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GENERALIZED NARAYANA SEQUENCES AND QUADRATIC SEQUENCES

R. SIVARAMAN

ABSTRACT. Among several amusing sequences that exist in mathematics, Fibonacci sequence is the most common and famous sequence that is known to everyone. An equally absorbing sequence was described by Indian mathematician Narayana Pandita. In this paper, we try to generalize Narayana sequence using Quadratic sequences coefficients in its recurrence relations and try to determine the limiting ratios of such sequences. In this sense, this paper explores the interesting mathematical relationship between Generalized Narayana sequences and Quadratic sequences.

1. INTRODUCTION

Around 14th century CE, notable Indian mathematician Narayana Pandita introduced a wonderful sequence using immortal cows resembling immortal rabbits of Fibonacci sequence. The behavior of Narayana sequence and the ratio of its successive terms are well known. The generalizations of Narayana sequence in various forms are dealt by several authors. In this paper, we shall consider generalizations of Narayana sequence using Quadratic sequence as coefficients. The main objective of this paper is to obtain interesting results regarding the limiting ratios of such generalized Narayana sequences. First, we begin with some definitions (for details see [1-10]).

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2. Definition

Narayana Pandita described the Narayana sequence in such a way that the number of cows present each year is equal to the number of cows in previous year plus the number of cows present three years ago. We assume that no cow die during this process. Using this convention, we form the following Recurrence Relation describing Narayana Sequence

$$N_{n+1} = N_n + N_{n-2}; n \geq 2, N_0 = 0, N_1 = 1, N_2 = 1. \tag{2.1}$$

Here n may be considered as number of years. Then according to (2.1), the number of cows in the year $n+1$ will be equal to number of cows in the year n (previous year) plus the number of cows three years ago (since $n+1-3 = n-2$). Hence, equation (2.1) describes the condition exactly as stated by Narayana Pandita. Using (2.1), if we compute the other terms of the sequence we get:

$$0, 1, 1, 1, 2, 3, 4, 6, 9, 13, \ldots. \tag{2.2}$$

The sequence in (2.2) is called "Narayana Sequence" named in the honor of its proposer Narayana Pandita.

2.1. Limiting Ratio and Characteristic Equation.

The ratio of successive terms of a sequence is called its "Limiting Ratio". In particular the ratio of the $(n+1)^{th}$ term to nth term of a sequence as $n \to \infty$ is defined as the Limiting Ratio of that sequence. In this paper, we denote the limiting ratio by λ. A polynomial equation whose roots are limiting ratios is called as "Characteristic Equation".

It is well known that the limiting ratio of Narayana sequence given by the numbers in (2.2) is a number given by $\lambda = 1.46557$. This number 1.46557 is called "Supergolden Ratio" (see [1]).

2.2. Quadratic Sequence.

The sequence defined by:

$$\left\{Ak^2 + Bk + C\right\}_{k=0}^{\infty} \tag{2.3}$$

is called the Quadratic sequence because the k^{th} term is described through a second degree expression. The coefficients A, B, C are real numbers. For any Quadratic sequence as defined in (2.3), the second forward differences will always be constant. We will use this sequence as coefficients of generalized Narayana sequences to explore further.

3. Generalizing through Natural Numbers and Quadratic Sequences

We consider the generalization of Narayana sequence by considering coefficients which are k^{th} terms of natural numbers and quadratic sequences. Let k be a positive integer. We define the generalized Narayana sequence through the recurrence relation

$$\begin{aligned} N_{k,n+1} &= kN_{k,n} + \left(Ak^2 + Bk + C\right) N_{k,n-2};\ n \geq 2, \\ N_{k,0} &= 0, N_{k,1} = 1, N_{k,2} = k, A + B + C = 1. \end{aligned} \tag{3.1}$$

We notice that for $k = 1$, this will reduce to usual Narayana sequence defined in (2.1). The coefficients of generalized Narayana sequence defined in (3.1) are k and $Ak^2 + Bk + C$ which represent natural numbers and quadratic sequences for each value of $k = 1, 2, 3. \dots$

If we assume that the limiting ratio of generalized Narayana sequence is λ then by definition we have $\lim\left(\frac{N_{k,n+1}}{N_{k,n}}\right) = \lambda$ as $n \to \infty$.

Now for any integer r, we have the following equation:

$$\begin{aligned} \lim\left(\frac{N_{k,n+r}}{N_{k,n}}\right) = \lim\Bigg(&\frac{N_{k,n+r}}{N_k n + r - 1} \times \frac{N_{k,n+r-1}}{N_{k,n+r-2}} \times \frac{N_{k,n+r-2}}{N_{k,n+r-3}} \times \\ &\cdots \times \frac{N_{k,n+1}}{N_{k,n}}\Bigg) \lambda \times \ldots \times \lambda = \lambda^r. \end{aligned} \tag{3.2}$$

Thus from (3.1), we get:

$$\begin{aligned} \lim\left(\frac{N_{k,n+1}}{N_{k,n}}\right) &= \lim\left(\frac{kN_{k,n} + (Ak^2 + Bk + C)\, N_{k,n-2}}{N_{k,n}}\right) \\ &= \lim\left(k + \left(Ak^2 + Bk + C\right) \frac{N_{k,n-2}}{N_{k,n}}\right). \end{aligned}$$

Now using (3.2), as $n \to \infty$ we get:

$$\lambda = k + \frac{Ak^2 + Bk + C}{\lambda^2}. \tag{3.3}$$

This leads us to the characteristic equation:

$$\lambda^3 - k\lambda^2 - \left(Ak^2 + Bk + C\right) = 0. \tag{3.4}$$

If $\lambda = O(k)$, then $\frac{Ak^2+Bk+C}{\lambda^2} \to A$ as $k \to \infty$. Thus if k is very large, then from (3.3) we get $\lambda = k + A$. Hence, the limiting ratio of the generalized Narayana sequence defined in (3.1) is:

$$\lambda = k + A \tag{3.5}$$

for large values of k. We notice that A is a real number which is coefficient of k^2 in the quadratic sequence.

4. Generalizing through two Quadratic Sequences

We consider the generalization of Narayana sequence by considering coefficients which are two quadratic sequences. Let k be a positive integer. We define the generalized Narayana sequence through the recurrence relation:

$$\begin{aligned}&N_{k,n+1} = \left(Pk^2 + Qk + R\right) N_{k,n} + \left(Ak^2 + Bk + C\right) N_{k,n-2};\\ &n \geq 2, N_{k,0} = 0, N_{k,1} = 1, N_{k,2} = k\\ &P + Q + R = 1, A + B + C = 1\end{aligned} \tag{4.1}$$

We notice that for $k = 1$, this will reduce to usual Narayana sequence defined in (2.1). The coefficients of generalized Narayana sequence defined in (4.1) are $Pk^2 + Qk + R$ and $Ak^2 + Bk + C$ which represent two quadratic sequences for each value of $k = 1, 2, 3 \ldots$.

If we now try to determine the limiting ratio, then from (4.1), we get

$$\begin{aligned}\lim \left(\frac{N_{k,n+1}}{N_{k,n}}\right) &= \lim \left(\frac{\left(Pk^2 + Qk + R\right) N_{k,n} + \left(Ak^2 + Bk + C\right) N_{k,n-2}}{N_{k,n}}\right)\\ &= \lim \left(\left(Pk^2 + Qk + R\right) + \left(Ak^2 + Bk + C\right) \frac{N_{k,n-2}}{N_{k,n}}\right).\end{aligned}$$

Using (3.2), as $n \to \infty$, we have:

$$\lambda = \left(Pk^2 + Qk + R\right) + \frac{Ak^2 + Bk + C}{\lambda^2}. \tag{4.2}$$

This leads to the characteristic equation:

$$\lambda^3 - \left(Pk^2 + Qk + R\right) \lambda^2 - \left(Ak^2 + Bk + C\right) = 0 \tag{4.3}$$

If $\lambda = O(k)$, then $\frac{Ak^2+Bk+C}{\lambda^2} \to A$ as $k \to \infty$ Thus if k is very large, then from (4.2) we get $\lambda = Pk^2 + Qk + R + A$. Hence, the limiting ratio of the generalized Narayana sequence defined in (4.1) is:

$$\lambda = Pk^2 + Qk + R + A \tag{4.4}$$

for large values of k.

5. Verification

We now consider couple of illustrations to verify the results obtained in sections 3 and 4.

5.1. Corresponding to the characteristic equation of section 3 given by (3.4), we consider $k = 1000, A = 4, B = -2, C = -1$. We notice k is large and $A + B + C = 1$ satisfying our assumptions. Thus the characteristic equation corresponding to (3.1), is given by:

$$\lambda^3 - 1000\lambda^2 - 3997999 = 0. \tag{5.1}$$

First, using Descarte's rule of signs, we find that there is only one positive real root and two imaginary roots. Using Newton-Raphson method, we find that the positive real root of equation (5.1) is $\lambda = 1003.96647$. Thus the limiting ratio for this case is $\lambda = 1003.96647$. We notice that $\lambda = 1003.96647 \approx 1004 = 1000 + 4 = k + A$ as obtained in (3.5).Thus the limiting ratio value verifies with our mathematical proof for this case.

5.2. Now corresponding to the characteristic equation of section 4 given by (4.3), we consider: $k = 1000, P = 1, Q = -1000, R = 1000, A = 3, B = -4, C = 2$. We notice that $P + Q + R = 1$ and $A + B + C = 1$. Thus the coefficients of our quadratic sequences satisfy our assumption in (4.1). Thus the characteristic equation corresponding to (4.1) is:

$$\lambda^3 - 1000\lambda^2 - 2996002 = 0. \tag{5.2}$$

Using Descarte's rule of signs, we find that there is only one positive real root and two imaginary roots. Using Newton-Raphson method, we find that the positive real root of equation (5.2) is $\lambda = 1002.97824$. Thus the limiting ratio for this case is $\lambda = 1002.97824$. We observe that $Pk^2 + Qk + R = 1000, \quad A = 3$. Thus, $\lambda = 1002.97824 \approx 1003 = 1000 + 3 = (Pk^2 + Qk + R) + A$ as obtained in (4.4). Hence the limiting ratio value verifies with our mathematical proof for this case.

Conclusion

By generalizing the usual Narayana sequence using quadratic sequences we have obtained interesting results for their limiting ratios as in equations (3.5) and (4.4) of sections 3 and 4 respectively. In particular, if we consider natural numbers and a quadratic sequence as coefficients then in section 3 we proved that the limiting ratio is $\lambda = k + A$ and if we consider two quadratic sequences as coefficients as in section 4, we proved that the limiting ratio is $\lambda = (Pk^2 + Qk + R) + A$. We have verified these results through two suitable illustrations presented in 5.1 and 5.2 respectively. We notice that these results are valid only if k is very large. In the limiting case when $k \to \infty$. the results obtained in (3.5) and (4.4) will be certainly true as derived in sections 3 and 4 of this paper.

Equations (3.4) and (4.3) provide with many choices of forming characteristic equations by altering the coefficients of the quadratic sequences considered in their recurrence relation. This gives us several limiting ratios accordingly. For example, by choosing $A = \frac{m-2}{2}, B = \frac{4-m}{2}, C = 0$ we get the quadratic sequence representing figurate numbers of order m. The limiting ratio according to (3.5) would be $\lambda = k + \frac{m-2}{2}$ and according to (4.4) would be $\lambda = (Pk^2 + Qk + R) + \frac{m-2}{2}$ whenever $k \to \infty$. Likewise, we can modify our quadratic sequences to represent several interesting quadratic sequences and obtain the limiting ratios according to (3.5) and (4.4). But we have to remember that the limiting ratios are true only in the limiting case as $k \to \infty$, that is, k is practically a very large number.

REFERENCES

[1] R. SIVARAMAN: *Behaviour of Generalized Narayana Sequences*, Journal of Advanced Research in Dynamical and Control Systems, **12**(7) (2020), 2558-2564.

[2] P. SINGH: *The so-called Fibonacci numbers in ancient and medieval India*, Historia Mathematia, **12** (1985), 229-244.

[3] M. GARDNER: *Catalan Numbers: An Integer sequence that materializes in unexpected places*, Scientific American, **234**(132) (1976), 120-125.

[4] T.V. DIDKIVSKA, M.V. STÓPOCHKINA: *Properties of Fibonacci-Narayana numbers*, In the World of Mathematics, **9**(1) (2003), 29-36.

[5] M. BONA, B.E. SAGAN: *On divisibility of Narayana numbers by primes*, J. Integer Sequences (2005).

[6] C. FLAUT, V. SHPAKIVSKYI: *On generalized Fibonacci quaternions and Fibonacci-Narayana quaternions*, Adv. Appl. Clifford Algebras, **23** (2013), 673-688.

[7] Y. HORIBE: *An entropy view of Fibonacci trees*, The Fibonacci Quarterly (1982).

[8] J.P. ALLOUCHE, T. JOHNSON: *Narayana's cows and delayed morphisms*, http://kalvos.org/johness1.html; N.J.A. Sloane, The online encyclopedia of integer sequences, (2008).

[9] T. KOSHY: *Discrete Mathematics with Applications*, Elsevier Academic Press, 2004.

[10] K. VENGATESAN, S.B. MAHAJAN, P. SANJEEVIKUMAR, S. MOIN: *The Performance Enhancement of Statistically Significant Bicluster Using Analysis of Variance*, In: Konkani A., Bera R., Paul S. (eds) Advances in Systems, Control and Automation. Lecture Notes in Electrical Engineering, **442** (2017), 671-678. Springer, Singapore. https://doi.org/10.1007/978-981-10-4762-6_64

ASSOCIATE PROFESSOR,
DEPARTMENT OF MATHEMATICS,
D. G. VAISHNAV COLLEGE,
CHENNAI, INDIA.
Email address: rsivaraman1729@yahoo.co.in