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EXISTENCE OF NONOSCILLATORY SOLUTIONS OF NONLINEAR NEUTRAL DELAY DIFFERENCE EQUATION OF FRACTIONAL ORDER

A. GEORGE MARIA SELVAM[1], MARY JACINTHA, AND R. JANAGARAJ

ABSTRACT. The objective of this article is to develop certain criteria for the existence of nonoscillatory solutions of a nonlinear neutral delay difference equation of fractional order of the form

$$\Delta\left[r(\ell)\Delta^{\beta}\left[u(\ell)+c(\ell)u(\ell-\tau)\right]\right]+p(\ell)u(\ell-\sigma_1)-q(\ell)u(\ell-\sigma_2)=0,\ \ell\geq\ell_0,$$

where Δ^{β} is the RL difference operator of the derivative of the order $\beta,\ 0<\beta\leq 1$ and $\tau>0,\ \sigma_1,\ \sigma_2\geq 0,\ c,\ p,\ q,\ r\in C\left([\ell_0,\infty),\mathbb{R}\right)$, with the aid of Banach's Contraction Mapping Principle.

1. INTRODUCTION

The natural extension of integer order calculus over real or complex domains can be termed as Fractional Calculus (FC), categorically the super-set of integer order calculus. In mathematical analysis, the study dealing with fractional order derivatives and integral operators is considered as Fractional Calculus.

The origin of fractional calculus can be traced back to 1695 and its novelty intrigued several scientists and the literature of pioneering research in this area can be found in [15,16,19]. But it is only since 1985, that fractional calculus has entered the arena of applied mathematics as novel and innovative applications

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[1]*corresponding author*

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of fractional differential equations came to fore in the most diverse areas of science, technology and engineering [8, 12, 13, 17, 25].

The mathematical theory of discrete fractional calculus (DFC) is under developing stage in comparison to continuous fractional calculus and only in recent decades there is an renewed interest in developing the theory of DFC which is focused around the fractional sum and difference operators. The contributions of Atici and Eloe [2, 3], Goodrich [10], Miller and Ross [16] and M.Holm [11] led to the progress of the theory of DFC and more specifically the discrete delta fractional calculus [10].

Oscillation theory is vital to gather relevant knowledge pertaining to the qualitative properties of fractional difference equations. In recent years, the study of the oscillation theory of fractional difference equations is been remarkably constructive, advancing rapidly and being the focus of research for many scientists, see [1, 4, 6, 14, 21, 22] and the references therein. The oscillatory criteria in [7, 20, 23] are obtained with the help of Riccati technique and and in [14, 24] with the assistance of Stirling formula.

Zhou et al. [26], obtained the sufficiency criteria for the existence of nonoscillatory solutions of the fractional neutral differential equation, while Zhou et al [27] and Muthulakshmi et al [18] , discussed the sufficiency criteria for the existence of nonoscillatory solutions of the fractional neutral functional differential equation with the help of certain new techniques and fixed point theorems.

In this paper, we derive sufficiency criteria for the existence of nonoscillatory solution of the following fractional neutral delay difference equation

$$\Delta\left[r(\ell)\Delta^{\beta}\left[u(\ell)+c(\ell)u(\ell-\tau)\right]\right]+p(\ell)u(\ell-\sigma_1)-q(\ell)u(\ell-\sigma_2)=0,\ \ell\geq\ell_0, \tag{1.1}$$

where Δ^{β} is the RL difference operator of the derivative of the order $\beta,\ 0<\beta\leq 1$ and $\tau>0,\ \sigma_1,\ \sigma_2\geq 0,\ c,\ p,\ q,\ r\in C\left([\ell_0,\infty),\mathbb{R}\right)$.

2. Basic lemmas and preliminaries

Definition 2.1. *[9] A nontrivial solution of equation (1.1) is said to be nonoscillatory if it is either eventually positive or eventually negative and oscillatory otherwise. Equation (1.1) is oscillatory if all of its solutions are oscillatory.*

Definition 2.2. *[2] The RL β^{th} fractional sum of f for $\beta > 0$, is defined as*

$$\Delta^{-\beta} f(\ell) = \frac{1}{\Gamma(\beta)} \sum_{\xi=a}^{\ell-\beta} (\ell - \xi - 1)^{(\beta-1)} f(\xi), \quad \text{for} \quad \mathbb{N}_{a+\beta},$$

where f is defined for $\xi \equiv a \; mod(1)$ and $\Delta^{-\beta} f$ is defined for $\ell \equiv a + \beta \; mod(1)$. The falling factorial is given by

$$\ell^{(\beta)} = \frac{\Gamma(\ell+1)}{\Gamma(\ell+1-\beta)},$$

where Γ is the gamma function.

Definition 2.3. *[2] The RL μ^{th} order fractional difference Δ^{μ} is defined as*

$$\Delta^{\mu} f(\ell) = \Delta^{\beta} \Delta^{-(\beta-\mu)} f(\ell), \;\; \ell \in \mathbb{N}_a$$

and so

$$\Delta^{\mu} f(\ell) = \frac{\Delta^{\beta}}{\Gamma(\beta-\mu)} \sum_{\xi=a}^{\ell-\beta+\mu} (\ell - \xi - 1)^{(\beta-1)} f(\xi), \;\; \ell \in \mathbb{N}_a.$$

Hence, the law of exponent for fractional sum is

$$\Delta^{-\mu} \left[\Delta^{-\beta} f(\ell) \right] = \Delta^{-(\mu+\beta)} f(\ell) = \Delta^{-\beta} \left[\Delta^{-\mu} f(\ell) \right].$$

Lemma 2.1. *[5] (Banach's Contraction Mapping Principle)*
A contraction mapping on a complete metric space has a unique fixed point.

3. EXISTENCE OF NONOSCILLATORY SOLUTIONS

In this segment, we establish sufficient conditions for the existence of nonoscillatory solutions of equation (1.1) using Banach contraction mapping principle.

Theorem 3.1. *Suppose that $r(\ell) > 0$. If*

$$\sum_{\xi=\ell_0}^{\infty} \xi^{(\beta)} \; R(\xi) \; p(\xi) < \infty, \tag{3.1}$$

and

$$\sum_{\xi=\ell_0}^{\infty} \xi^{(\beta)} \; R(\xi) \; q(\xi) < \infty, \tag{3.2}$$

where $R(\ell) = \sum\limits_{\xi=\ell_0}^{\ell-1} \frac{1}{r(\xi)}$ and $c(\ell)$ satisfies one of the below mentioned ranges:

(i) $c(\ell) \equiv 1$, *(ii)* $0 < c(\ell) \le c_1 < 1$, *(iii)* $1 < c_2 \le c(\ell) \le c_1$,
(iv) $-1 < -c_2 \le c(\ell) < 0$, *(v)* $-c_2 \le c(\ell) \le -c_1 < -1$,

then equation (1.1) has a bounded nonoscillatory solution.

Proof. Let L denote Lipschitz constant, $\alpha_1 = \max_{u \in A}\{u\}$ and $M = max\ \{L, \alpha_1\}$. Also $\sigma = max\{\sigma_1, \sigma_2\}$.

Case (i): $c(\ell) \equiv 1$.

From conditions (3.1) and (3.2), we choose a $\ell_1 > \ell_0 + \sigma$ sufficiently large such that

$$\frac{M}{\Gamma(\beta+1)}\sum_{i=1}^{\infty}\left[\sum_{\xi=\ell+i\tau}^{\infty}\frac{(\ell-\xi-1-i\tau)^{(\beta)}}{r(\xi)}\sum_{k=\ell_1+i\tau}^{\xi-1}p(k)\right] < \frac{1}{3}, \tag{3.3}$$

and

$$\frac{M}{\Gamma(\beta+1)}\sum_{i=1}^{\infty}\left[\sum_{\xi=\ell+i\tau}^{\infty}\frac{(\ell-\xi-1-i\tau)^{(\beta)}}{r(\xi)}\sum_{k=l_1+i\tau}^{\xi-1}q(k)\right] < \frac{1}{3} \tag{3.4}$$

hold for $\ell \ge \ell_1$.

Let U be the set of all bounded real sequences $\{u(\ell)\}$ defined for $\ell \ge \ell_0$ with the Supremum norm $\|u\| = \sup_{\ell\, \ge\, \ell_0} |u(\ell)|$. Set $A = \{u \in U : 1 \le u \le 3,\ \ell \ge \ell_0\}$. It is clear that A is a bounded, closed and convex subset of U.

Define a mapping $T : A \to U$ as follows:

$$(Tu)(\ell) = \begin{cases} 2 - \dfrac{1}{\Gamma(\beta+1)}\sum\limits_{i=1}^{\infty}\left[\sum\limits_{\xi=\ell+i\tau}^{\infty}\dfrac{(\ell-\xi-1-i\tau)^{(\beta)}}{r(\xi)}\right. \\ \left.\times\sum\limits_{k=\ell_1+i\tau}^{\xi-1}p(k)u(k-\sigma_1)-q(k)u(k-\sigma_2)\right], \quad \text{for} \quad \ell \ge \ell_1, \\ (Tu)(\ell_1), \quad \text{for} \quad \ell_0 \le \ell \le \ell_1. \end{cases}$$

Clearly Tu is continuous.

For every $u \in U$ and $\ell \ge \ell_1$ using (3.3), we get

$$(Tu)(\ell) \ge 2 - \frac{1}{\Gamma(\beta+1)}\sum_{i=1}^{\infty}\left[\sum_{\xi=\ell+i\tau}^{\infty}\frac{(\ell-\xi-1-i\tau)^{(\beta)}}{r(\xi)}\sum_{k=\ell_1+i\tau}^{\xi-1}p(k)u(k-\sigma_1)\right],$$

$$(Tu)(\ell) \ge 2 - \frac{M}{\Gamma(\beta+1)}\sum_{i=1}^{\infty}\left[\sum_{\xi=\ell+i\tau}^{\infty}\frac{(\ell-\xi-1-i\tau)^{(\beta)}}{r(\xi)}\sum_{k=\ell_1+i\tau}^{\xi-1}p(k)\right] \ge 1.$$

Furthermore, using (3.4) we have

$$(Tu)(\ell) \leq 2 + \frac{1}{\Gamma(\beta+1)} \sum_{i=1}^{\infty} \left[\sum_{\xi=\ell+i\tau}^{\infty} \frac{(\ell-\xi-1-i\tau)^{(\beta)}}{r(\xi)} \sum_{k=\ell_1+i\tau}^{\xi-1} q(k) u(k-\sigma_2) \right],$$

$$(Tu)(\ell) \leq 2 + \frac{M}{\Gamma(\beta+1)} \sum_{i=1}^{\infty} \left[\sum_{\xi=\ell+i\tau}^{\infty} \frac{(\ell-\xi-1-i\tau)^{(\beta)}}{r(\xi)} \sum_{k=\ell_1+i\tau}^{\xi-1} q(k) \right] \leq 3.$$

Thus $TA \subset A$. In order to employ the Contraction Mapping Principle, we show that T is a contraction mapping on A. Now for $u, v \in A$ and $\ell \geq \ell_1$, we have

$$\begin{aligned} |(Tu)(\ell) - (Tv)(\ell)| \leq & \frac{1}{\Gamma(\beta+1)} \sum_{i=1}^{\infty} \left[\sum_{\xi=\ell+i\tau}^{\infty} \frac{(\ell-\xi-1-i\tau)^{(\beta)}}{r(\xi)} \right. \\ & \left. \sum_{k=\ell_1+i\tau}^{\xi-1} L[p(k)\,|u-v| + q(k)\,|u-v|] \right], \\ \leq & \frac{M}{\Gamma(\beta+1)} \sum_{i=1}^{\infty} \left[\sum_{\xi=\ell+i\tau}^{\infty} \frac{(\ell-\xi-1-i\tau)^{(\beta)}}{r(\xi)} \right. \\ & \left. \sum_{k=\ell_1+i\tau}^{\xi-1} p(k) + q(k) \right] \|u-v\| \leq \left[\frac{1}{3} + \frac{1}{3} \right] \|u-v\| \\ |(Tu)(\ell) - (Tv)(\ell)| \leq & \frac{2}{3} \|u-v\|. \end{aligned}$$

This implies that T is a contraction mapping on A as $\frac{2}{3} < 1$ and by Lemma 2.1, T has a unique fixed point which is a positive and bounded solution of (1.1).

Case (ii): $0 < c(\ell) \leq c_1 < 1$.

From conditions (3.1) and (3.2), we can choose a $\ell_1 > \ell_0 + \sigma$ sufficiently large such that

$$\frac{1}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\infty} (\ell-\xi-1)^{(\beta)} R(\xi)\ [p(\xi)+q(\xi)] < \frac{1-c_1}{2L},$$

$$\frac{1}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\infty} \alpha_1\ (\ell-\xi-1)^{(\beta)}\ R(\xi)\ p(\xi) \leq N_1 - 1 \tag{3.5}$$

and

$$\frac{1}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\infty} \alpha_1\ (\ell-\xi-1)^{(\beta)} R(\xi)\ q(\xi) \leq 1 - c_1 N_1 - M_1, \tag{3.6}$$

where M_1 and N_1 are positive constants such that $1 < N_1 < \dfrac{1-M_1}{c_1}$.

Let U be the set of all bounded real sequences $\{u(\ell)\}$ defined for $\ell \geq \ell_0$ with Supremum norm $\|u\| = \sup\limits_{\ell \,\geq\, \ell_0} |u(\ell)|$. Set $A = \{u \in U : M_1 \leq u \leq N_1, \ell \geq \ell_0\}$. It is clear that A is a bounded, closed and convex subset of U.

Define a mapping $T : A \to U$ as follows:

$$(Tu)(\ell) = \begin{cases} 1 - c(\ell)u(\ell-\tau) + \dfrac{R(\ell)}{\Gamma(\beta+1)} \sum\limits_{\xi=\ell}^{\infty} (\ell-\xi-1)^{(\beta)} \\ \times \left[p(\xi)u(\xi-\sigma_1) - q(\xi)u(\xi-\sigma_2)\right] + \dfrac{1}{\Gamma(\beta+1)} \sum\limits_{\xi=\ell_1}^{l} (\ell-\xi-1)^{(\beta)} R(\xi) \\ \times \left[p(\xi)u(\xi-\sigma_1) - q(\xi)u(\xi-\sigma_2)\right] \quad \text{for} \quad \ell \geq \ell_1, \\ (Tu)(\ell_1) \quad \text{for} \quad \ell_0 \leq \ell \leq \ell_1. \end{cases}$$

Clearly Tu is continuous.

For every $u \in U$ and $\ell \geq \ell_1$ using (3.5) we get

$$\begin{aligned} (Tu)(\ell) &\leq 1 + \frac{1}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\infty} (\ell-\xi-1)^{(\beta)} R(\xi)\, p(\xi)\, u(\xi-\sigma_1), \\ &\leq 1 + \frac{1}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\infty} \alpha_1 (\ell-\xi-1)^{(\beta)}\, R(\xi)\, p(\xi) \leq N_1. \end{aligned}$$

Furthermore, using (3.6) we have

$$(Tu)(\ell) \geq 1 - c_1 N_1 - \frac{1}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\infty} (\ell-\xi-1)^{(\beta)} R(\xi)\, q(\xi)\, u(\xi-\sigma_2),$$

$$(Tu)(\ell) \geq 1 - c_1 N_1 - \frac{1}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\infty} \alpha_1 (\ell-\xi-1)^{(\beta)} R(\xi)\, q(\xi) \geq M_1.$$

Thus $TA \subset A$. We will show that T is a contraction mapping on A. Now for $u, v \in A$ and $\ell \geq \ell_1$, we have

$$|(Tu)(\ell) - (Tv)(\ell)| \leq \left[c_1 + \frac{L}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\infty} (\ell-\xi-1)^{(\beta)} R(\xi) \left[p(\xi) + q(\xi)\right]\right] \|u-v\|$$

$$|(Tu)(\ell) - (Tv)(\ell)| \leq \left[c_1 + L \ \frac{1-c_1}{2L}\right] \|u-v\| \leq \left[\frac{c_1+1}{2}\right] \|u-v\| \leq \lambda_1 \|u-v\|.$$

Since $\lambda_1 = \dfrac{c_1+1}{2} < 1$, T is a contraction mapping on A. By Contraction Mapping Principle stated in Lemma 2.1, T has a unique fixed point which is a positive and bounded solution of equation (1.1).

Case (iii): $1 < c_2 \leq c(\ell) \leq c_1 < \infty$.

From conditions (3.1) and (3.2), we can choose a $\ell_1 > \ell_0 + \sigma$, sufficiently large such that

$$\frac{1}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\infty} (\ell-\xi-1-\tau)^{(\beta)} R(\xi)\,[p(\xi)+q(\xi)] < \frac{c_2-1}{2L},$$

$$\frac{1}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\infty} (\ell-\xi-1-\tau)^{(\beta)} \alpha_1\, R(\xi)\, p(\xi) \leq c_2 N_2 - 1, \tag{3.7}$$

$$\frac{1}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\infty} (\ell-\xi-1-\tau)^{(\beta)}\, \alpha_1\, R(\xi)\, q(\xi) \leq 1 - \frac{c_1 N_1}{c_2} - c_1 M_2, \tag{3.8}$$

where M_2 and N_2 are positive constants such that $1 < c_1 N_1 < c_2\,(1 - c_1 M_2)$.

Let U be the set of all bounded real sequences $\{u(\ell)\}$ defined for $\ell \geq \ell_0$ with the Supremum norm $\|u\| = \sup_{\ell \,\geq\, \ell_0} |u(\ell)|$. Set $A = \{u \in U : M_2 \leq u \leq N_2,\ \ell \geq \ell_0\}$. It is clear that A is a bounded, closed and convex subset of U.

Define a mapping $T : A \to U$ as follows:

$$(Tu)(\ell) = \begin{cases} \dfrac{1}{c(\ell+\tau)} - \dfrac{u(\ell+\tau)}{c(\ell+\tau)} + \dfrac{R(\ell+\tau)}{c(\ell+\tau)} \dfrac{1}{\Gamma(\beta+1)} \sum\limits_{\xi=\ell+\tau}^{\infty} (\ell-\xi-1-\tau)^{(\beta)} \\ \times [p(\xi)u(\xi-\sigma_1) - q(\xi)u(\xi-\sigma_2)] + \dfrac{1}{c(\ell+\tau)}\, \dfrac{1}{\Gamma(\beta+1)} \sum\limits_{\xi=\ell_1}^{l+\tau} R(\xi) \\ \times (\ell-\xi-1-\tau)^{(\beta)}\, [p(\xi)u(\xi-\sigma_1) - q(\xi)u(\xi-\sigma_2)] \quad \text{for } \ell \geq \ell_1, \\ (Tu)(\ell_1) \quad \text{for } \ell_0 \leq \ell \leq \ell_1. \end{cases}$$

Clearly Tu is continuous.

For every $u \in U$ and $\ell \geq \ell_1$ using (3.7), we get

$$\begin{aligned} (Tu)(\ell) &\leq \frac{1}{c_2} + \frac{1}{c_2}\, \frac{1}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\infty} R(\xi)\, (\ell-\xi-1-\tau)^{(\beta)} p(\xi) u(\xi-\sigma_1), \\ &\leq \frac{1}{c_2} + \frac{1}{c_2}\, \frac{1}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\infty} \alpha_1\, R(\xi)(\ell-\xi-1-\tau)^{(\beta)} p(\xi) \leq N_2. \end{aligned}$$

Furthermore using (3.8), we have

$$(Tu)(\ell) \geq \frac{1}{c_1} - \frac{N_2}{c_2} - \frac{1}{c_1}\,\frac{1}{\Gamma(\beta+1)}\sum_{\xi=\ell_1}^{\infty} R(\xi)(\ell-\xi-1-\tau)^{(\beta)}q(\xi)u(\xi-\sigma_2),$$
$$\geq \frac{1}{c_1} - \frac{N_2}{c_2} - \frac{1}{c_1}\,\frac{1}{\Gamma(\beta+1)}\sum_{\xi=\ell_1}^{\infty} \alpha_1 R(\xi)(\ell-\xi-1-\tau)^{(\beta)}q(\xi) \geq M_2.$$

Thus $TA \subset A$. Let us show that T is a contraction mapping on A. Now for $u, v \in A$ and $\ell \geq \ell_1$, we have

$$|(Tu)(\ell) - (Tv)(\ell)| \leq \frac{1}{c_2}\left[1 + \frac{L}{\Gamma(\beta+1)}\sum_{\xi=\ell_1}^{\infty}(\ell-\xi-1-\tau)^{(\beta)}R(\xi)\right.$$
$$\left.\times\left[p(\xi)+q(\xi)\right]\right]\|u-v\| \leq \frac{1}{c_2}\left[1 + L\ \frac{c_2-1}{2L}\right]\|u-v\|,$$
$$|(Tu)(\ell) - (Tv)(\ell)| \leq \frac{1}{c_2}\left[\frac{1+c_2}{2}\right]\|u-v\| \leq \lambda_2\|u-v\|.$$

Since $\lambda_2 = \dfrac{1+c_2}{2c_2} < 1$, T is a contraction mapping on A . Lemma 2.1 implies that T has a unique fixed point which is a positive and bounded solution of equation (1.1).

Case (iv): $-1 < -c_2 \leq c(\ell) < 0$.

From conditions (3.1) and (3.2), we can choose a $\ell_1 > \ell_0 + \sigma$, sufficiently large such that

$$\frac{1}{\Gamma(\beta+1)}\sum_{\xi=\ell_1}^{\infty}(\ell-\xi-1)^{(\beta)}R(\xi)\left[p(\xi)+q(\xi)\right] < \frac{1-c_2}{2L},$$

$$\frac{1}{\Gamma(\beta+1)}\sum_{\xi=\ell_1}^{\infty}\alpha_1(\ell-\xi-1)^{(\beta)}R(\xi)\ p(\xi) \leq N_3(1-c_2)-1, \tag{3.9}$$

and

$$\frac{1}{\Gamma(\beta+1)}\sum_{\xi=\ell_1}^{\infty}\alpha_1\ (\ell-\xi-1)^{(\beta)}R(\xi)q(\xi) \leq 1-M_3, \tag{3.10}$$

where M_3 and N_3 are positive constants such that $M_3 < 1 < N_3(1-c_2)$. Let U be the set of all bounded real sequences $\{u(\ell)\}$ defined for $\ell \geq \ell_0$ with Supremum norm $\|u\| = \sup\limits_{\ell\,\geq\,\ell_0}|u(\ell)|$. Set $A = \{u \in U : M_3 \leq u \leq N_3,\ \ell \geq \ell_0\}$. It is clear that A is a bounded, closed and convex subset of U.

Define a mapping $T : A \to U$ as follows:

$$(Tu)(\ell) = \begin{cases} 1 - c(\ell)u(\ell-\tau) + \dfrac{R(\ell)}{\Gamma(\beta+1)} \sum\limits_{\xi=\ell}^{\infty} (\ell-\xi-1)^{(\beta)} \\ \times \left[p(\xi)u(\xi-\sigma_1) - q(\xi)u(\xi-\sigma_2)\right] + \dfrac{1}{\Gamma(\beta+1)} \sum\limits_{\xi=\ell_1}^{\ell} (\ell-\xi-1)^{(\beta)} \\ \times R(\xi)\left[p(\xi)u(\xi-\sigma_1) - q(\xi)u(\xi-\sigma_2)\right] \quad \text{for} \quad \ell \geq \ell_1, \\ (Tu)(\ell_1) \quad \text{for} \quad \ell_0 \leq \ell \leq \ell_1. \end{cases}$$

Clearly Tu is continuous.

For every $u \in U$ and $\ell \geq \ell_1$ using (3.9) we get

$$(Tu)(\ell) \leq 1 + c_2 N_3 + \frac{1}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\ell-1} (\ell-\xi-1)^{(\beta)} R(\xi)\, p(\xi)\, u(\xi-\sigma_1),$$

$$\leq 1 + c_2 N_3 + \frac{1}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\ell} \alpha_1 (\ell-\xi-1)^{(\beta)} R(\xi)\, p(\xi) \leq N_3.$$

Furthermore, using (3.10), we have

$$(Tu)(\ell) \geq 1 - \frac{1}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\infty} (\ell-\xi-1)^{(\beta)} R(\xi)\, q(\xi) u(\xi-\sigma_2),$$

$$\geq 1 - \frac{1}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\infty} \alpha_1 (\ell-\xi-1)^{(\beta)} R(\xi)\, q(\xi) \geq M_3.$$

Thus $TA \subset A$. We show that T is a contraction mapping on A. Now for $u, v \in A$ and $\ell \geq \ell_1$, we have

$$|(Tu)(\ell) - (Tv)(\ell)| \leq \left[c_2 + \frac{L}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\infty} (\ell-\xi-1)^{(\beta)} R(\xi) \right.$$

$$\left. [p(\xi) + q(\xi)]\right] \|u-v\| \leq \left[c_2 + L \ \frac{1-c_2}{2L}\right] \|u-v\|,$$

$$|(Tu)(\ell) - (Tv)(\ell)| \leq \frac{c_2+1}{2} \|u-v\| \leq \lambda_3 \|u-v\|.$$

Since $\lambda_3 = \dfrac{1+c_2}{2} < 1$, we conclude that T is a contraction mapping on A . By Lemma 2.1, T has a unique fixed point which is a positive and bounded solution of equation (1.1).

Case (v): $-\infty < -c_2 \leq c(\ell) \leq -c_1 < -1$

From condition (3.1) and (3.2), we can choose a $\ell_1 > \ell_0 + \sigma$ sufficiently large such that

$$\frac{1}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\infty} (\ell - \xi - 1 - \tau)^{(\beta)} R(\xi) \left[p(\xi) + q(\xi)\right] < \frac{c_1 - 1}{2L},$$

$$\frac{1}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\infty} \alpha_1 \, (\ell - \xi - 1 - \tau)^{(\beta)} R(\xi) \, p(\xi) \leq \frac{c_1}{c_2} \left[1 - (c_2 - 1) M_4\right], \tag{3.11}$$

and

$$\frac{1}{\Gamma(\beta+1)} \sum_{\xi=\ell_1}^{\infty} \alpha_1 (\ell - \xi - 1 - \tau)^{(\beta)} R(\xi) \, q(\xi) \leq N_4(c_1 - 1) - 1, \tag{3.12}$$

where M_4 and N_4 are positive constants such that $N_4(c_1 - 1) > 1 > (c_2 - 1) M_4$.

Let U be the set of all bounded real sequences $\{u(\ell)\}$ defined for $\ell \geq \ell_0$ with Supremum norm $\|u\| = \sup\limits_{l \geq l_0} |u(\ell)|$. Set $A = \{u \in U : M_4 \leq u \leq N_4, \ \ell \geq \ell_0\}$. It is clear that A is a bounded, closed and convex subset of U.

Define a mapping $T : A \to U$ as follows:

$$(Tu)(\ell) = \begin{cases} -\dfrac{1}{c(\ell+\tau)} - \dfrac{u(\ell+\tau)}{c(\ell+\tau)} + \dfrac{R(\ell+\tau)}{c(\ell+\tau)} \dfrac{1}{\Gamma(\beta+1)} \sum\limits_{\xi=\ell+\tau}^{\infty} (\ell - \xi - 1 - \tau)^{(\beta)} \\ \times \left[p(\xi) u(\xi - \sigma_1) - q(\xi) u(\xi - \sigma_2)\right] + \dfrac{1}{c(\ell+\tau)} \dfrac{1}{\Gamma(\beta+1)} \sum\limits_{\xi=\ell_1}^{\ell+\tau} R(\xi) \\ \times (\ell - \xi - 1 - \tau)^{(\beta)} \left[p(\xi) u(\xi - \sigma_1) - q(\xi) u(\xi - \sigma_2)\right] \quad \text{for } \ \ell \geq \ell_1, \\ (Tu)(\ell_1) \quad \text{for } \ \ell_0 \leq \ell \leq \ell_1. \end{cases}$$

Clearly Tu is continuous.

For every $u \in U$ and $\ell \geq \ell_1$ using (3.11) we get

$$\begin{aligned} (Tu)(\ell) &\geq \frac{1}{c_2} + \frac{M_4}{c_2} - \frac{1}{c_1} \sum_{\xi=\ell_1}^{\infty} (\ell - \xi - 1 - \tau)^{(\beta)} R(\xi) \, p(\xi) u(\xi - \sigma_1), \\ &\geq \frac{1}{c_2} + \frac{M_4}{c_2} - \frac{1}{c_1} \sum_{\xi=\ell_1}^{\infty} \alpha_1 \, (\ell - \xi - 1 - \tau)^{(\beta)} R(\xi) \, p(\xi) \geq M_4. \end{aligned}$$

Furthermore, using (3.12) we have

$$
\begin{aligned}
(Tu)(\ell) &\leq \frac{1}{c_1} + \frac{N_4}{c_1} + \frac{1}{c_1}\sum_{\xi=\ell_1}^{\infty}(\ell-\xi-1-\tau)^{(\beta)}R(\xi)\ q(\xi)\ u(\xi-\sigma_2),\\
&\leq \frac{1}{c_1} + \frac{N_4}{c_1} + \frac{1}{c_1}\sum_{\xi=\ell_1}^{\infty}\alpha_1(\ell-\xi-1-\tau)^{(\beta)}R(\xi)\ q(\xi) \leq N_4.
\end{aligned}
$$

Thus $TA \subset A$. To apply the Contraction Principle, we show that T is a contraction mapping on A. Now for $u, v \in A$ and $\ell \geq \ell_1$, we have

$$
\begin{aligned}
|(Tu)(\ell)-(Tv)(\ell)| \leq& \frac{1}{c_1}\left[1 + \frac{L}{\Gamma(\beta+1)}\sum_{\xi=1}^{\infty}(\ell-\xi-1-\tau)^{(\beta)}\right.\\
&\left. R(\xi)\left[p(\xi)+q(\xi)\right]\right]\|u-v\| \leq \frac{1}{c_1}\left[1 + L\ \frac{c_1-1}{2L}\right]\|u-v\|\\
|(Tu)(\ell)-(Tv)(\ell)| \leq& \frac{c_1+1}{2c_1}\|u-v\| \leq \lambda_4\|u-v\|.
\end{aligned}
$$

Hence T is a contraction mapping on A, since $\lambda_4 = \dfrac{1+c_1}{2c_1} < 1$. By Lemma 2.1, T has a unique fixed point which is a positive and bounded solution of equation (1.1). This completes the proof. $\square$

4. CONCLUDING REMARKS

The authors in this article established the existence of nonoscillatory solutions of a nonlinear neutral delay difference equation of fractional order with the aid of Banach's Contraction Mapping Principle. These results are new in the literature and opens new arena for the researchers to explore. In future, we intend to extend the results further for the following equation

$$
\begin{aligned}
\Delta\left[r(\ell)\Delta^{\beta}\left[u(\ell)+m(\ell)u(\ell-\tau_1+m(\ell)u(\ell+\tau_2))\right]\right] + p(\ell)u(\ell-\sigma_1)&\\
-q(\ell)u(\ell-\sigma_2) = h(\ell),\ \ell \geq \ell_0.&
\end{aligned}
$$

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DEPARTMENT OF MATHEMATICS
SACRED HEART COLLEGE (AUTONOMOUS)
TIRUPATTUR - 635 601, TAMIL NADU, INDIA
E-mail address: agmshc@gmail.com

DEPARTMENT OF MATHEMATICS
SACRED HEART COLLEGE (AUTONOMOUS)
TIRUPATTUR - 635 601, TAMIL NADU, INDIA
E-mail address: sayhai2014@gmail.com

DEPARTMENT OF MATHEMATICS
SACRED HEART COLLEGE (AUTONOMOUS)
TIRUPATTUR - 635 601, TAMIL NADU, INDIA
E-mail address: janagarajtk@gmail.com