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A NEW TWO-STEP SIXTH-ORDER ITERATIVE METHOD WITH HIGH-EFFICIENCY INDEX

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ABSTRACT. There are many sixth-order iterative methods to solve the non-linear equations. Here we develop a new two-step iterative method to solve the non-linear equation. For this new method, convergent analysis shows the sixth order convergence and finally, we tested with several problems to show the efficiency of the method over the existing methods.

1. INTRODUCTION

Many complications in engineering and science require solving a nonlinear scalar equation. There are several methods available in the literature for finding the root of non -linear equations.

Newton's method (NR) is one of the well-known methods [2] to obtain the zero of a non-linear equation

$$h(t) = 0\,, \tag{1.1}$$

and is given by

$$t_{n+1} = t_n - \frac{h(t_n)}{h'(t_n)}\,, \quad n = 0, 1, 2, \dots\,, \tag{1.2}$$

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This method converges quadratically and its efficiency index is $\sqrt{2} = 1.414$.

A sixth-order Iterative method (NT) proposed by Neta [1] is given by,

$$\begin{aligned} y_n &= t_n - \frac{h(t_n)}{h'(t_n)} \\ z_n &= y_n - \frac{h(t_n)h(y_n)}{h(t_n) - 2h(y_n)h'(t_n)} \\ x_{n+1} &= z_n - \frac{h(t_n) - h(y_n)h(z_n)}{h(t_n) - 3h(y_n)h'(t_n)}. \end{aligned}$$

A sixth-order convergent method (SG) proposed by Sharma and Guha [2] is given by

$$\begin{aligned} y_n &= t_n - \frac{h(t_n)}{h'(t_n)} \\ z_n &= y_n - \frac{h(t_n)h(y_n)}{h(t_n) - 2h(y_n)h'(t_n)} \\ x_{n+1} &= z_n - \frac{h(z_n) + h(y_n)h(z_n)}{h(z_n) + 3h(y_n)h'(t_n)}. \end{aligned}$$

A Newton-type method with sixth-order convergent (PG) proposed by Parhi and Gupta [7] is given by

$$\begin{aligned} y_n &= t_n - \frac{h(t_n)}{h'(t_n)} \\ z_n &= t_n - \frac{2h(t_n)}{h'(t_n) + h'(y_n)} \\ x_{n+1} &= z_n - \frac{h'(t_n) + h'(y_n)h(z_n)}{3h'(y_n) - h'(x_n)h'(t_n)}. \end{aligned}$$

A new sixth-order method (MF) proposed by Rafiullah [4] is given by

$$\begin{aligned} y_n &= t_n - \frac{h(t_n)}{h'(t_n)} \\ z_n &= y_n - \frac{h(t_n)(h'(t_n) - h'(y_n))}{2(h'(t_n))^2} \\ x_{n+1} &= z_n - \frac{2(h'(t_n)h(z_n)}{4h'(t_n)h'(y_n) - (h'(t_n))^2 - (h'(y_n))^2}. \end{aligned}$$

A sixth-order Newton-type method (YC) proposed by Ham and Chun [9] is given by

$$
\begin{aligned}
y_n &= t_n - \frac{h(t_n)}{h'(t_n)} \\
z_n &= x_n - \frac{h(t_n)}{2}\left(\frac{1}{h'(t_n)} + \frac{1}{h'(y_n)}\right) \\
x_{n+1} &= z_n - \frac{2(h'(t_n)h(z_n)}{h'^2(t_n) - 4h'(t_n)h'(y_n) + h'^2(y_n)}\,.
\end{aligned}
$$

In section 2, we derived the new two-step iterative method and in section 3, its convergence is carried out. In section 4, various comparisons are given with other schemes.

2. Sixth Order Convergent (SRK) Method

Consider t^* is an exact root of (1.1) where $h(t)$ is continuous and has well defined first derivatives. Let t_n be the root of nth approximation of (1.1) and is

$$t^* = t_n + \epsilon_n\,, \tag{2.1}$$

where ϵ_n is the error. Thus, we get

$$h(t^*) = 0\,. \tag{2.2}$$

Writing $h(t*)$ by Taylor's series about t_n, we have

$$h(t^*) = h(t_n) + (t^* - t_n)h'(t_n) + \frac{(t^* - t_n)^2}{2!}h''(t_n) + \dots$$

$$h(t^*) = h(t_n) + \epsilon_n h'(t_n) + \frac{\epsilon_n^2}{2!}h''(t_n) + \dots \tag{2.3}$$

Here higher powers of ϵ_n are neglected that to from ϵ_n^3 onwards. Using (2.2) and (2.3) , we have

$$\epsilon_n^2 h''(t_n) + 2\epsilon_n h^{'}(t_n) + 2h(t_n) = 0$$

$$\epsilon_n = [-2h'(t_n) \pm \sqrt{4h'(t_n) - 8h(t_n)h''(t_n)]} \div h''(t_n)\,. \tag{2.4}$$

On substituting t^* by t_{n+1} in (2.1) and from (2.4), we get $t_{n+1} = t_n - \frac{2h(t_n)}{h'(t_n)}\left(\frac{1}{1+\sqrt{1-2\mu}}\right)$, where, $\mu = \frac{h(t_n)h''(t_n)}{[h'(t_n)]^2}$. Here, the second derivative is considered by Solaiman and Hashim [5] and is given by

$$h''(t_n) = \frac{2}{t_n - 1 - t_n}[3\frac{h(t_n-1) - h(t_n)}{t_n - 1 - t_n} - 2h'(t_n) - h'(t_n - 1)]\,. \tag{2.5}$$

We develop the algorithm by taking (1.2) as the first step and (2.5) as the second step.

2.1. **Algorithm.** The iterative scheme is computed by as x_{n+1}

$$\begin{aligned} z_n &= t_n - \frac{h(t_n)}{h'(t_n)} \\ x_{n+1} &= z_n - \frac{2h(z_n)}{h'(z_n)}(\frac{1}{1+\sqrt{1-2\mu}})\,, \end{aligned}$$

where, $\mu = \frac{h(z_n)h''(z_n)}{[h'(z_n)]^2}$ and

$$h''(z_n) = \frac{2}{t_n - z_n}[3\frac{h(t_n) - h(z_n)}{t_n - z_n} - 2h'(z_n) - h'(t_n)]\,. \tag{2.6}$$

This method (2.6) requires 2 functional evaluations and 2 of its first derivatives.

3. CONVERGENCE CRITERIA

Theorem 3.1. *Let $t_0 \in D$ be a single zero of a sufficiently differentiable function h for an open interval D. If is in the neighborhood of t^*. Then the algorithm (2.6) has sixth-order convergence.*

Proof. Let the single zero of (1.1) be t^* and $t^* = t_n + \epsilon_n$. Thus, $h(t^*) = 0$. By Taylor's series, writing $h(t^*)$ about t_n, we obtain

$$h(t_n) = h'(t^*)(\epsilon_n + c_2\epsilon_n^2 + c_3\epsilon_n^3 + c_4\epsilon_n^4 + ...) \tag{3.1}$$

$$h'(t_n) = h'(t^*)(1 + 2c_2\epsilon_n + c_3\epsilon_n^2 + 4c_4\epsilon_n^3 + ...)\,. \tag{3.2}$$

Dividing (3.1) by (3.2), we get

$$\frac{h(t_n)}{h'(t_n)} = (\epsilon_n - c_2\epsilon_n^2 - (2c_3 - 2c_2^2)\epsilon_n^3 - (3c_4 - 7c_2c_3 + 4c_2^3)\epsilon_n^4 + ...)$$

Now, $z_n = t_n - \frac{h(t_n)}{h'(t_n)}$, we get $z_n = t^* + \omega_n$, where $\omega_n = c_2\epsilon_n^2 + (2c_3 - 2c_2^2)\epsilon_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)\epsilon_n^4 + ...$ Here

$$h''(z_n) = h'(t^*)(2c_2 + 2(3c_2c_3 - c_4)\epsilon_n^2 - 4(3c_2^2c_3 - c_4)\epsilon_n^2 - 4(3c_2^2c_3 - 3c_3^2 - c_2c_4 + c_5)\epsilon_n^3 + ...)$$

and x

$$\frac{h(z_n)}{h'(z_n)} = L_1\epsilon_n^2 + L_2\epsilon_n^3 + L_3\epsilon_n^4 + ... \tag{3.3}$$

where, $L_1 = c_2$, $L_2 = (2c_3 - 2c_2^2)$, $L_3 = (3c_2^3 - 7c_2c_3 + 3c_4)$. From $\mu = \frac{h(z_n)h''(z_n)}{[h'(z_n)]^2}$, we get

$$2\mu = P_1\epsilon_n^2 + P_2\epsilon_n^2 + P_3\epsilon_n^3 + ... , \tag{3.4}$$

where, $P_1 = 4c_2^2$, $P_2 = 4(6c_2c_3^2 - 2c_3c_4 - 6c_2^3c_3 + c_2^2c_4)$, $P_3 = 2(-8c_2^2c_3 + 4c_2c_4 + 2c_2^4)$. From (3.4), on simplification

$$(1 + \sqrt{1 - 2\mu})^{-1} = 2(1 + M_1\epsilon_n^2 + M_2\epsilon_n^3 + M_3\epsilon_n^4 + ..), \tag{3.5}$$

where $M_1 = c_2^2$, $M_2 = 6c_2c_3^2 - 2c_3c_4 - 6c_2^3c_3 + c_2^2c_4$, $M_3 = -4c_2^2c_3 + 4c_2c_4 + 6c_2^4$. Using (3.3) and (3.5) , we get

$$\frac{2h(z_n)}{h'(z_n)}(\frac{1}{1 + \sqrt{1 - 2\mu}}) = L_1\epsilon_n^2 + L_2\epsilon_n^3 + L_3\epsilon_n^4 + L_4\epsilon_n^5 + (L_1M_3 + L_3M_1 + L_2M_3)\epsilon_n^6 + 0(\epsilon_n^7) .$$

Now from (2.6) , we have $x_{n+1} = (L_1M_3 + L_3M_1 + L_2M_3)\epsilon_n^6 + 0(\epsilon_n^7)$. Thus, we derived the convergence of this method which is of sixth-order and its efficiency index is $\sqrt[4]{6} = 1.565$. □

4. ITERATIVE METHOD

We consider some examples proposed by Vatti [8] and Mylapalli [6] and compared our method SRK with NR, NT, SG, PG, MR, and YC methods. The computations are carried out by using mpmath-PYTHON and the number of iterations for these methods is obtained for comparisons such that $|x_{n+1} - x_n| < 10^{-201}$ and $|h(x_{n+1})| < 10^{-201}$.

The test functions and simple zeros are given below:

$$\begin{aligned}
h_1(x) &= \sin(2cosx) - 1 - x^2 + e^{\sin(x^3)} \,, t^* = -0.784895 \\
h_2(x) &= \sin x + \cos x + x \,, t^* = -0.456624 \\
h_3(x) &= x^2 + \sin(\frac{x}{5}) - \frac{1}{4} \,, t^* = 0.409992 \\
h_4(x) &= \cos x - x \,, t^* = 0.739085 \\
h_5(x) &= x^3 - 10 \,, t^* = 2.154434 \\
h_6(x) &= e^x + \cos x \,, t^* = 1.746139 \\
h_7(x) &= \sin^2 x - x^2 + 1 \,, t^* = 1.465577 \\
h_8(x) &= e^{\sin x} - x + 1 \,, t^* = 2.630664 \,.
\end{aligned}$$

Table-IV(a): ANALOGY OF EFFICIENCY

Methods	P	N	EI
NR	2	2	1.414
NT	6	4	1.565
SG	6	4	1.565
PG	6	4	1.565
MR	6	4	1.565
YC	6	4	1.565
SRK	6	4	1.565

Where P is the convergence order, N is the number of functional values per iteration and EI is the Efficiency Index.

Where x_0 is the initial approximation, n is the number of iterations, er is the error, and fv is the functional value.

5. Conclusion

In this method, we introduced a new two-step sixth-order convergent method with efficiency index 1.565. Table IV(a) compares the efficiency of different methods and the computational results in table IV(b) show the dominance of SRK over the three-step methods of NR, NT, SG, PG, MR, and YC in terms of the number of iterations.

h	Method	x_0	n	er	fv	x_0	n	er	fv
h_1	NR	-1	9	1.6(201)	4.1(201)	-1.5	DIVERGENT		
	NT		5	0	4.1(201)		7	0	4.1(201)
	SG		5	8.1(201)	4.1(201)		8	1.6(201)	2.4(201)
	PG		4	3.2(201)	4.1(201)		6	3.2(201)	4.1(201)
	MR		5	2.4(201)	4.1(201)		DIVERGENT		
	YC		6	4.1(201)	4.1(201)		5	4.1(201)	4.1(201)
	SRK		4	2.4(201)	4.1(201)		6	2.4(201)	4.1(201)
h_2	NR	0.1	9	2.4(201)	5.3(201)	-1	8	2.4(201)	5.3(201)
	NT		5	0	5.3(200)		4	4.5(201)	1.8(201)
	SG		5	1.2(201)	5.3(201)		5	1.2(201)	5.3(201)
	PG		5	5.3(201)	1.8(201)		4	4.9(201)	5.3(201)
	MR		5	4.9(201)	5.3(201)		4	5.3(201)	5.3(201)
	YC		7	0	1.8(199)		7	0	5.3(198)
	SRK		4	3.2(201)	5.3(201)		4	5.7(201)	1.8(200)
h_3	NR	0.2	10	2(201)	2.2(201)	1	10	2(201)	2.2(201)
	NT		5	0(201)	2.2(201)		5	4.9(201)	7.7(201)
	SG		6	5.8(201)	7.7(201)		6	1.2(201)	2.2(201)
	PG		5	5.7(201)	2.2(201)		5	5.7(201)	7.7(201)
	MR		5	4.0(201)	2.2(201)		5	5.7(201)	2.2(201)
	YC		6	2.2(201)	2.2(201)		7	0	2.2(201)
	SRK		4	2.8(201)	2.2(201)		4	2.8(201)	2.2(201)
h_4	NR	1.4	9	1.6(201)	2.4(200)	2	9	1.6(201)	2.4(201)
	NT		5	0	2.4(201)		5	0	2.4(201)
	SG		5	3.2(201)	2.4(201)		5	8.1(201)	2.4(201)
	PG		5	6.5(201)	2.4(201)		5	3.2(201)	2.4(201)
	MR		5	4.1(201)	2.4(201)		5	2.4(201)	2.4(201)
	YC		7	0	2.4(201)		7	0	2.4(201)
	SRK		4	2.4(201)	2.4(201)		4	6.5(202)	1.3(200)
h_5	NR	1.9	9	1.6(200)	2.2(199)	5	9	1.6(200)	2.0(199)
	NT		4	6.2(200)	1.2(198)		5	0	2.1(201)
	SG		5	6.5(201)	2.1(199)		6	1.0(200)	2.1(201)
	PG		4	3.2(200)	2.1(199)		5	3.2(200)	2.1(199)
	MR		5	3.2(201)	2.1(201)		5	3.2(201)	2.1(201)
	YC		6	2.1 (201)	2.1(201)		7	0	1.2(198)
	SRK		4	2.2(200)	2.1(199)		5	2.0 (201)	2.1(199)
h_6	NR	1	5	4.9(201)	6.5(201)	2.9	8	4.9(201)	6.5(201)
	NT		5	0	6.5(201)		6	0	6.5(201)
	SG		5	3.2(201)	6.5(201)		6	3.2(201)	6.5(201)
	PG		5	9.7(201)	6.5(201)		5	9.7(201)	6.5(201)
	MR		5	9.7(201)	6.5(201)		7	9.7(201)	6.5(201)
	YC		7	0	6.5(201)		8	0	6.5(201)
	SRK		4	6.5(201)	6.5(201)		5	6.5(201)	6.5(201)

REFERENCES

[1] B. NETA: *A sixth-order family of methods for nonlinear equations*, Int. J.Comput. Math. **7** (1979), 157-161.

[2] J. F. TRAUB: *Iterative Methods for the Solution of Equations*, Chelsea Publishing Company, New York, 1977.

[3] J. R. SHARMA, R. K. GUHA: *A family of modified Ostrowski methods with accelerated sixth-order convergence*, Appl.Math. Comput., **190** (2007), 111-115.

[4] M. RAFIULLAH: *Three-Step Iterative Method with Sixth Order Convergence for Solving Nonlinear Equations*, Int. Journal of Math. Analysis, **4**(50) (2010), 2459 - 2463.

[5] O. S. SOLAIMAN, I. HASHIM: *Efficacy of Optimal Methods for Nonlinear Equations with Chemical Engineering Applications*, Mathematical Problems in Engineering, Volume 2019, 1-11.

[6] R. K. PALLI, M. S. K. MYLAPALLI, C. PRAGATHI, S. RAMADEVI: *An Optimal Three-Step Method for solving non-linear equations*, Journal of Critical Reviews, **7**(6) (2020), 100-103.

[7] S. K. PARHI, D. K. GUPTA: *A sixth order method for nonlinear equations*, Appl. Math.Comput., **203** (2008), 50-55.

[8] V. B. K. VATTI, M. S. K. MYLAPALLI, R. SRI, S. DEB: *Two-step Extrapolated Newton's Method with High-Efficiency Index*, Jour of Adv Research in Dynamical and Control Systems, **9**(5) (2017), 8-15.

[9] Y. HAM, C. CHUN, S. LEE: *Some higher-order modifications of Newton's method for solving nonlinear equations*, Journal of Computational and Applied Mathematics, **222** (2008), 477-486.

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