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ON M-PROJECTIVE CURVATURE TENSOR OF SASAKIAN MANIFOLDS ADMITTING ZAMKOVOY CONNECTION

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ABSTRACT. The purpose of the present paper is to study some properties of Sasakian manifold admitting Zamkovoy connection. We study $M-$ Projectively flat, as well as $\phi-M-$Projectively flat Sasakian manifolds admitting Zamkovoy connection. Moreover, we discuss locally $M-$Projectively $\phi-$symmetric Sasakian manifold with respect to Zamkovoy connection. Besides these, we discuss Sasakian manifolds satisfying $\overline{M}(\xi,U)\circ\overline{R}=0$, where $\overline{M}$ and $\overline{R}$ are $M-$Projective curvature tensor and Riemannian curvature tensor with respect to Zamkovoy connection respectively.

1. INTRODUCTION

The notion of Sasakian structure [12] was introduced by Japanese mathematician S. Sasaki in the year 1960. If a contact metric manifold is normal then the manifold is said to be a Sasakian manifold. In some respect, Sasakian manifolds may be viewed as an odd dimensional analogues of Kâhler manifolds.

In 1971, Pokhariyal and Mishra [9] introduced the notion of $M-$Projective curvature tensor on Riemannian manifold. Properties of the $M-$projective curvature tensor in Sasakian manifolds were studied by R.H. Ojha [7]. Also, in [11], R.H. Ojha studied some properties of $M-$Projective curvature tensor in

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Sasakian manifolds and showed that a $M-$Projective recurrent Sasakian manifold is $M-$Projectively flat iff it is an Einstein manifold. This curvature tensor was further studied by many authors. For instance, see ([5, 6, 8, 10, 14]). The $M-$Projective curvature tensor of rank 3 is given by

$$\begin{aligned} M(X,Y)Z =& R(X,Y)Z - \frac{1}{2(n-1)}\left[S(Y,Z)X - S(X,Z)Y\right] \\ & - \frac{1}{2(n-1)}\left[g(Y,Z)QX - g(X,Z)QY\right] \end{aligned} \tag{1.1}$$

for all $X, Y, Z \in \chi(\mathbf{M})$, set of all vector fields of manifold $\mathbf{M}$, where $R(X,Y)Z$ is the Riemannian curvature tensor of type $(0,3)$, S denotes the Ricci tensor of type $(0,2)$ and Q denotes Ricci operator.

In 2008, The notion of Zamkovoy connection was introduced by S. Zamkovoy [17] for a paracontact manifold. And this connection was defined as a canonical paracontact connection whose torsion is the obstruction of paracontact manifold to be a para Sasakian manifold. For an n-dimensional almost contact metric manifold $\mathbf{M}$ equipped with an almost contact metric structure (ϕ,ξ,η,g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, the Zamkovoy connection is defined by

$$\overline{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi + \eta(X)\phi Y \tag{1.2}$$

for all $X, Y \in \chi(\mathbf{M})$. This connection was further studied by A.M. Blaga [3] in para Kenmotsu manifolds and A. Biswas, K.K. Baishya ([1, 2]) in Sasakian manifolds.

In a Sasakian manifold $\mathbf{M}$ of dimension $n > 2$, the $M-$projective curvature tensor $\overline{M}$ with respect to the Zamkovoy connection $\overline{\nabla}$ is given by

$$\begin{aligned} \overline{M}(X,Y)Z =& \overline{R}(X,Y)Z - \frac{1}{2(n-1)}\left[\overline{S}(Y,Z)X - \overline{S}(X,Z)Y\right] \\ & - \frac{1}{2(n-1)}\left[g(Y,Z)\overline{Q}X - g(X,Z)\overline{Q}Y\right] \end{aligned} \tag{1.3}$$

where $\overline{R}$, $\overline{S}$ and $\overline{Q}$ are Riemannian curvature tensor, Ricci tensor and Ricci Operator with respect to Zamkovoy connection $\overline{\nabla}$ respectively.

Definition 1.1. *An n-dimensional Sasakian manifold M is said to be $\eta-$Einstein manifold if the Ricci tensor is of the form: $S(X,Y) = k_1 g(X,Y) + k_2 \eta(X)\eta(Y)$, for all $X, Y \in \chi(\mathbf{M})$, where k_1 and k_2 are scalars.*

Definition 1.2. *An $n-$dimensional Sasakian manifold $\mathbf{M}$ is said to be M-projectively flat if the $M-$Projective curvature tensor vanishes identically (that is, $\overline{M}=0$).*

Definition 1.3. *An n-dimensional Sasakian manifold $\mathbf{M}$ is said to be $\phi-M-$projectively flat if $g\left(\overline{M}\left(\phi X,\phi Y\right)\phi Z,\phi V\right)=0$ for all X,Y,Z and $V\in\chi\left(\mathbf{M}\right)$.*

Definition 1.4. *An n-dimensional Sasakian manifold $\mathbf{M}$ is said to be $\xi-M-$projectively flat if $\overline{M}\left(X,Y\right)\xi=0$ for all $X,Y\in\chi\left(\mathbf{M}\right)$.*

This paper is organised as follows:

After introduction, a short description of Sasakian manifold is given in Section 2. In Section 3, we have discussed Sasakian manifold admitting Zamkovoy connection $\overline{\nabla}$ and obtained Riemannian curvature tensor $\overline{R}$, Ricci tensor $\overline{S}$, Scalar curvature tensor $\overline{r}$, Ricci operator $\overline{Q}$ with respect to Zamkovoy connection. In Section 4, we have discussed M-Projectively flat Sasakian manifold with respect to the connection. In Section 5, we have discussed $\phi-M$-Projectively flat Sasakian manifold with respect to Zamkovoy connection. In Section 6, we have discussed locally $M-$Projectively $\phi-$symmetric Sasakian manifold. In Section 7, we have discussed Sasakian manifold satisfying $\overline{M}\left(\xi,U\right)\circ\overline{R}=0$. Finally, Section 8 contains a 5-dimensional Sasakian manifold admitting Zamkovoy connection.

2. Preliminaries

Let M be an n-dimensional almost contact metric manifold equipped with an almost contact metric structure (ϕ,ξ,η,g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying

$$\begin{aligned}\phi^2Y &= -Y+\eta\left(Y\right)\xi,\eta(\xi)=1,\eta\left(\phi X\right)=0,\ \phi\xi=0\\ g(\phi X,\phi Y) &= g(X,Y)-\eta(X)\eta(Y)\\ g(X,\phi Y) &= -g(\phi X,Y),\eta(Y)=g(Y,\xi),\forall X,Y\in\chi\left(\mathbf{M}\right).\end{aligned}$$

Then, the almost contact metric manifold M is said to be a contact metric manifold if $g(X,\phi Y)=d\eta(X,Y),\forall X,Y\in\chi\left(\mathbf{M}\right)$.

Also, a contact manifold M is said to be Sasakian manifold if the following conditions hold:

$$\begin{aligned}\nabla_X\xi &= -\phi X,(\nabla_X\phi)Y=g(X,Y)\xi-\eta(Y)X\\ R\left(X,Y\right)\xi &= \eta\left(Y\right)X-\eta\left(X\right)Y,\forall X,Y\in\chi\left(\mathbf{M}\right).\end{aligned}\tag{2.1}$$

In a Sasakian manifold equipped with the structure (ϕ,ξ,η,g), the following relations also hold ([4, 13, 16]):

$$
\begin{aligned}
(\nabla_X \eta) Y &= g(X, \phi Y), R(\xi, X)Y = g(X,Y)\xi - \eta(Y)X \\
S(X,\xi) &= (n-1)\eta(X), R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi \\
Q\xi &= (n-1)\xi, S(X,Y) = g(QX,Y).
\end{aligned} \tag{2.2}
$$

Using (2.1) and (2.2) in (1.2), we get

$$\overline{\nabla}_X Y = \nabla_X Y + g(X,\phi Y)\xi + \eta(Y)\phi X + \eta(X)\phi Y \tag{2.3}$$

with torsion tensor $\overline{T}(X,Y) = 2g(X,\phi Y)\xi$.

Proposition 2.1. *Zamkovoy connection on Sasakian manifold is a metric connection and its torsion is of the form* $\overline{T}(X,Y) = 2g(X,\phi Y)\xi$.

Proposition 2.2. *In Sasakian manifold* ξ *and* g *are parallel with respect to Zamkovoy connection.*

Proposition 2.3. *In Sasakian manifold integral curve of* ξ *is a geodesic with respect to Zamkovoy connection.*

3. Some properties of Sasakian manifold admitting Zamkovoy connection

Let $\overline{R}$ denote the Riemannian curvature tensor with respect to Zamkovoy connection, defined as:

$$\overline{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z,$$

for all $X,Y,Z \in \chi(\mathbf{M})$.

By the help of (2.3), (2.1), and the above equation, we get the Riemannian curvature tensor with respect to Zamkovoy connection as

$$
\begin{aligned}
\overline{R}(X,Y)Z = R(X,Y)Z &- g(Z,\phi X)\phi Y - g(Y,\phi Z)\phi X \\
&- 2g(Y,\phi X)\phi Z + g(X,Z)\eta(Y)\xi - \eta(X)g(Y,Z)\xi \\
&+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X.
\end{aligned} \tag{3.1}
$$

Consequently, one can easily bring out the following results:

$$\overline{S}(Y,Z) = S(Y,Z) + 2g(Y,Z) - (1+n)\eta(Y)\eta(Z) \tag{3.2}$$

$$\overline{S}(Y,\xi) = 0 = \overline{S}(\xi,Z) \tag{3.3}$$

$$\overline{Q}Y = QY + 2Y - (1+n)\eta(Y)\xi \ , \ \overline{Q}\xi = 0 \tag{3.4}$$

$$\overline{r} = r + n - 1 \tag{3.5}$$

$$\overline{R}(X,Y)\xi = 0, \overline{R}(\xi,Y)Z = 0, \overline{R}(X,\xi)Z = 0 \tag{3.6}$$

Proposition 3.1. *Let* M *be an* n*-dimensional Sasakian manifold admitting Zamkovoy connection* $\overline{\nabla}$, *Then*

(i) *The curvature tensor* $\overline{R}$ *of* $\overline{\nabla}$ *is given by (3.1)*
(ii) *The Ricci tensor* $\overline{S}$ *of* $\overline{\nabla}$ *is given by (3.2)*
(iii) *The scalar curvature* $\overline{r}$ *of* $\overline{\nabla}$ *is given by (3.5)*
(iv) *The Ricci tensor* $\overline{S}$ *of* $\overline{\nabla}$ *is symmetric.*

Theorem 3.1. *If a Sasakian manifold* M *is Ricci flat with respect to the Zamkovoy connection, then* M *is an* η*-Einstein manifold.*

Proof. Suppose that the Sasakian manifold is Ricci flat with respect to the Zamkovoy connection. Then from (3.2), we get

$$S(Y,Z) = -2g(Y,Z) + (1+n)\eta(Y)\eta(Z).$$

Which shows that M is an $\eta-$Einstein manifold. $\square$

4. M-PROJECTIVELY FLAT SASAKIAN MANIFOLD WITH RESPECT TO THE ZAMKOVOY CONNECTION $\overline{\nabla}$

Theorem 4.1. *A* $M-$*projectively flat Sasakian manifold* M $(n>2)$ *admitting Zamkovoy connection* $\overline{\nabla}$ *is an* η*-Einstein manifold.*

Proof. Let M be an $n-$ dimensional M-projectively flat Sasakian manifold with respect to Zamkovoy connection, i.e $\overline{M}=0$, then from (1.3) we have

$$\begin{aligned}\overline{R}(X,Y)Z &= \frac{1}{2(n-1)}\left[\overline{S}(Y,Z)X - \overline{S}(X,Z)Y\right] \\ &+ \frac{1}{2(n-1)}\left[g(Y,Z)\overline{Q}X - g(X,Z)\overline{Q}Y\right].\end{aligned} \tag{4.1}$$

Taking inner product with a vector field V in (4.1), we get

$$\overline{R}(X,Y,Z,V) = \frac{1}{2(n-1)}\left[\overline{S}(Y,Z)g(X,V) - \overline{S}(X,Z)g(Y,V)\right] + \frac{1}{2(n-1)}\left[g(Y,Z)\overline{S}(X,V) - g(X,Z)\overline{S}(Y,V)\right].$$

Taking a frame field of $\mathbf{M}$, and contracting over X and V, we get

$$n\overline{S}(Y,Z) = g(Y,Z)\overline{r}. \tag{4.2}$$

Using (3.2), (3.5) in (4.2), we get

$$S(Y,Z) = \frac{1}{n}(r-n-1)g(Y,Z) + (1+n)\eta(Y)\eta(Z).$$

Therefore, M is an η-Einstein manifold. □

Theorem 4.2. *A* $\xi - M-$*projectively flat Sasakian manifold* M $(n>2)$ *admitting Zamkovoy connection* $\overline{\nabla}$ *is an* η*-Einstein manifold.*

Proof. If M $(n>2)$ be $\xi - M-$Projectively flat with respect to Zamkovoy connection, i.e $\overline{M}(X,Y)\xi = 0$ then (1.3) gives

$$0 = \overline{M}(X,Y)\xi = \eta(Y)\overline{Q}X - \eta(X)\overline{Q}Y.$$

Now, taking inner product with a vector field V, we get

$$0 = \eta(Y)\overline{S}(X,V) - \eta(X)\overline{S}(Y,V). \tag{4.3}$$

Setting $Y=\xi$ and using (3.2), (3.3) in (4.3), we have

$$S(X,V) = -2g(X,V) + (1+n)\eta(X)\eta(V). \tag{4.4}$$

Therefore, M is an η-Einstein manifold. □

Corollary 4.1. *If a Sasakian manifold* M $(n>2)$ *admitting Zamkovoy connection* $\overline{\nabla}$ *is* $\xi - M-$*projectively flat then its scalar curvature is constant.*

Let $\{e_i\}$ $(1 \le i \le n)$ be an orthonormal basis of the tangent space at any point of the manifold $\mathbf{M}$. Then putting $Y = Z = e_i$ in the equation (4.4) and taking summation over i, $1 \le i \le n$, we get: $r = -(n-1) =$Constant.

Theorem 4.3. *An* $n-$*dimensional Sasakian manifold is* $\xi - M-$*projectively flat with respect to Zamkovoy connection iff it is so with respect to Levi-Civita connection, provided that the vector fields are horizontal vector fields.*

Proof. From (1.1), (1.3), (3.2) and (3.4), we get

$$
\begin{aligned}
&\overline{M}(X,Y)Z \\
&= M(X,Y)Z - g(Z,\phi X)\phi Y - g(Y,\phi Z)\phi X - 2g(Y,\phi X)\phi Z \\
&+ g(X,Z)\eta(Y)\xi - \eta(X)g(Y,Z)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
&- \frac{1}{2(n-1)}[4g(Y,Z)X - 4g(X,Z)Y] + \frac{(1+n)}{2(n-1)}[\eta(Y)\eta(Z)X \\
&-\eta(X)\eta(Z)Y] + \frac{1}{2(n-1)}[(1+n)g(Y,Z)\eta(X)\xi \\
&- (1+n)g(X,Z)\eta(Y)\xi].
\end{aligned} \tag{4.5}
$$

Setting $Z=\xi$ in (4.5) we have

$$
\begin{aligned}
\overline{M}(X,Y)\xi &= M(X,Y)\xi + \eta(X)Y - \eta(Y)X \\
&+ \frac{(n-3)}{2(n-1)}[\eta(X)Y - \eta(Y)X].
\end{aligned} \tag{4.6}
$$

If X and Y are horizontal vector fields then from (4.6), it follows that $\overline{M}(X,Y)\xi = M(X,Y)\xi$. Hence, the theorem. □

5. $\phi-$M-PROJECTIVELY FLAT SASAKIAN MANIFOLD WITH RESPECT TO THE CONNECTION $\overline{\nabla}$

Theorem 5.1. *A $\phi-$ M-projectively flat Sasakian manifold* $\mathbf{M}$ $(n>2)$ *admitting Zamkovoy connection is an η-Einstein manifold.*

We assume that a Sasakian manifold $\mathbf{M}(n>2)$ is $\phi - M$-Projectively flat with respect to Zamkovoy connection, i.e, $g\left(\overline{M}(\phi X,\phi Y)\phi Z,\phi V\right) = 0$, for all $X,Y,Z,V \in \chi(\mathbf{M})$. Then, in view of (1.3), we have

$$
\begin{aligned}
0 &= \overline{R}(\phi X,\phi Y,\phi Z,\phi V) - \frac{1}{2(n-1)}\left[\overline{S}(\phi Y,\phi Z)g(\phi X,\phi V)\right. \\
&\left.-\overline{S}(\phi X,\phi Z)g(\phi Y,\phi V)\right] - \frac{1}{2(n-1)}\left[g(\phi Y,\phi Z)\overline{S}(\phi X,\phi V)\right. \\
&\left.-g(\phi X,\phi Z)\overline{S}(\phi Y,\phi V)\right],
\end{aligned} \tag{5.1}
$$

where $\overline{R}(\phi X,\phi Y,\phi Z,\phi V) = g\left(\overline{R}(\phi X,\phi Y)\phi Z,\phi V\right)$.

Let $\{e_i,\xi\}$ $(1\le i\le n-1)$ be a local orthonormal basis of the tangent space at any point of the manifold M. Using the fact that $\{\phi e_i,\xi\}$, $(1\le i\le n-1)$ is also a local orthonormal basis and setting $X = V = e_i$ and taking summation over $i(1 \leq i \leq n-1)$ in (5.1), we get

$$\begin{aligned} 0 &= \sum_{i=1}^{n-1} \overline{R}\left(\phi e_i, \phi Y, \phi Z, \phi e_i\right) \\ &- \frac{1}{2(n-1)}\left[\sum_{i=1}^{n-1} \overline{S}\left(\phi Y, \phi Z\right) g\left(\phi e_i, \phi e_i\right) - \sum_{i=1}^{n-1} \overline{S}\left(\phi e_i, \phi Z\right) g\left(\phi Y, \phi e_i\right)\right] \\ &- \frac{1}{2(n-1)}\left[\sum_{i=1}^{n-1} g\left(\phi Y, \phi Z\right) \overline{S}\left(\phi e_i, \phi e_i\right) - \sum_{i=1}^{n-1} g\left(\phi e_i, \phi Z\right) \overline{S}\left(\phi Y, \phi e_i\right)\right]. \end{aligned}$$

From which we obtain: $S(Y,Z) = \left[\frac{n-r+3}{n+1}\right] g(Y,Z) + \left[\frac{n^2-n+r-4}{n+1}\right] \eta(Y)\eta(Z)$. Therefore, M is an η-Einstein manifold.

6. LOCALLY $M-$PROJECTIVELY $\phi-$SYMMETRIC SASAKIAN MANIFOLDS ADMITTING ZAMKOVOY CONNECTION

The notion of local $\phi-$symmetry for Sasakian manifolds was introduced by Takahashi [15]. In this section we consider a locally $M-$Projectively $\phi-$symmetric Sasakian manifolds with respect to Zamkovoy connection.

Definition 6.1. *A Sasakian manifold* $\mathbf{M}$ *is said to be locally* $M-$*Projectively* ϕ-*symmetric with respect to Zamkovoy connection* $\overline{\nabla}$ *if the* $M-$*Projective curvature tensor with respect to Zamkovoy connnection satisfies:* $\phi^2\left(\overline{\nabla}_V \overline{M}\right)(X,Y)Z = 0$, *where* X, Y, Z *and* V *are horizontal vector fields on* $\mathbf{M}$.

Theorem 6.1. *A Sasakian manifold* $\mathbf{M}\,(n>2)$ *is locally* M-*projectively* ϕ-*symmetric with respect to Zamkovoy connection iff it is so with respect to Levi-Civita connection.*

Proof. In view of (2.3), we have

$$\begin{aligned} \left(\overline{\nabla}_V \overline{M}\right)(X,Y)Z &= \left(\nabla_V \overline{M}\right)(X,Y)Z + g\left(V, \phi\overline{M}(X,Y)Z\right)\xi \\ &\quad + \eta\left(\overline{M}(X,Y)Z\right)\phi V + \eta(V)\phi\overline{M}(X,Y)Z. \end{aligned} \tag{6.1}$$

Taking covariant derivative of (4.5), in the direction of V, we get

$$\left(\nabla_V \overline{M}\right)(X,Y)Z = \left(\nabla_V M\right)(X,Y)Z. \tag{6.2}$$

Taking inner product of (4.5) with ξ, we get

$$\begin{aligned} \eta\left(\overline{M}(X,Y)Z\right) &= \eta\left(M(X,Y)Z\right) + g(X,Z)\eta(Y) - \eta(X)g(Y,Z) \\ &\quad + \frac{(n-3)}{2(n-1)}\left[4g(Y,Z)\eta(X) - 4g(X,Z)\eta(Y)\right]. \end{aligned} \tag{6.3}$$

Using (6.2), (6.3) in (6.1), and applying ϕ^2 on both sides, we have: $\phi^2\left(\overline{\nabla}_V\overline{M}\right)(X,Y)Z = \phi^2\left(\nabla_V M\right)(X,Y)Z$, where X, Y, Z and V are horizontal vector fields on $\mathbf{M}$. The last equation shows that the locally $M-$Projectively $\phi-$symmetries in Sasakian manifold with respect to Zamkovoy connection $\overline{\nabla}$ and Levi-Civita connection ∇ are equivalent. $\square$

7. Sasakian manifold admitting Zamkovoy connection $\overline{\nabla}$ satisfying $\overline{M}(\xi,U)\circ\overline{R}=0$

Theorem 7.1. *In an n-dimensional $(n>2)$ Sasakian manifold $\mathbf{M}$ admitting Zamkovoy connection $\overline{\nabla}$, if the condition $\overline{M}(\xi,U)\circ\overline{R}=0$ holds, then the equation: $S^2(Y,U) = -4S(Y,U) - 4g(Y,U) + (1+n)^2\eta(Y)\eta(U)$ is satisfied on $\mathbf{M}$.*

Proof. We consider a Sasakian manifold $\mathbf{M}$ satisfying the condition:

$$\left(\overline{M}(\xi,U)\circ\overline{R}\right)(X,Z)V = 0,$$

where $\overline{M}$, $\overline{R}$ are M-Projective curvature tensor and Riemannian curvature tensor with respect to Zamkovoy connection respectively and $U, Y, Z \in \chi(\mathbf{M})$. Then, we have

$$\begin{aligned} \overline{M}(\xi,U)\overline{R}(X,Z)V &= \overline{R}\left(\overline{M}(\xi,U)X,Z\right)V + \overline{R}\left(X,\overline{M}(\xi,U)Z\right)V \\ &\quad + \overline{R}(X,Z)\overline{M}(\xi,U)V. \end{aligned}$$

Replacing X by ξ and using (3.6), we get

$$0 = \overline{R}\left(\overline{M}(\xi,U)\xi,Z\right)V. \tag{7.1}$$

By the help of (1.3),(3.6) and (7.1), we get: $\overline{R}(QU,Z)V + 2\overline{R}(U,Z)V = 0$.

Taking inner product with Y in the above equation and considering a frame field of $\mathbf{M}$, and contracting over Z and V, we get

$$S^2(Y,U) = -4S(Y,U) - 4g(Y,U) + (1+n)^2\eta(Y)\eta(U).$$

Hence, the theorem is proved. $\square$

8. Example of 5-dimensional Sasakian manifold admitting Zamkovoy connection.

Consider 5-dimensional manifold $\mathbf{M}^5 = \{(x, y, z, u, v) \in R^5\}$, where (x, y, z, u, v) are the std. co-ordinates in R^5

We choose the linearly independent vector fields $I_1 = e^z \frac{\partial}{\partial x}, I_2 = e^z \frac{\partial}{\partial y}, I_3 = \frac{\partial}{\partial z}, I_4 = e^z \frac{\partial}{\partial u}, I_5 = e^z \frac{\partial}{\partial v}$.

Let g be the Riemannian metric defined by $g(I_i, I_j) = 0$, if $i \neq j$ and $g(I_i, I_j) = 1, if\ i = j$, for $i, j = 1, 2, 3, 4, 5$.

Let η be the $1 - form$ defined by $\eta(X) = g(X, I_3)$ for any $X \in \chi(\mathbf{M}^5)$. Let ϕ be the $(1, 1)$ tensor field defined by:

$\phi I_1 = I_1, \phi I_2 = I_2, \phi I_3 = 0, \phi I_4 = I_4, \phi I_5 = I_5$.

Let $X, Y, Z \in \chi(\mathbf{M}^5)$ be given by

$X = x_1 I_1 + x_2 I_2 + x_3 I_3 + x_4 I_4 + x_5 I_5, Y = y_1 I_1 + y_2 I_2 + y_3 I_3 + y_4 I_4 + y_5 I_5,$

$Z = z_1 I_1 + z_2 I_2 + z_3 I_3 + z_4 I_4 + z_5 I_5$.

Then we have

$g(X, Y) = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4 + x_5 y_5, \eta(X) = x_3,$

$g(\phi X, \phi Y) = x_1 y_1 + x_2 y_2 + x_4 y_4 + x_5 y_5$.

Using the linearity of g and ϕ, we get: $\eta(I_3) = 1, \phi^2 X = -X + \eta(X) I_3$ and $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ for all $X, Y \in \chi(M^5)$.

We have: $[I_1, I_3] = -I_1, [I_2, I_3] = -I_2, [I_4, I_3] = -I_4, [I_5, I_3] = -I_5$ and $[\epsilon_I, \epsilon_j] = 0$ for all others I and j.

Let the Levi-Civita connection with respect to g be ∇, then using Koszul formula we get the following

$\nabla_{I_1} I_1 = I_3, \nabla_{I_1} I_3 = -I_1, \nabla_{I_2} I_2 = I_3, \nabla_{I_2} I_3 = -I_2, \nabla_{I_4} I_3 = -I_4,$

$\nabla_{I_4} I_4 = I_3, \nabla_{I_5} I_3 = -I_5, \nabla_{I_5} I_5 = I_3$.

From the above results we see that the structure (ϕ, ξ, η, g) satisfies: $(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \forall X, Y \in \chi(M^5)$, where $\eta(\xi) = \eta(I_3) = 1$. Hence (ϕ, ξ, η, g) is a Sasakian structure and $M^5(\phi, \xi, \eta, g)$ is a 5-dimensional Sasakian manifold.

The non zero components of Riemannian curvature with respect to Levi-Civita connection ∇ are given by:

$R(I_1, I_2) I_1 = I_2, R(I_1, I_2) I_2 = -I_1, R(I_1, I_3) I_1 = I_3, R(I_1, I_3) I_3 = -I_1$

$R(I_1, I_4) I_1 = I_4, R(I_1, I_4) I_4 = -I_1, R(I_1, I_5) I_1 = I_5, R(I_1, I_5) I_5 = -I_1$

$R(I_2, I_1) I_2 = I_1, R(I_2, I_1) I_1 = -I_2, R(I_2, I_3) I_2 = I_3, R(I_2, I_3) I_3 = -I_2$

$R(I_2, I_4) I_2 = I_4, R(I_2, I_4) I_4 = -I_2, R(I_2, I_5) I_2 = I_5, R(I_2, I_5) I_5 = -I_2$

$R(I_3,I_1)I_3=I_1, R(I_3,I_1)I_1=-I_3, R(I_3,I_2)I_3=I_2, R(I_3,I_2)I_2=-I_3$
$R(I_3,I_4)I_3=I_4, R(I_3,I_4)I_4=-I_3, R(I_3,I_5)I_3=I_5, R(I_3,I_5)I_5=-I_3$
$R(I_4,I_1)I_4=I_1, R(I_4,I_1)I_1=-I_4, R(I_4,I_2)I_4=I_2, R(I_4,I_2)I_2=-I_4$
$R(I_4,I_3)I_4=I_3, R(I_4,I_3)I_3=-I_4, R(I_4,I_5)I_4=I_5, R(I_4,I_5)I_5=-I_4$
$R(I_5,I_1)I_5=I_1, R(I_5,I_1)I_1=-I_5, R(I_5,I_2)I_5=I_2, R(I_5,I_2)I_2=-I_5$
$R(I_5,I_3)I_5=I_3, R(I_5,I_3)I_3=-I_5, R(I_5,I_4)I_5=I_4, R(I_5,I_4)I_4=-I_5$

Using (2.3), we obtain

$\overline{\nabla}_{I_1}I_1=2I_3, \overline{\nabla}_{I_2}I_2=2I_3, \overline{\nabla}_{I_3}I_1=I_1, \overline{\nabla}_{I_3}I_2=I_2$
$\overline{\nabla}_{I_3}I_4=I_4, \overline{\nabla}_{I_3}I_5=I_5, \overline{\nabla}_{I_4}I_4=2I_3, \overline{\nabla}_{I_5}I_5=2I_3$

The non zero components of Riemannian curvature tensor with respect to Zamkovoy connection are given by

$\overline{R}(I_1,I_3)I_1=4I_3, \overline{R}(I_2,I_3)I_2=4I_3, \overline{R}(I_4,I_3)I_4=4I_3,$
$\overline{R}(I_5,I_3)I_5=4I_3, \overline{R}(I_3,I_1)I_1=-4I_3, \overline{R}(I_3,I_2)I_2=-4I_3$

Using the above curvature tensors the Ricci tensors with respect to ∇ and ∇^* are: $S(I_1,I_1)=S(I_2,I_2)=S(I_3,I_3)=S(I_4,I_4)=S(I_5,I_5)=-4$. The relation between two scalar curvatures $\overline{r}$ and r in M^5 is obtained as follows $\overline{r}=\sum_{I=1}^{5}\overline{S}(I_i,I_i)=-16=\sum_{I=1}^{5}S(I_i,I_i)+5-1=r+n-1$, which implies the relation (3.5). Similarly, all the results can be verified.

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