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QUALITATIVE ANALYSIS OF EQUATIONS OF THE REGULATORY OF LIVER CELLS IN HEPATITIS B

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ABSTRACT. The purpose of this article is to present a system of functional differential equations with a delay argument for understanding the general patterns of the formation of infectious viral hepatitis B based on the interrelated activity of liver cells (LC) and hepatitis B viruses (HBV) at the molecular genetic level. The results of the qualitative analysis show that it is possible to analyze important features of the model without reaching explicit solutions to this differential equation.

1. INTRODUCTION

The processes inside a living system are often non-linear. The mathematical models that represent their activities include non-linear expressions. For this reason, there are mathematical difficulties in solving. This applies to the theory of differential equations. The methods of qualitative analysis of the theory of differential equations make it possible to analyze important features of the model without reaching explicit solutions to this differential equation [1, 2].

We use a system of functional differential equations with a delay argument to analyze the regulatorika of liver cells in hepatitis B based on a mathematical model. The mathematical model of the regulatory of liver cells in hepatitis B is presented as follows [3, 4]:

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$$\begin{aligned} &\varepsilon_1 \frac{dX(t)}{dt} = \frac{aX^2(t-1)}{1+X^2(t-1)+cY^2(t-1)} - X(t); \\ &\varepsilon_2 \frac{dY(t)}{dt} = \frac{bX(t-1)Y(t-1)}{1+dX^2(t-1)+Y^2(t-1)} - Y(t);\ \ t > 1; \\ &X(t) = \varphi_1(t),\ \ Y(t) = \varphi_2(t),\ \ t \in [0,\ 1]\,. \end{aligned} \tag{1.1}$$

where $X(t),\, Y(t)$ - the values defining the activity of the molecular genetic systems of the LC and HBV; $a,\, b$ - constant product formation rates of molecular genetic systems of LC and HBV; $c,\, d$ - parameters of the degree of inter-repression of molecular genetic systems of hepatocytes and hepatitis B viruses; - parameters of the regulatory of LC and HBV; $\varphi_1(t),\ \varphi_2(t)$ - continuous functions on $[0, 1]$. All parameters are positive.

Analysis of the mathematical model of the regulatory of liver cells in hepatitis B allows us to understand the general patterns of the formation of infectious viral hepatitis B based on the interrelated activity of LC and HBV at the molecular genetic level and to identify the dynamic types of their behavior.

2. Qualitative Analysis

The analytical solutions of (1.1) are difficult to find. Therefore, a qualitative analysis of this system is required. We find solutions of (1.1) in the interval $(1,\ 2]$ for $t > 1$, satisfying the initial conditions $\varphi_1(t),\ \varphi_2(t)$, specified in the interval $[0,\ 1]$, using the method of successive integration [5, 6]. $X(t)$ and $Y(t)$ are equal to continuous functions $\varphi_1(t)$ and $\varphi_2(t)$ in the interval $[0,\ 1]$, respectively. This implies $X\,(t-1) = \varphi_1\,(t-1)\,,\ Y\,(t-1) = \varphi_2\,(t-1)$. We put these conditions into (1.1) and obtain the following equations:

$$\begin{aligned} &\varepsilon_1 \frac{dX(t)}{dt} = \frac{a{\varphi_1}^2(t-1)}{1+{\varphi_1}^2(t-1)+c{\varphi_2}^2(t-1)} - X\,(t)\,; \\ &\varepsilon_2 \frac{dY(t)}{dt} = \frac{b\varphi_1(t-1)\varphi_2(t-1)}{1+d{\varphi_1}^2(t-1)+{\varphi_2}^2(t-1)} - Y\,(t)\,. \end{aligned} \tag{2.1}$$

Let the solution we are looking for be as follows:

$$X\,(t) = g\,(t)\,e^{-\frac{t}{\varepsilon_1}}, \qquad Y\,(t) = q\,(t)\,e^{-\frac{t}{\varepsilon_2}}. \tag{2.2}$$

From here we get these expressions

$$\frac{dX(t)}{dt} = \frac{dg(t)}{dt}e^{-\frac{t}{\varepsilon_1}} - \frac{1}{\varepsilon_1}g\left(t\right)e^{-\frac{t}{\varepsilon_1}};$$

$$\frac{dY(t)}{dt} = \frac{dq(t)}{dt}e^{-\frac{t}{\varepsilon_2}} - \frac{1}{\varepsilon_2}q\left(t\right)e^{-\frac{t}{\varepsilon_2}}.$$

Substitute in equation (2.1) and we can write be as follows:

(2.3)
$$\begin{aligned} \varepsilon_1\frac{dg(t)}{dt}e^{-\frac{t}{\varepsilon_1}} - g\left(t\right)e^{-\frac{t}{\varepsilon_1}} &= \frac{a{\varphi_1}^2(t-1)}{1+{\varphi_1}^2(t-1)+c{\varphi_2}^2(t-1)} - g\left(t\right)e^{-\frac{t}{\varepsilon_1}};\\ \varepsilon_2\frac{dq(t)}{dt}e^{-\frac{t}{\varepsilon_2}} - q\left(t\right)e^{-\frac{t}{\varepsilon_2}} &= \frac{b\varphi_1(t-1)\varphi_2(t-1)}{1+d{\varphi_1}^2(t-1)+{\varphi_2}^2(t-1)} - q\left(t\right)e^{-\frac{t}{\varepsilon_2}}.\end{aligned}$$

After simplifying the obtained equations (2.3)

(2.4)
$$\begin{aligned} \frac{dg(t)}{dt} &= \frac{a}{\varepsilon_1}e^{\frac{t}{\varepsilon_1}}\frac{{\varphi_1}^2(t-1)}{1+{\varphi_1}^2(t-1)+c{\varphi_2}^2(t-1)};\\ \frac{dq(t)}{dt} &= \frac{b}{\varepsilon_2}e^{\frac{t}{\varepsilon_2}}\frac{\varphi_1(t-1)\varphi_2(t-1)}{1+d{\varphi_1}^2(t-1)+{\varphi_2}^2(t-1)},\end{aligned}$$

we obtain (2.4). We find $g(t)$ and $q(t)$ and replace them with the sought $X(t)$ and $Y(t)$, i.e., based on (2.2), we have the following solution:

(2.5)
$$\begin{aligned} X\left(t\right) &= X\left(1\right)e^{\frac{1-t}{\varepsilon_1}} + \frac{a}{\varepsilon_1}\int\limits_1^t \frac{{\varphi_1}^2(\tau-1)}{1+{\varphi_1}^2(\tau-1)+c{\varphi_2}^2(\tau-1)}e^{\frac{\tau-t}{\varepsilon_1}}d\tau;\\ Y\left(t\right) &= Y\left(1\right)e^{\frac{1-t}{\varepsilon_2}} + \frac{b}{\varepsilon_2}\int\limits_1^t \frac{\varphi_1(\tau-1)\varphi_2(\tau-1)}{1+d{\varphi_1}^2(\tau-1)+{\varphi_2}^2(\tau-1)}e^{\frac{\tau-t}{\varepsilon_2}}d\tau.\end{aligned}$$

The found (2.5) are the solution of (1.1) in the interval $(1, 2]$ and the initial function for the interval $(2, 3]$. If we continue the process of successive integration for $t > 1$, then we can verify the existence, continuity, non-negativeness of the solution of (1.1).

We check the uniqueness of solutions to (1.1) in the interval $(1, 2]$. Suppose that (1.1) satisfies solutions $X_1(t)$ and $X_2(t)$ for $X(t)$, and $Y_1(t)$ and $Y_2(t)$ for $Y(t)$ in the interval $(1, 2]$ and conditions $X(t) = \varphi_1(t)$ and $Y(t) = \varphi_2(t)$ in the interval $[0,\ 1]$. In this case, we have (2.6) for $X(t)$

$$\varepsilon_1 \frac{dX_1(t)}{dt} = \frac{aX_1^2(t-1)}{1+X_1^2(t-1)+cY_1^2(t-1)} - X_1(t); \tag{2.6}$$
$$\varepsilon_1 \frac{dX_2(t)}{dt} = \frac{aX_2^2(t-1)}{1+X_2^2(t-1)+cY_2^2(t-1)} - X_2(t),$$

and for $Y(t)$ to the following (2.7)

$$\varepsilon_2 \frac{dY_1(t)}{dt} = \frac{bX_1(t-1)Y_1(t-1)}{1+dX_1^2(t-1)+Y_1^2(t-1)} - Y_1(t); \tag{2.7}$$
$$\varepsilon_2 \frac{dY_2(t)}{dt} = \frac{bX_2(t-1)Y_2(t-1)}{1+dX_2^2(t-1)+Y_2^2(t-1)} - Y_2(t).$$

We separate equations (2.6) and (2.7) from each other, respectively, and obtain the following equality in the interval $(1, 2]$

$$\varepsilon_1 \frac{d(X_1(t) - X_2(t))}{dt} = X_2(t) - X_1(t);$$
$$\varepsilon_2 \frac{d(Y_1(t) - Y_2(t))}{dt} = Y_2(t) - Y_1(t).$$

Since the first additions, the right-hand sides of (2.6) and (2.7) are equal

$$\frac{a\varphi_1{}^2(t-1)}{1+\varphi_1{}^2(t-1)+c\varphi_2{}^2(t-1)};$$
$$\frac{b\varphi_1(t-1)\varphi_2(t-1)}{1+d\varphi_1{}^2(t-1)+\varphi_2{}^2(t-1)},$$

their difference is 0. Since the initial condition for (2.6) and (2.7) has the form (2.5), we obtain the following equality

$$d(X_1(t) - X_2(t)) = \frac{1}{\varepsilon_1} dt\,(X_2(t) - X_1(t));$$
$$d(Y_1(t) - Y_2(t)) = \frac{1}{\varepsilon_2} dt\,(Y_2(t) - Y_1(t)).$$

From that we move on to equations in which the variables are separated:

$$\frac{d(X_1(t) - X_2(t))}{X_1(t) - X_2(t)} = -\frac{1}{\varepsilon_1} dt;$$
$$\frac{d(Y_1(t) - Y_2(t))}{Y_1(t) - Y_2(t)} = -\frac{1}{\varepsilon_2} dt.$$

We integrate the left and right sides at the appropriate intervals:

$$\ln\left(X_1\left(t\right)-X_2\left(t\right)\right)-\ln\left(X_1\left(1\right)-X_2\left(1\right)\right)=-\frac{1}{\varepsilon_1}(t-1);$$

$$\ln\left(Y_1\left(t\right)-Y_2\left(t\right)\right)-\ln\left(Y_1\left(1\right)-Y_2\left(1\right)\right)=-\frac{1}{\varepsilon_2}(t-1),$$

and calculate the values for the new time:

$$X_1\left(t\right)-X_2\left(t\right)=\left(X_1\left(1\right)-X_2\left(1\right)\right)e^{\frac{1-t}{\varepsilon_1}};$$
$$Y_1\left(t\right)-Y_2\left(t\right)=\left(Y_1\left(1\right)-Y_2\left(1\right)\right)e^{\frac{1-t}{\varepsilon_1}};$$
$$t\in\left(1,2\right].$$

Continuous solutions $X_1\left(t\right)$ and $X_2\left(t\right)$, and $Y_1(t)$ and $Y_2(t)$ are equal to each other in the interval $[0,1]$, that is, $X_1(1)=X_2(1)=\varphi_1(1)$ and $Y_1(1)=Y_2(1)=\varphi_2(1)$. This results in $X_1\left(t\right)=X_2\left(t\right)$ and $Y_1(t)=Y_2(t)$ in the interval $(1,2]$.

Consequently, our assumption turned out to be incorrect. This implies that the solution to the equation (1.1) in the interval $(1,2]$ is unique.

The limited resources of the studied object from a biological point of view and the passage of processes in a limited period of time implies the limitation of the solution of equations.

If conditions $0\leq\varphi_1(t)<\infty$ and $0\leq\varphi_2(t)<\infty$ are satisfied on interval $[0,1]$, we check that the solution to equation (1.1) is bounded. For very large values $X\left(t\right)$ and $Y\left(t\right)$, equation (1.1) can be written as follows:

$$\varepsilon_1\tfrac{dX(t)}{dt}=n-X\left(t\right);$$
(2.8)
$$\varepsilon_2\tfrac{dY(t)}{dt}=m-Y\left(t\right).$$

Where

$$n=\lim_{X(t),Y(t)\to\infty}\frac{aX^2\left(t-1\right)}{1+X^2\left(t-1\right)+cY^2\left(t-1\right)};$$

$$m=\lim_{X(t),Y(t)\to\infty}\frac{bX\left(t-1\right)Y\left(t-1\right)}{1+dX^2\left(t-1\right)+Y^2\left(t-1\right)}.$$

From these equations we obtain

$$n=\lim_{X,Y\to\infty}\frac{a}{\frac{1}{X^2}+1+c\frac{Y^2}{X^2}};$$

$$m = \lim_{X,Y\to\infty} \frac{b}{\frac{1}{XY} + d\frac{X}{Y} + \frac{Y}{X}}.$$

Here $\frac{1}{X^2} = 0$; $\frac{1}{XY} = 0$, and using $\frac{X}{Y} = 1$ taking into account the equal velocity levels X and Y, and on $X(t) \to \infty$ and $Y(t) \to \infty$ we have

$$n = \frac{a}{1+c};$$

$$m = \frac{b}{1+d}.$$

Consequently, if conditions $0 \le \varphi_1(t) < \infty$ and $0 \le \varphi_2(t) < \infty$ are satisfied on interval $[0,1]$, then the solution to equation (2.8) is as follows

$$X(t) = \tfrac{a}{1+c} + \varphi_1(1)\, e^{\frac{1-t}{\varepsilon_1}},$$

(2.9)

$$Y(t) = \tfrac{b}{1+d} + \varphi_2(1)\, e^{\frac{1-t}{\varepsilon_2}}$$

and the obtained solution (2.9) means that the solution of the system of equations (1.1) is bounded for very large values of the variables.

Thus, the results of the qualitative analysis showed that the solutions of the system of functional differential equations with a delay argument, describing the functioning of LC and HBV, have the characteristics of existence, continuity, non-negativity, uniqueness and boundedness.

REFERENCES

[1] B. N. HIDIROV : *Selected works on mathematical modeling of the regulatory of living systems*, Publishing House, Moscow, Izhevsk, (2014). 304.

[2] Y. F. DOLGIY: *Mathematical models of dynamical systems with delay*, Textbook. manual / Yu.F. Dolgiy, P.G. Surkov. - Yekaterinburg: Publishing house Ural, (2012), 122.

[3] M. SAIDALIEVA, M. B. HIDIROVA, A. R. SHAKAROV, A. M. TURGUNOV, A. A. HASANOV, Z. DJ. YUSUPOVA: *Dynamics of Regulatory Mechanisms of the Human Organism on the Basic Hierarchical Levels of the Organization*, IOP Conf. Series: J. Phys.: Conf. Series **1210**(012123) (2019), 1–8.

[4] M. SAIDALIEVA, M. B. HIDIROVA, A. M. TURGUNOV, Z.DJ. YUSUPOVA: *Quantitative study of the regulatory mechanisms of cardiac activity and liver function in pathogenesis*, IOP Conf. Series: J. Phys.: Conf. Series **1260** (102016) (2019), 1–9.

[5] M. B. HIDIROVA: *Solutions of the functional-differential equation for adjustment of human systems*, Vest. Mosk. Univer., **1** (2004), 50–52.

[6] R. BELLMAN, K. COOKE : *Differential Difference Equations*, Publishing House, Academic Press, (1963), 548 p.

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