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OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR FORCED IMPULSIVE DIFFERENTIAL EQUATIONS WITH DAMPING TERM UNDER VARIABLE DELAY

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ABSTRACT. In this paper, the oscillation criteria for second order nonlinear forced impulsive differential equations with damping term under variable delay are studied. We use arithmetic-geometric mean inequality, Riccati transformation to obtain the oscillation criteria. In literature there are no results for second order impulsive differential equations involving damping term with variable delay. The results obtained in this paper extend some of the existing result. An example is provided to illustrate the main result.

1. INTRODUCTION

The theory of impulsive differential equation have applications in control theory, physics, population dynamics, industrial robotics etc. The oscillation of solutions of second order impulsive differential equations are systematically studied by several authors [2–6]. In [7–9],the authors studied the oscillation of solutions of second order differential equations with constant delay and in [10–12] authors studied the oscillation of solutions of second order differential equations with variable delay. Motivated by the work of [8, 11], we obtain the oscillation criteria for second order nonlinear forced impulsive differential equation with damping term under variable delay. The results obtained in this paper extend some of the existing results and are illustrated by an example.

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Consider the second-order impulsive differential equation,

$$(1.1)\quad \begin{aligned} &\big(r(t)\Phi_\alpha(x'(t))\big)' + p(t)\Phi_\alpha(x'(t)) + q_0(t)\Phi_\alpha(x(t)) \\ &\qquad + \sum_{i=1}^{n} q_i(t)\Phi_{\beta_i}(x(t-\sigma(t))) = e(t), \quad t \neq \tau_k, \\ &\qquad x(\tau_k{}^+) = a_k x(\tau_k), \quad x'(\tau_k{}^+) = b_k x'(\tau_k), \quad t = \tau_k, \end{aligned}$$

where $\Phi_\star(s) = |s|^{\star-1}s$, $k \in \mathbb{N}$, $t \geq t_0$, τ_k is the impulse moments sequence with

$$\begin{aligned} 0 \leq t_0 &= \tau_0 < \tau_1 < \cdots < \tau_k < \ldots \lim_{k\to\infty} \tau_k = \infty, x(\tau_k{}^+) = \lim_{t\to\tau_k^{+0}} x(t), \\ x'(\tau_k{}^+) &= \lim_{h\to 0^+} \frac{x(\tau_k+h) - x(\tau_k{}^+)}{h}, x'(\tau_k{}^-) = \lim_{h\to 0^-} \frac{x(\tau_k+h) - x(\tau_k)}{h} = x'(\tau_k), \\ x(\tau_k^-) &= \lim_{t\to\tau_k^{-0}} x(t) = x(\tau_k). \end{aligned}$$

Let $J \subset \mathbb{R}$ be an interval and define

$$PLC(J,\mathbb{R}) = \{x : J \to \mathbb{R} : x(t) \text{ is piecewise-left continuous and has discontinuity of first kind at } \tau_k' s\}.$$

Define a delay function $D_k(t) = t - \tau_k - \sigma(t), t \in [\tau_k, \tau_{k+1}], k \in \mathbb{N}$.

Throughout this paper, we always assume the following conditions hold:

(A1) $r \in C^1([t_0,\infty),(0,\infty))$, $p, q_i, e \in PLC([t_0,\infty),\mathbb{R})$, $i = 0, 1, 2\ldots, n$;

(A2) $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$ are constants;

(A3) $b_k \geq a_k > 0$, $k \in \mathbb{N}$ are constants.

(A4) $\sigma(t) \in C([t_0,\infty))$,there exists a nonnegative constant σ such that $0 \leq \sigma(t) \leq \sigma$ for all $t \geq t_0$ and $\tau_{k+1} - \tau_k > \sigma$ for all $k \in \mathbb{N}$

(A5) There is one zero point $t_k \in (\tau_k, \tau_{k+1}]$ such that $D_k(t) < 0$ for $t \in (\tau_k, t_k)$, $D_k(t) > 0$ for $t \in (t_k, t_{k+1}]$ and $D_k(t_k) = 0$.

By a solution of (1.1), we mean a function $x \in PC([t_0,\infty),\mathbb{R})$ such that $x' \in PC([t_0,\infty),\mathbb{R})$ and $x(t)$ satisfies (1.1) for $t \geq t_0$. A nontrivial solution is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. An equation is called oscillatory if all its solutions are oscillatory.

2. MAIN RESULTS

We begin with the following notation. Let $k(s) = \max\{i : t_0 < \tau_i < s\}$, let $r_j = \max\{r(t) : t \in [c_j, d_j]\}$ and $\mathcal{F}(c_j, d_j) = \{u \in C^1\big([c_j, d_j],\mathbb{R})\big) : u(t) \not\equiv$

$0, u(c_j) = u(d_j) = 0\}, j = 1, 2$. For two constants $c, d \notin \{\tau_k\}$ with $c < d$ and a function $\phi \in C([c,d],\mathbb{R})$, we define an operator $\Omega : C([c,d],\mathbb{R}) \to \mathbb{R}$ by

$$\Omega_c^d[\phi] = \begin{cases} 0, & \text{for } k(c) = k(d), \\ \phi(\tau_{k(c)+1})\theta(c) + \sum_{i=k(c)+2}^{k(d)} \phi(\tau_i)\varepsilon(\tau_i), & \text{for } k(c) < k(d), \end{cases}$$

where

$$\theta(c) = \frac{(b_{k(c)+1})^\alpha - (a_{k(c)+1})^\alpha}{(a_{k(c)+1})^\alpha(\tau_{k(c)+1} - c)^\alpha}, \quad \varepsilon(\tau_i) = \frac{(b_i)^\alpha - (a_i)^\alpha}{(a_i)^\alpha(\tau_i - \tau_{i-1})^\alpha}.$$

For the discusion of the impulse moments of $x(t)$ and $x(t-\sigma(t))$, we need to consider the following four cases for $k(c_j) < k(d_j), j = 1, 2$.

(C1) $\tau_{k(c_j)} + \sigma < c_j$ and $\tau_{k(d_j)} + \sigma < d_j$
(C2) $\tau_{k(c_j)} + \sigma < c_j$ and $\tau_{k(d_j)} + \sigma > d_j$
(C3) $\tau_{k(c_j)} + \sigma > c_j$ and $\tau_{k(d_j)} + \sigma < d_j$
(C4) $\tau_{k(c_j)} + \sigma > c_j$ and $\tau_{k(d_j)} + \sigma > d_j$,

and three cases for $k(c_j) = k(d_j), j = 1, 2$

($\bar{C}1$) $\tau_{k(c_j)} + \sigma < c_j$
($\bar{C}2$) $c_j < \tau_{k(c_j)} + \sigma < d_j$
($\bar{C}3$) $\tau_{k(c_j)} + \sigma > d_j$.

Combining (C*) with ($\bar{C}*$), we get 12 cases. Throughout the paper we consider (C1) with ($\bar{C}1$) only. The discussions for other cases are similar and omitted.

The following preparatory lemmas will be useful to prove main theorem.

Lemma 2.1. *Let* $\{\beta_i\}$*,* $i = 1, 2, \ldots, n$*, be the n-tuple satisfying* $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$*. Then there exist an* n*-tuple* $(\eta_1, \eta_2, \ldots, \eta_n)$ *satisfying*

$$\sum_{i=1}^{n} \beta_i \eta_i = \alpha, \tag{2.1}$$

which also satisfies

$$\sum_{i=1}^{n} \eta_i \le 1, \quad 0 < \eta_i < 1, \tag{2.2}$$

The proof of Lemma 2.1 can be obtained easily from Lemma 1 of [2] by taking $\alpha_i = \beta_i/\alpha$.

The Lemma below can be found in [1].

Lemma 2.2. *Let X and Y be non-negative real numbers.Then*

$$\lambda XY^{\lambda-1} - X^{\lambda} \leq (\lambda-1)Y^{\lambda}, \lambda > 1,$$

where inequality holds if and only if $X = Y$.

Let $\alpha > 0, A > 0, B \geq 0$ and $y \geq 0$. Put $X = B^{\frac{\alpha}{\alpha+1}}, Y = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha} A^{\alpha} B^{\frac{-\alpha^2}{\alpha+1}}$, $\lambda = 1 + \frac{1}{\alpha}$ in Lemma 2.2, we have $Ay - By^{\frac{\alpha+1}{\alpha}} \leq \left(\frac{A}{\alpha+1}\right)^{\alpha+1} \left(\frac{\alpha}{B}\right)^{\alpha}$.

Theorem 2.1. *Suppose that for any $T \geq t_0$, there exist $c_j, d_j \notin \{\tau_k\}$, $j = 1, 2$ such that $T < c_1 - \sigma < c_1 < d_1 \leq c_2 - \sigma < c_2 < d_2$, and*

(2.3) $$p(t), q_i(t) \geq 0, (-1)^j e(t) \geq 0, t \in [c_j - \sigma, d_j] \backslash \{\tau_k\}, i = 0, 1, 2 \ldots n, j = 1, 2.$$

Let $\{\eta_i\}$, $i = 1, 2, \ldots, n$, be an n tuple satisfying (2.1)*and* (2.2)*. If there exist $u \in \mathcal{F}(c_j, d_j)$ such that,*

(2.4) $$\begin{aligned} &\int_{c_j}^{\tau_{k(c_j)+1}} \frac{(t - \tau_{k(c_j)} - \sigma(t))^{\alpha}}{(t - \tau_{k(c_j)})^{\alpha}} \bar{Q}(t) dt \\ &+ \sum_{l=k(c_j)+1}^{k(d_j)-1} \Big(\int_{\tau_l}^{t_l} \frac{(t - \tau_l)^{\alpha}}{b_l^{\alpha}(t + \sigma(t) - \tau_l)^{\alpha}} \bar{Q}(t) dt + \int_{t_l}^{\tau_{l+1}} \frac{(t - \tau_l - \sigma(t))^{\alpha}}{(t - \tau_l)^{\alpha}} \bar{Q}(t) dt \Big) \\ &+ \int_{\tau_{k(d_j)}}^{t_{k(d_j)}} \frac{(t - \tau_{k(d_j)})^{\alpha} \bar{Q}(t) dt}{b_{k(d_j)}^{\alpha}(t + \sigma(t) - \tau_{k(d_j)})^{\alpha}} + \int_{t_{k(d_j)}}^{d_j} \frac{(t - \tau_{k(d_j)} - \sigma(t))^{\alpha}}{(t - \tau_{k(d_j)})^{\alpha}} \bar{Q}(t) dt \\ &- \int_{c_j}^{d_j} \frac{r(t)}{(\alpha+1)^{\alpha+1}} \left[(\alpha+1) u'(t) - \frac{p(t) u(t)}{r(t)} \right]^{\alpha+1} dt + \int_{c_j}^{d_j} q_0(t) |u(t)|^{\alpha+1} dt \\ &> r_j \Omega_{c_j}^{d_j} [|u(t)|^{\alpha+1}] \end{aligned}$$

where, $\bar{Q}(t) = Q(t)|u(t)|^{\alpha+1}$ and $Q(t) = {\eta_0}^{-\eta_0} |e(t)|^{\eta_0} \prod_{i=1}^{n} \eta_i^{-\eta_i} (q_i(t))^{\eta_i}$, $\eta_0 = 1 - \sum_{i=1}^{n} \eta_i$, then (1.1) *is oscillatory.*

Proof. Let us suppose that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t) > 0$ for $t \in [c_1, d_1]$. Define

$$w(t) = \frac{r(t) \Phi_{\alpha}(x'(t))}{\Phi_{\alpha}(x(t))}.$$

Then for $t \in [c_1, d_1]$ and $t \neq \tau_k$, we have

$$w'(t) = \Big[-\sum_{i=1}^{n} q_i(t)\Phi_{\beta_i-\alpha}(x(t-\sigma(t))) - \frac{|e(t)|}{\Phi_\alpha(x(t-\sigma(t)))}\Big] \frac{\Phi_\alpha(x(t-\sigma(t)))}{\Phi_\alpha(x(t))} - \frac{p(t)}{r(t)}w(t) - \frac{\alpha}{(r(t))^{\frac{1}{\alpha}}}|w(t)|^{\frac{\alpha+1}{\alpha}} - q_0(t). \tag{2.5}$$

By arithmetic-geometric mean inequality, $\sum_{i=0}^{n}\eta_i v_i \geq \prod_{i=0}^{n} v_i^{\eta_i}, v_i \geq 0$. Take $v_0 = \eta_0^{-1}\frac{|e(t)|}{\Phi_\alpha(x(t-\sigma(t)))}$ and $v_i = \eta_i^{-1}q_i(t)\Phi_{\beta_i-\alpha}(x(t-\sigma(t)))$ and from (2.1) and (2.2), we get

$$-\sum_{i=1}^{n} q_i(t)\Phi_{\beta_i-\alpha}(x(t-\sigma(t))) - \frac{|e(t)|}{\Phi_\alpha(x(t-\sigma(t)))} \leq -{\eta_0}^{-\eta_0}|e(t)|^{\eta_0}\prod_{i=1}^{n}\eta_i^{-\eta_i}(q_i(t))^{\eta_i}.$$

Now equation (2.5) becomes

$$w'(t) \leq -Q(t)\frac{\Phi_\alpha(x(t-\sigma(t)))}{\Phi_\alpha(x(t))} - q_0(t) - \frac{p(t)}{r(t)}w(t) - \frac{\alpha}{(r(t))^{\frac{1}{\alpha}}}|w(t)|^{\frac{\alpha+1}{\alpha}}, \tag{2.6}$$

where $Q(t) = {\eta_0}^{-\eta_0}|e(t)|^{\eta_0}\prod_{i=1}^{n}\eta_i^{-\eta_i}(q_i(t))^{\eta_i}$.

If $k(c_1) < k(d_1)$, then there are impulsive moments $\tau_{k(c_1)+1}, \tau_{k(c_1)+2}, \ldots, \tau_{k(d_1)}$ in $[c_1, d_1]$ and zero point t_l of $D_l(t)$ in each (τ_l, τ_{l+1}) for $l = k(c_1)+1, k(c_1)+2, ..., k(d_1)-1$. Multiplying both sides of (2.6) by $|u(t)|^{\alpha+1}$,where $u(t) \in \mathcal{F}(c_1, d_1)$ and integrating over $[c_1, d_1]$, then using integration by parts and the fact that $u(c_1) = u(d_1)$, we obtain

$$\sum_{l=k(c_1)+1}^{k(d_1)} |u(\tau_l)|^{\alpha+1}[w(\tau_l) - w(\tau_l^+)]$$
$$\leq -\left[\int_{c_1}^{\tau_{k(c_1)+1}} + \sum_{l=k(c_1)+1}^{k(d_1)-1}\left(\int_{\tau_l}^{t_l} + \int_{t_l}^{\tau_{l+1}}\right) + \int_{\tau_{k(d_1)}}^{t_{k(d_1)}} + \int_{t_{k(d_1)}}^{d_1}\right]$$
$$Q(t)|u(t)|^{\alpha+1}\frac{\Phi_\alpha(x(t-\sigma(t)))}{\Phi_\alpha(x(t))}dt + \int_{c_1}^{d_1}\Bigg[(\alpha+1)|u(t)|^\alpha u'(t)|w(t)|$$
$$-\frac{p(t)}{r(t)}|u(t)|^{\alpha+1}|w(t)| - \frac{\alpha}{(r(t))^{\frac{1}{\alpha}}}|w(t)|^{\frac{\alpha+1}{\alpha}}|u(t)|^{\alpha+1} - q_0(t)|u(t)|^{\alpha+1}\Bigg]dt$$

$$\leq -\Bigg[\int_{c_1}^{\tau_{k(c_1)+1}} + \sum_{l=k(c_1)+1}^{k(d_1)-1}\left(\int_{\tau_l}^{t_l} + \int_{t_l}^{\tau_{l+1}}\right) + \int_{\tau_{k(d_1)}}^{t_{k(d_1)}} + \int_{t_{k(d_1)}}^{d_1}\Bigg] Q(t)|u(t)|^{\alpha+1}$$
$$\frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)}dt + \int_{c_1}^{d_1}\Bigg[\left(\left|(\alpha+1)u'(t) - \frac{p(t)}{r(t)}|u(t)|\right|\right)|w(t)||u(t)|^\alpha$$
$$- \frac{\alpha}{(r(t))^{\frac{1}{\alpha}}}|w(t)|^{\frac{\alpha+1}{\alpha}}|u(t)|^{\alpha+1}\Bigg]dt - \int_{c_1}^{d_1} q_0(t)|u(t)|^{\alpha+1}dt$$

Use Lemma 2.2 with $A = \left|(\alpha+1)u'(t) - \frac{p(t)|u(t)|}{r(t)}\right|, B = \frac{\alpha}{(r(t))^{\frac{1}{\alpha}}}, y = |w(t)||u(t)|^\alpha,$

$$\left(\left|(\alpha+1)u'(t) - \frac{p(t)}{r(t)}|u(t)|\right|\right)|w(t)||u(t)|^\alpha - \frac{\alpha}{(r(t))^{\frac{1}{\alpha}}}|w(t)|^{\frac{\alpha+1}{\alpha}}|u(t)|^{\alpha+1}$$
$$\leq \frac{r(t)}{(\alpha+1)^{\alpha+1}}\left[(\alpha+1)u'(t) - \frac{p(t)u(t)}{r(t)}\right]^{\alpha+1}. \tag{2.7}$$

Apply (2.7) in above inequality we get

$$\sum_{l=k(c_1)+1}^{k(d_1)} |u(\tau_l)|^{\alpha+1}[w(\tau_l) - w(\tau_l^+)] \tag{2.8}$$
$$\leq -\Big[\int_{c_1}^{\tau_{k(c_1)+1}} + \sum_{l=k(c_1)+1}^{k(d_1)-1}\Big(\int_{\tau_l}^{t_l} + \int_{t_l}^{\tau_{l+1}}\Big) + \int_{\tau_{k(d_1)}}^{t_{k(d_1)}} + \int_{t_{k(d_1)}}^{d_1}\Big]$$
$$Q(t)|u(t)|^{\alpha+1}\frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)}dt$$
$$+ \int_{c_1}^{d_1} \frac{r(t)}{(\alpha+1)^{\alpha+1}}\left[(\alpha+1)u'(t) - \frac{p(t)u(t)}{r(t)}\right]^{\alpha+1}dt - \int_{c_1}^{d_1} q_0(t)|u(t)|^{\alpha+1}dt$$

For $t = \tau_k$, $k = 1, 2, \ldots$, we have $w(\tau_k{}^+) = \frac{b_k{}^\alpha}{a_k{}^\alpha}w(\tau_k)$.

Therefore from (2.8) we get

$$\text{(2.9)} \quad \Big[\int_{c_1}^{\tau_{k(c_1)+1}} + \sum_{l=k(c_1)+1}^{k(d_1)-1} \Big(\int_{\tau_l}^{t_l} + \int_{t_l}^{\tau_{l+1}}\Big) + \int_{\tau_{k(d_1)}}^{t_{k(d_1)}} + \int_{t_{k(d_1)}}^{d_1}\Big]\bar{Q}(t)\frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)}dt$$
$$- \int_{c_1}^{d_1} \frac{r(t)}{(\alpha+1)^{\alpha+1}}\Big[(\alpha+1)u'(t) - \frac{p(t)u(t)}{r(t)}\Big]^{\alpha+1}dt + \int_{c_1}^{d_1} q_0(t)|u(t)|^{\alpha+1}dt$$
$$\leq \sum_{l=k(c_1)+1}^{k(d_1)} |u(\tau_l)|^{\alpha+1}\Big[\frac{{b_l}^\alpha - {a_l}^\alpha}{{a_l}^\alpha}\Big]w(\tau_l), \quad \text{where } \bar{Q}(t) = Q(t)|u(t)|^{\alpha+1}.$$

Now for $t \in [c_1, d_1]\backslash\{\tau_l\}$, from (1.1), it is clear that

$$\big(r(t)\Phi_\alpha(x'(t))\big)' + p(t)\Phi_\alpha(x'(t)) = e(t) - q_0(t)\Phi_\alpha(x(t)) - \sum_{i=1}^{n} q_i(t)\Phi_{\beta_i}(x(t)) \leq 0.$$

Multiplying both side of above inequality by $\bar{p}(t) = \exp\big(\int^t \frac{p(s)}{r(s)}ds\big)$, we get

$$\Big(\bar{p}(t)r(t)\Phi_\alpha(x'(t))\Big)' = \bar{p}(t)\Big(e(t) - q_0(t)\Phi_\alpha(x(t)) - \sum_{i=1}^{n} q_i(t)\Phi_{\beta_i}(x(t))\Big) \leq 0,$$

which implies that $\big[\bar{p}(t)r(t)\Phi_\alpha(x'(t))\big]$ is non-increasing on $[c_1, d_1]\backslash\{\tau_l\}$.Because there are different integrations in (2.9), we will estimate $\frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)}$ in each interval of t.

Case (i): If $t_l < t \leq \tau_{l+1}$, for $l = k(c_1)+1, \ldots, k(d_1)-1$, then $(t-\sigma(t), t) \subset \big(\tau_l, \tau_{l+1}\big]$. Thus there is no impulsive moment in $\big(t-\sigma(t), t\big)$. Therefore for any $s \in \big(t-\sigma(t), t\big)$, there exists a $\xi_l \in (\tau_l, s)$ such that $x(s) > x(s) - x(\tau_l^+) = x'(\xi_l)(s-\tau_l)$. Since $x(\tau_l^+) > 0$, the function $\Phi_\alpha(.)$ is an increasing function and $\big[\bar{p}(t)r(t)\Phi_\alpha(x'(t))\big]$ is non-increasing on (τ_l, τ_{l+1}), we have

$$\Phi_\alpha(x(s)) > \Phi_\alpha[x'(\xi_l)(s-\tau_l)] = \frac{\bar{p}(\xi_l)r(\xi_l)}{\bar{p}(\xi_l)r(\xi_l)}\Phi_\alpha(x'(\xi_l))(s-\tau_l)^\alpha$$
$$\geq \frac{\bar{p}(s)r(s)}{\bar{p}(\xi_l)r(\xi_l)}\Phi_\alpha(x'(s))(s-\tau_l)^\alpha.$$

Therefore, $\Phi_\alpha(x'(s)(s-\tau_l)) < \dfrac{\bar{p}(\xi_l)r(\xi_l)}{\bar{p}(s)r(s)}\Phi_\alpha(x(s)) \le \Phi_\alpha(x(s)), \xi_l \in (\tau_l, s)$. Thus $\dfrac{x'(s)}{x(s)} < \dfrac{1}{s-\tau_l}$. Integrating both sides from $t-\sigma(t)$ to t, we obtain

$$\frac{x(t-\sigma(t))}{x(t)} > \frac{t-\tau_l-\sigma(t)}{t-\tau_l}, t \in (t_l, \tau_{l+1}].$$

Case (ii): If $\tau_l < t < t_l$, for $l = k(c_1)+1, \dots, k(d_1)$, then $\tau_l - \sigma < t - \sigma(t) < \tau_l < t$. There is an impulsive moment τ_l in $(t-\sigma(t), t)$. For any $t \in (\tau_l, t_l)$, we have $x(t) - x(\tau_l^+) = x'(\zeta_l)(t-\tau_l), \zeta_l \in (\tau_l, t)$. Using the impulsive conditions and the monotone properties of $r(t), \Phi_\alpha(.)$ and $\left[\bar{p}(t)r(t)\Phi_\alpha(x'(t))\right]$, we get

$$\begin{aligned}\Phi_\alpha(x(t)-a_ix(\tau_l)) = \frac{\bar{p}(\zeta_l)r(\zeta_l)}{\bar{p}(\zeta_l)r(\zeta_l)}\Phi_\alpha(x'(\zeta_l))(t-\tau_l)^\alpha &\le \frac{\bar{p}(\tau_i^+)r(\tau_i^+)}{\bar{p}(\zeta_l)r(\zeta_l)}\Phi_\alpha(x'(\tau_i^+))(t-\tau_l)^\alpha \\ &= \frac{\bar{p}(\tau_i)r(\tau_i)}{\bar{p}(\zeta_l)r(\zeta_l)}\Phi_\alpha(b_ix'(\tau_i)(t-\tau_l)).\end{aligned}$$

Since $x(\tau_i) > 0$, we have

$$\Phi_\alpha\Big(\frac{x(t)}{x(\tau_l)} - a_i\Big) \le \frac{\bar{p}(\tau_i)r(\tau_i)}{\bar{p}(\zeta_l)r(\zeta_l)}\Phi_\alpha\Big(b_i\frac{x'(\tau_l)}{x(\tau_l)}(t-\tau_l)\Big). \tag{2.10}$$

In addition, $x(\tau_l) > x(\tau_l) - x(\tau_l - \sigma(t)) = x'(\delta_l)\sigma(t), \quad \delta_l \in (\tau_l - \sigma(t), \tau_l)$.

Similar to the analysis in case(i), we have

$$\frac{x'(\tau_l)}{x(\tau_l)} < \frac{1}{\sigma(t)}. \tag{2.11}$$

From (2.10) and (2.11) and note that the monotone properties of $\Phi_\alpha(.), \bar{p}(t)$ and $r(t)$, we get $\dfrac{x(t)}{x(\tau_l)} < a_l + \dfrac{b_l}{\sigma(t)}(t-\tau_l)$. In view of assumption (A_3), we have

$$\frac{x(\tau_l)}{x(t)} > \frac{\sigma(t)}{\sigma(t)a_l + b_l(t-\tau_l)} \ge \frac{\sigma(t)}{b_i(t+\sigma(t)-\tau_l)} > 0. \tag{2.12}$$

On the other hand, using similar analysis of case(i), we get

$$\frac{x'(s)}{x(s)} < \frac{1}{s-\tau_l+\sigma(t)}, \quad s \in (\tau_l - \sigma(t), \tau_l). \tag{2.13}$$

Integrating (2.13) from $t-\sigma(t)$ to τ_l, where $t \in (\tau_l, \tau_l + \sigma(t))$, we have

$$\frac{x(t-\sigma(t))}{x(\tau_l)} > \frac{t-\tau_l}{\sigma(t)} \ge 0. \tag{2.14}$$

From (2.12) to (2.14), we obtain $\dfrac{x(t-\sigma(t))}{x(t)} > \dfrac{t-\tau_l}{b_l(t+\sigma(t)-\tau_l)}$, $t \in (\tau_l, t_l)$. Using same as the proof of Case(i) and Case(ii), we can prove the following cases.

Case (iii): If $c_1 \leq t \leq \tau_{k(c_1+1}$, then $\dfrac{x(t-\sigma(t))}{x(t)} > \dfrac{t-\tau_{k(c_1)}-\sigma(t)}{t-\tau_{k(c_1)}}$.

Case (iv): If $t_{k(d_1)} < t \leq d_1$, then $\dfrac{x(t-\sigma(t))}{x(t)} > \dfrac{t-\tau_{k(d_1)}-\sigma(t)}{t-\tau_{k(d_1)}}$.

On the other hand, for $t \in (\tau_{l-1}, \tau_l) \subset [c_1, d_1], l = k(c_1)+2, ..., k(d_1)$, there exists $\gamma_l \in (\tau_{l-1}, t)$ such that $x(t) - x(\tau_{l-1}^+) = x'(\gamma_l)(t-\tau_l)$.

In view of $x(\tau_{l-1}^+) > 0$ and the monotone properties of $\Phi_\alpha(.), \bar{p}(t)r(t)\Phi_\alpha(t)$ we obtain $\Phi_\alpha(x(t)) > \Phi_\alpha(x'(\gamma_l))(t-\tau_{l-1})^\alpha \geq \dfrac{\bar{p}(t)r(t)}{\bar{p}(\gamma_l)r(\gamma_l)}\Phi_\alpha(x'(t))(t-\tau_{l-1})^\alpha$ which implies, $\dfrac{r(t)\Phi_\alpha(x'(t))}{\Phi_\alpha(x(t))} < \dfrac{\bar{p}(\gamma_l)r(\gamma_l)}{\bar{p}(t)(t-\tau_{l-1})^\alpha} \leq \dfrac{r(\gamma_l)}{(t-\tau_{l-1})^\alpha}$.

This is $w(t) \leq \dfrac{r(\gamma_l)}{(t-\tau_{l-1})^\alpha}$. Letting t tends to τ_l^-, we obtain

$$w(\tau_l) \leq \frac{r_1}{(\tau_l-\tau_{l-1})^\alpha}, \text{for } \tau_l \in [c_1, d_1], \quad l = k(c_1)+2, ..., k(d_1); \tag{2.15}$$

Using similar analysis we can get

$$w(\tau_l) \leq \frac{r_1}{(\tau_l-c_1)^\alpha}, \text{for } \tau_l \in [c_1, d_1], \quad l = k(c_1)+1. \tag{2.16}$$

Using case(i)-(iv), (2.9), (2.15), (2.16) and (A3), we obtain

$$\begin{aligned}
&\int_{c_1}^{\tau_{k(c_1)+1}} \frac{(t-\tau_{k(c_1)}-\sigma(t))^\alpha}{(t-\tau_{k(c_1)})^\alpha}\bar{Q}(t)dt \\
&+ \sum_{l=k(c_1)+1}^{k(d_1)-1}\Big(\int_{\tau_l}^{t_l} \frac{(t-\tau_l)^\alpha}{b_l^\alpha(t+\sigma(t)-\tau_l)^\alpha}\bar{Q}(t)dt + \int_{t_l}^{\tau_{l+1}} \frac{(t-\tau_l-\sigma(t))^\alpha}{(t-\tau_l)^\alpha}\bar{Q}(t)dt\Big) \\
&+ \int_{\tau_{k(d_1)}}^{t_{k(d_1)}} \frac{(t-\tau_{k(d_1)})^\alpha}{b_{k(d_1)}^\alpha(t+\sigma(t)-\tau_{k(d_1)})^\alpha}\bar{Q}(t)dt + \int_{t_{k(d_1)}}^{d_1} \frac{(t-\tau_{k(d_1)}-\sigma(t))^\alpha}{(t-\tau_{k(d_1)})^\alpha}\bar{Q}(t)dt \\
&- \int_{c_1}^{d_1} \frac{r(t)}{(\alpha+1)^{\alpha+1}}\Big[(\alpha+1)u'(t) - \frac{p(t)u(t)}{r(t)}\Big]^{\alpha+1}dt + \int_{c_1}^{d_1} q_0(t)|u(t)|^{\alpha+1}dt
\end{aligned}$$

$$\leq \sum_{l=k(c_1)+1}^{k(d_1)} \Big(\frac{{b_l}^{\alpha} - {a_l}^{\alpha}}{{a_l}^{\alpha}}\Big) |u(\tau_l)|^{\alpha+1} w(\tau_l)$$

$$\leq \Big(\frac{b_{k(c_1)+1}^{\alpha} - a_{k(c_1)+1}^{\alpha}}{a_{k(c_1)+1}^{\alpha}}\Big) |u(\tau_{k(c_1)+1})|^{\alpha+1} \frac{r_1}{(\tau_{k(c_1)+1} - c_1)^{\alpha}} + \sum_{l=k(c_1)+2}^{k(d_1)} \Big(\frac{{b_l}^{\alpha} - {a_l}^{\alpha}}{{a_l}^{\alpha}}\Big) |u(\tau_l)|^{\alpha+1} \frac{r_1}{(\tau_l - \tau_{l-1})^{\alpha}}$$

$$= r_1 \Omega_{c_1}^{d_1}[|u(t)|^{\alpha+1}],$$

which contradicts (2.4) for $j = 1$.

If $k(c_1) = k(d_1)$ then by condition $(\bar{C}_1)$ there is no impulsive moments in $[c_1, d_1]$.By similar method as used above we obtain

$$\int_{c_1}^{d_1} \frac{r(t)}{(\alpha+1)^{\alpha+1}} \Big[(\alpha+1)u'(t) - \frac{p(t)u(t)}{r(t)}\Big]^{\alpha+1} dt - \int_{c_1}^{d_1} q_0(t)|u(t)|^{\alpha+1} dt + \int_{c_1}^{d_1} \frac{(t - c_1 - \sigma(t))^{\alpha}}{(t - c_1)^{\alpha}} \bar{Q}(t) dt \leq 0.$$

It is again a contradiction with (2.4). This completes the proof when $x(t)$ is positive.The proof when $x(t)$ is eventually negative is analogous by repeating a similar argument on the interval $[c_2, d_2]$. □

Remark 2.1. *When $p(t) = 0$ and $\sigma(t) = \sigma$, Theorem 2.1 reduces to Theorem 2.3 of [7].*

Remark 2.2. *When $p(t) = 0$, result is reduces to the result of [11].*

3. Example

In this section, we give an example to illustrate our results.

Example 1. *Consider the impulsive differential equation*

$$\begin{aligned} &x''(t) + (\sin t)x'(t) + v_1 \Phi_{\frac{5}{2}}(x(t - \frac{\pi}{12}\sin^2 t)) \\ &\qquad + v_2 \Phi_{\frac{1}{2}}(x(t - \frac{\pi}{12}\sin^2 t)) = -\sin 2t, t \neq \tau_k \qquad (3.1) \\ &x({\tau_k}^+) = a_k x(\tau_k), \quad x'({\tau_k}^+) = b_k x'(\tau_k), \quad t = \tau_k \end{aligned}$$

where $\tau_k : \tau_{n,i} = 2n\pi + \frac{2\pi}{9} + (i-1)\frac{\pi}{3}, i = 1,2, n \in \mathbb{N}$, v_i, $i = 1,2$ *are positive constants,* $b_k \geq a_k, k \in \mathbb{N}$.

We see that $r(t) = 1, p(t) = \sin t, q_0(t) = 0, q_1(t) = v_1, q_2(t) = v_2, e(t) = -\sin 2t, \alpha = 1, \beta_1 = 5/2, \beta_2 = 1/2$. *Choose* $\eta_1 = 1/3, \eta_2 = 1/3$ *and* $\eta_0 = 1/3$. *So the conditions of Lemma 2.1 are satisfied. For any* $T \geq 0$,*we can choose* n_0 *large enough such that* $T < c_1 = 2n\pi + \frac{\pi}{6}, d_1 = 2n\pi + \frac{\pi}{3}, c_2 = 2n\pi + \frac{\pi}{2} d_2 = 2n\pi + \frac{2\pi}{3}, n = 1,2,\ldots$. *There are impulsive moments* $\tau_{n,1}$ *in* $[c_1, d_1]$ *and* $\tau_{n,2}$ *in* $[c_2, d_2]$. *The variable delay* $\sigma(t) = \frac{\pi}{12}\sin^2 t$ *satisfies* $0 \leq \sigma(t) \leq \sigma = \pi/12$. *From* $\tau_{n,2} - \tau_{n,1} = \pi/3 > \pi/12$ *and* $\tau_{n+1,1} - \tau_{n,2} = 5\pi/3 > \pi/12$ *for all* $n > n_0$.*Therefore* $\tau_{k+1} - \tau_k > \sigma$. *Let* $D_k(t) = t - \tau_k - \sigma(t) = t - \left(2n\pi + \frac{2\pi}{9} + (i-1)\frac{\pi}{3}\right) - \frac{\pi}{12}\sin^2 t$ *and there exist zero points* $t_1 \approx 0.8445 \in (\tau_{n,1}, d_1]$ *and* $t_2 \approx 1.9680 \in (\tau_{n,2}, d_2]$. *Moreover, conditions (C1) and* (2.3) *are satisfied.*

For $t \in [c_1, d_1]$,*let* $u(t) = \sin 6t$. *Then* $\bar{Q}(t) = 3(v_1v_2)^{1/3}|\sin 2t|^{1/3}\sin^2 6t$ *and the left side of* (2.4) *is*

$$\begin{aligned}
&\int_{2n\pi+\pi/6}^{2n\pi+2\pi/9} \frac{t - (2n\pi - \frac{13\pi}{9}) - \frac{\pi}{12}\sin^2 t}{t - (2n\pi - \frac{13\pi}{9})}\bar{Q}(t)dt \\
&+ \int_{2n\pi+2\pi/9}^{2n\pi+0.8445} \frac{t - (2n\pi + 2\pi/9)}{b_{n,1}(t - (2n\pi + \frac{2\pi}{9}) + \frac{\pi}{12}\sin^2 t)}\bar{Q}(t)dt \\
&+ \int_{2n\pi+0.8445}^{2n\pi+\pi/3} \frac{t - (2n\pi + \frac{2\pi}{9}) - \frac{\pi}{12}\sin^2 t}{t - (2n\pi + \frac{2\pi}{9})}\bar{Q}(t)dt \\
&- \int_{2n\pi+\pi/6}^{2n\pi+\pi/3} \frac{[12\cos 6t - \sin t \sin 6t]^2}{4}dt \\
&\approx 3(v_1v_2)^{1/3}\Big[0.04958 + 0.01358 + \frac{0.04604}{b_{n,1}}\Big] - 9.5496 \\
&\approx 3(v_1v_2)^{1/3}\Big[0.06316 + \frac{0.04604}{b_{n,1}}\Big] - 9.5496.
\end{aligned}$$

On the other hand, the right side of (2.4) *is*

$$\Omega_{c_1}^{d_1}[|u(t)|^{\alpha+1}] = \sin^2 6(2n\pi + 2\pi/9)\frac{b_{n,1} - a_{n,1}}{a_{n,1}(\pi/18)} = \frac{27}{2\pi}\Big[\frac{b_{n,1} - a_{n,1}}{a_{n,1}}\Big].$$

Thus condition (2.4) *is satisfied if*

$$3(v_1v_2)^{1/3}\Big[0.06316 + \frac{0.04604}{b_{n,1}}\Big] - 9.5496 > \frac{27}{2\pi}\Big[\frac{b_{n,1} - a_{n,1}}{a_{n,1}}\Big].$$

Similarly for $t \in [c_2, d_2]$, *let* $u(t) = \sin 6t$. *Then the left side of* (2.4) *is*

$$\approx 3(v_1v_2)^{1/3}\Big[0.02665 + \frac{0.04421}{b_{n,2}}\Big] - 9.4516.$$

Also the right side is $\Omega_{c_2}^{d_2}[|u(t)|^{\alpha+1}] = \dfrac{27}{2\pi}\Big[\dfrac{b_{n,2}-a_{n,2}}{a_{n,2}}\Big]$. *Thus condition* (2.4) *is satisfied if*

$$3(v_1v_2)^{1/3}\Big[0.02665 + \frac{0.04421}{b_{n,2}}\Big] - 9.4516 > \frac{27}{2\pi}\Big[\frac{b_{n,2}-a_{n,2}}{a_{n,2}}\Big]$$

Hence by Theorem 2.1, equation (3.1) *is oscillatory if*

$$3(v_1v_2)^{1/3}\Big[0.06316 + \frac{0.04604}{b_{n,1}}\Big] - 9.5496 > \frac{27}{2\pi}\Big[\frac{b_{n,1}-a_{n,1}}{a_{n,1}}\Big]$$

and

$$3(v_1v_2)^{1/3}\Big[0.02665 + \frac{0.04421}{b_{n,2}}\Big] - 9.4516 > \frac{27}{2\pi}\Big[\frac{b_{n,2}-a_{n,2}}{a_{n,2}}\Big].$$

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