase diagrams of eardrum equation generated from the solution of Homotopy analysis Method with the exact solution

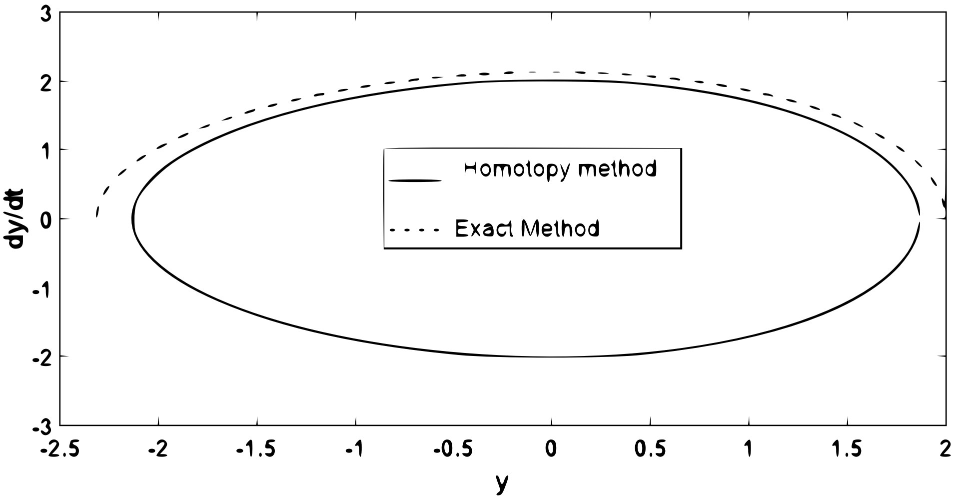

FIGURE 2. Comparison of the phase diagrams of eardrum equation generated from the solution of Homotopy analysis Method with the exact solution

$x(0) = 1$ represents the closed boundary having periodicity. It is very interesting that the phase diagram generated for $x(0) = 5$ have the close magnitudes of positive and negative amplitudes. Whereas the separatrix indicates the large difference in the magnitudes of positive and negative amplitudes. The homotopy solution accurately near to the region where the initial conditions are specified.

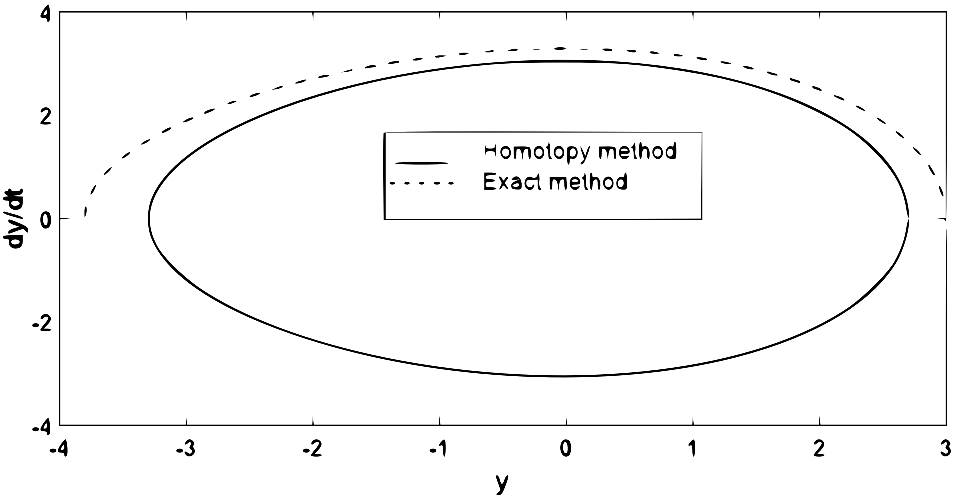

FIGURE 3. Comparison of the phase diagrams of eardrum equation generated from the solution of Homotopy analysis Method with the exact solution

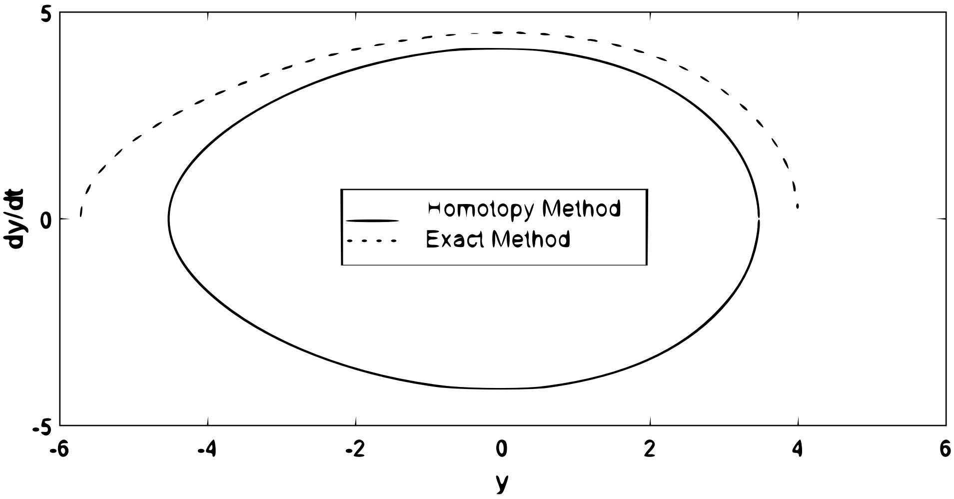

FIGURE 4. Comparison of the phase diagrams of eardrum equation generated from the solution of Homotopy analysis Method with the exact solution

When the values of $A = 2, 4, 5$ and 6 for the phase diagrams are generated in Figures 2 to 5 and compared with the solution obtained by homotopy method. It is very interesting to note that the phase diagram generated for $x(0) = 5$ from equation (3.10) has close magnitudes of positive and negative amplitudes,

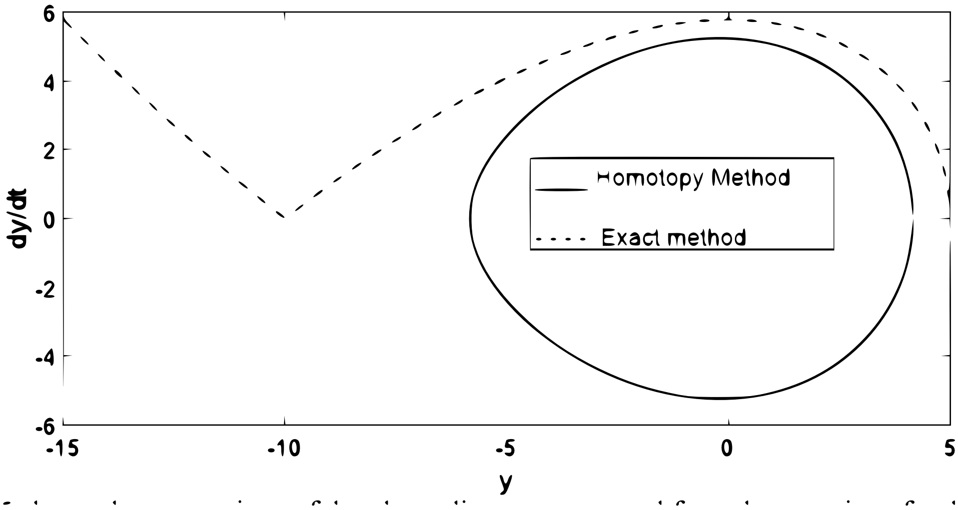

FIGURE 5. Comparison of the phase diagrams of eardrum equation generated from the solution of Homotopy analysis Method with the exact solution

whereas the separatrix indicates the large difference in the magnitudes of positive and negative amplitudes. The homotopy solution represents accurately close to the region at the specified initial conditions.

5. CONCLUDING REMARKS

The adequacy of the homotopy perturbation method (HPM) is examined by observing the oscillations of eardrum equation. For the specified large amplitudes large discrepancy is noted in magnitudes of positive and negative amplitudes using the homotopy method. From the phase plane diagrams it is observed that the maximum values that $x(0)$ can have for the eardrum to oscillate is -10 and 5. Beyond those values the eardrum does not oscillates.

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