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EMBEDDING OF CONCEPT LATTICE INTO ITS MIXED CONCEPT LATTICE

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ABSTRACT. Formal Concept Analysis (FCA) is a method of analysis of object-attribute relational data and knowledge representation. In this paper, the foundations of FCA are extended and in particular mixed concept lattices are studied in depth. We mainly focused on embedding property of a given lattice with respect to its mixed concept lattice. At the end of the article, we characterized the types of lattices which can be embedded in the corresponding mixed concept lattice.

1. INTRODUCTION

Formal Concept analysis (FCA) method was introduced by R.Wille, a German mathematician. Theortical foundations of FCA are built on applied lattice theory [1, 2]. FCA constitutes a very successful mathematical approach to knowledge representation with a rich theoy as well as numerous practical application. FCA is a method of analysis of object-attribute relational data and knowledge representation. For the last two decades, FCA has been used extensively in various disciplines such as software engineering, linguistics, information retrieval, bioinformatics and data mining [3–7].

FCA gives the fundamental data model in a binary relation between a set of objects and attributes, which indicates the presence of a property in an object.

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However some applications require to treat the absence of some property in an object as a negative information to be represented.

Although the concept of mixed concept lattice was initiated in [8], the complete characterization is given by J.M Rodriguez et.,al in [9]. Here, the foundations of FCA are extended and in particular mixed concept lattices are studied in depth. This paper is mainly focused on embedding property of a given lattice with respect to its mixed concept lattice. At the end of the article, we characterized the types of lattices which can be embedded in the corresponding mixed concept lattice.

2. NOTATIONS AND DEFINITIONS

A lattice is a partially ordered set in which every two elements have a supremum (also called a least upper bound or join) and an infimum (also called a greatest lower bound or meet). A lattice which contains both minimum and maximum element is called a bounded lattice [10].

An element a of a lattice L is called an atom of L, if L has the minimum element 0 and 0 is covered by a. A lattice L is said to be atomistic if every non zero element of L is join of atoms contained in it.

An element j is said to be $\vee$-irreducible element in L if $l_1 \vee l_2 = j$ implies $l_1 = j$ or $l_2 = j$ for all $l_1, l_1 \in L$. The set of all $\vee$-irreducible elements in L are denoted by $J(L)$. The $\wedge$ -irreducible element dually defined and $M(L)$ denotes the set of $\wedge$ -irreducible elements of in L. For a given $l \in L$, $(l] = \{x \in L : x \leq l\}$ and $[l) = \{x \in L : l \leq x\}$. Finally, l_J denotes $(l] \cap J(L)$ and l^M denotes $[l) \cap M(L)$.

A lattice L is said to be pseudocomplemented, for each $x \in L$ there is an element $x^* \in L$ such that $x \wedge y = 0$ iff $y \leq x^*$. A lattice L is said to be distributive, it has to satisfy the identity $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for any $x, y, z \in L$.

Let L be a finite lattice with minimum element $\perp$ and maximum element $\top$. For each $l \in L$,its opposite element l^{op} is defied as, $l^{op} = \vee\{x \in L : l \wedge x = \perp\}$ [9]. Observe that $l \wedge l^{op}$ need not be $\perp$.

Let $< L_1, \wedge_1, \vee_1 >$ and $< L_2, \wedge_2, \vee_2 >$ be two lattices. A map $f : L_1 \to L_2$ is said to be,

(i) a meet-homomorphism if $f(a \wedge_1 b) = f(a) \wedge_2 f(b), \forall a, b \in L$
(ii) a join-homomorphism if $f(a \vee_1 b) = f(a) \vee_2 f(b), \forall a, b \in L$

(iii) a homomorphism if f is both a meet-homomorphism and a join-homomorphism.

(iv) a one to one homomorphism is called as embedding.

A formal context or simply a context $T = (G, M, I)$ consists of two sets G and M and a relation I between G and M. The elements of G are called objects of T, the elements of M are called attributes of T [1]. If an objects a has attribute m, we denote it by aIm. A 1 in row a and column m means that the object a has attribute m. A context is represented in terms of binary matrix.

For a set $A \subseteq G$ of objects and $B \subseteq M$ of attributes, $A' = \{m \in M : aIm$ for all $a \in A\}$; $B' = \{g \in G : bIg$ for all $b \in B\}$. A formal concept of the context $T = (G, M, I)$ is a pair (A, B) with $A \subseteq G$, $B \subseteq M$, $A' = B$ and $B' = A$. $L(G, M, I)$ denotes the set of all concepts of the context T. The set of all concepts, when ordered by set-inclusion, satisfies the properties of a complete lattice. The lattice of all concepts is called concept lattice.The context table T is presented in Table 1 and corresponding formal concept lattice is shown in Figure 1.

	1	**2**	**3**	**4**
a	1	0	0	1
b	1	1	0	0
c	1	0	1	0
d	0	1	0	1

TABLE 1. Context table

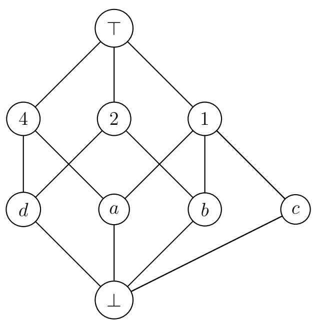

FIGURE 1. Concept lattice

A context (G, M, I) is called clarified, if for any $g, h \in G$ from $g^{'} = h^{'}$, it always follows that $g = h$ and correspondingly, $m^{'} = n^{'}$ implies $m = n$ for all $m, n \in M$. A clarified context (G, M, I) is called row reduced, if every object concept is $\vee - irreducible$ and column reduced, if every attribute concept is $\wedge - irreducible$. A context, which is both row reduced and column reduced is called reduced context.

3. Main Results

Definition 3.1. *A lattice L is said to be $\wedge$-semicomplemented, if for all $m \in M(L)$ we have $m^{op} \wedge m = 0$.*

Example 1. *The lattice depicted in FIGURE 2 is an example of non $\wedge$- semicomplemented lattice.*

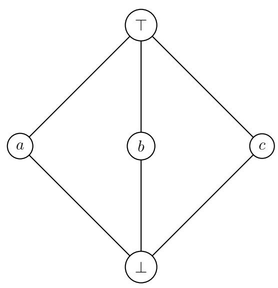

FIGURE 2. non $\wedge$- semicomplemented lattice

Proposition 3.1. *Let L be an atomistic lattice then L is $\wedge$-semicomplimented if and only if L is $\wedge$-complemented.*

Proof. Let L be an atomistic lattice. Clearly, $\wedge$-complemented implies $\wedge$-semicomplimented. Conversely, suppose that L is $\wedge$-semicomplimented. Let $m \in M(L)$. Then we observe that

$$m \vee m^{op} = (\vee\{a \in At(L) | a \leq m\}) \vee (\vee\{m \in At(L) | a \nleq m\}) = \top\,.$$

$\square$

Theorem 3.1. *A finite $\wedge$ -semicomplemented lattice can be embedded in its mixed concept lattice.*

Proof. Given a $\wedge$- semicomplemented lattice L, consider the context $K(L) = \langle J(L), M(L), \leq \rangle$ and the following mapping:
$h : L \to B(K(L))$ with $h(l) = \langle l_J, l^M \cup l^M_{op} \rangle$ where $l^M_{op} = \{\overline{m} | m \in M(L), l \leq m^{op}\}$.

First, we are going to prove that the mapping h is well defined, i.e, $h(l)$ is a concept in the lattice $B(K(L))$.

- To prove $l_J^{\Uparrow} = l^M \cup l^M_{op}$ it is enough to check the following equalities:
  i) $l^M = \{m \in M(L) | j \leq m, \forall j \in l_J\}$.
  Clearly, $l^M = \{m \in M(L) | j \leq m, \forall j \in l_J\}$.
  ii) $l^M_{op} = \{\overline{m} | m \in M(L), m \wedge l = \perp\}$.
  The inclusion $l^M_{op} = \{\overline{m} | m \in M(L), m \wedge l = \perp\} \subseteq \{\overline{m} | m \in M(L), l \leq m^{op}\}$ is straightforward. On the other hand suppose that $l \leq m^{op}$ and $l \wedge m = t$. Then $t \leq m$ and $t \leq l \leq m^{op}$. Since L is $\wedge$-semicomplemented $t \leq m \wedge m^{op} = \perp$, implies $t = \perp$.
- We prove that $l_J = (l^M \cup l^M_{op})^{\Downarrow}$. That is, any $j \in J(L)$satisfies $j \leq l$ if and only if $j \leq m$ for all $m \in l^M$ and $j \not\leq m$ for all $m \in M(L)$ with $l \leq m^{op}$.
  If $j \leq l$, by transitivity, $j \leq m$ for all $m \in l^M$ and, for all $m \in M(L)$ with $l \leq m^{op}$ we have $l \wedge m = \perp$ therefore $j \not\leq m$. Conversely if $j \leq m$ for all $m \in l^M$ then $j \leq \wedge l^M = l$.

Now we prove that h is a lattice homomorphism.

a) Consider $h(l_1 \wedge l_2)$ and focus on extents. We have $(l_1 \wedge l_2)_J = l_{1_J} \cap l_{2_J}$. therefore $h(l_1 \wedge l_2) = h(l_1) \wedge h(l_2)$.
b) To prove that $h(l_1) \vee h(l_2) = h(l_1 \vee l_2)$, for all $l_1, l_2 \in L$, we focus on intents.
   $(l_1^M \cup l^M_{1_{op}}) \cap (l_2^M \cup l^M_{2_{op}}) = (l_1^M \cap l_2^M) \cup (l_1^M \cap l^M_{2_{op}}) \cup (l^M_{1_{op}} \cap l_2^M) \cup (l^M_{1_{op}} \cap l^M_{2_{op}})$.
   As $(l_1^M \cap l^M_{2_{op}}) = \cup (l^M_{1_{op}} \cap l_2^M) = \emptyset$,

$$(l_1^M \cup l^M_{1_{op}}) \cap (l_2^M \cup l^M_{2_{op}}) = (l_1^M \cap l_2^M) \cup (l^M_{1_{op}} \cap l^M_{2_{op}}).$$

   Now we prove that $(l_1 \vee l_2)^M \cup (l_1 \vee l_2)^M_{op} = (l_1^M \cap l_2^M) \cup (l^M_{1_{op}} \cap l^M_{2_{op}})$.
   - If $m \in (l_1 \vee l_2)^M$ then $m \in l_1^M \cap l_2^M$ is straightforward.
     If $\overline{m} \in (l_1 \vee l_2)^M_{op}$ then $l_1 \vee l_2 \leq m^{op}$. Consequently, $l_1 \wedge m \leq m^{op} \wedge m = \perp$ and $l_2 \wedge m \leq m^{op} \wedge m = \perp$ implying $\overline{m} \in l^M_{1_{op}} \cap l^M_{2_{op}}$.
   - On the other hand, if $m \in (l_1^M \cap l_2^M)$ then $m \geq l_1 and m \geq l_2$ consequently $m \geq l_1 \vee l_2$. Therefore $m \in (l_1 \vee l_2)^M$

If $\overline{m} \in (l_{1_{op}}^{M} \cap l_{2_{op}}^{M})$ then $m \wedge l_1 = \perp$ and $m \wedge l_2 = \perp$ implies $l_1 \leq m^{op}$ and $l_2 \leq m^{op}$. Now $(l_1 \vee l_2) \wedge m \leq m^{op} \wedge m = \perp$, hence $\overline{m} \in (l_1 \vee l_2)_{op}^{M}$.

Finally, h is injective is straightforward. for, if $h(l_1) = h(l_2)$ then
$l_1 = \vee l_{1_J} = \vee l_{2_J} = l_2$. □

Theorem 3.2. *Pseudocomplemented lattices are always $\wedge$- semicomplemented lattices.*

Proof. Let L be a pseudocomplemented lattice, then by definition for each $x \in L$ there is an element $x^* \in L$ such that $x \wedge y = 0$ iff $y \leq x^*$.
In particular, for any $x \in M(L)$ we have $x^* \in L$ such that $x \wedge y = 0 \Leftrightarrow y \leq x^*$.
Equivalently, $x^* = \bigvee\{y \in L : x \wedge y = 0\} = x^{op}$.
In this case,clearly $x \wedge x^* = 0$. Proving L is $\wedge$- semicomplemented lattice. □

Remark 3.1. *The converse of the theorem is not true always.*

Corollary 3.1. *Pseudocomplemented lattice can be embedded in its mixed concept lattice.*

Proof. Clearly, by *Theorem 3.2*, pseudocomplemented lattices are $\wedge$- semicomplemented lattices. Hence by *Theorem 3.1*, it follows that pseudocomplemented lattice can be embedded in its mixed concept lattice. □

Corollary 3.2. *Any distributive concept lattice can be embedded in its mixed concept lattice.*

Proof. It is clear that, every finite distributive lattice is pseudocomplemented [11]. Therefore by *corollary 3.1*, it is clear that, any distributive concept lattice can be embedded in its mixed concept lattice. □

4. CONCLUSION

We constructed the concept lattice and its corresponding mixed concept lattice with respect to their context tables. We investigated its embedding property for different types of lattices. Overall, in this paper we characterized the lattices which are embedding into its mixed concept lattices.

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